J. Korean Math. Soc. **37** (2000), No. 3, pp. 391–410

ON BERNOULLI NUMBERS

MIN-SOO KIM AND JIN-WOO SON

ABSTRACT. In the complex case, we construct a q-analogue of the Riemann zeta function $\zeta_q(s)$ and a q-analogue of the Dirichlet L-function $L_q(s,\chi)$, which interpolate the q-analogue Bernoulli numbers. Using the properties of p-adic integrals and measures, we show that Kummer type congruences for the q-analogue Bernoulli numbers are the generalizations of the usual Kummer congruences for the ordinary Bernoulli numbers. We also construct a q-analogue of the p-adic L-function $L_p(s,\chi;q)$ which interpolates the q-analogue Bernoulli numbers at non positive integers.

0. Introduction

Throughout this paper p will denote a prime number, $\mathbb{Z}_p$ the ring of p-adic integer, $\mathbb{Q}_p$ the field of fractions of $\mathbb{Z}_p$, and $\mathbb{C}_p$ the p-adic completion of the algebraic closure $\overline{\mathbb{Q}}_p$. Let v_p be the p-adic valuation of $\mathbb{C}_p$ normalized so that $|p|_p = p^{-v_p(p)} = p^{-1}$. We denote by $\mathbb{R}_{\geq 0}$ the set consisting of all non-negative real numbers and by $\mathbb{Z}$ the ring of integers and by $\mathbb{Z}_{\geq 0}$ the set of non-negative integers.

The Bernoulli numbers B_k are defined by $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$, where the symbol B_k is interpreted to mean that B^k must be replaced by B_k when we expand the one on the left. This relation can also be written as $e^{(B+1)t} - e^{Bt} = t$ or, equating the same powers of t, as

$$B_0 = 1, \quad (B+1)^k - B_k = \begin{cases} 1, & \text{if } k = 1; \\ 0, & \text{if } k > 1. \end{cases}$$

Received March 31, 1999. Revised December 21, 1999.

2000 Mathematics Subject Classification: 11E95, 11M38.

Key words and phrases: Bernoulli number, Kummer Congruence, p-adic L-function.

The authors wish to acknowledge the financial support of the Korea Research Foundation made in the program year of 1998.

The Bernoulli numbers now may be computed recursively. One finds that $B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}$, etc. The Bernoulli numbers are rational numbers whose denominators are known by the theorem of von-Staudt and Clausen. In particular, if $p > 3$ the numbers $B_2, B_4, B_6, \cdots, B_{p-3}$ are p-integral. Also, Kummer proved the important congruences

$$\frac{B_m}{m} \equiv \frac{B_n}{n} \pmod{p}$$

for positive even integers m, n such that $m \equiv n \not\equiv 0 \pmod{p-1}$ and $p > 3$ (cf. [1], [9, p. 61 Corollary 5.14]). This congruence is cornerstone of the theory of p-adic L-functions. More general version of these congruences are given in the theorem 2 below.

The q-analogue Bernoulli numbers $\mathcal{B}_n(q)$ is defined in [7] by

$$\frac{t}{qe^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(q) \frac{t^n}{n!}$$

for $q \in \mathbf{T}_p$ (see 2). This relation can be determined inductively by

$$\mathcal{B}_0 = 0, \quad q(\mathcal{B}+1)^n - \mathcal{B}_n = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention of replacing $\mathcal{B}^n(q)$ by $\mathcal{B}_n$. And using the p-adic invariant measure on the p-adic integers, he found new relations for q-analogue Bernoulli numbers.

In section 1, we first give some properties of the ordinary Bernoulli numbers. In section 2, we shall define the generalized q-analogue Bernoulli numbers $\mathcal{B}_{n,\chi}(q)$ for any Dirichlet character χ, and we construct a q-analogue of the Riemann zeta function $\zeta_q(s)$ and the Dirichlet L-function $L_q(s, \chi)$. In the last section, we prove that Kummer type congruences for the q-analogue Bernoulli numbers, which are generalizations of the usual Kummer congruences. By using this congruence, we shall construct a p-adic interpolation function of the q-analogue Bernoulli numbers.

1. Ordinary and generalized Bernuolli numbers, and p-adic L-function

Let p be a prime number and let $\mathbb{Q}_p$ be the field of p-adic numbers, that is, the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic metric, given by the p-adic norm

$$|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0},$$

$$|x|_p = \begin{cases} \frac{1}{p^{v_p(x)}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases} \tag{1.1}$$

where $v_p(x) = \alpha$ if $x = p^\alpha \frac{m}{n}, (p,m) = (p,n) = 1,\ m,n \in \mathbb{Z}$. The function $|\cdot|_p$ is multiplicative since $v_p(a_1a_2) = v_p(a_1) + v_p(a_2)$, and satisfies the non-Archimedean property $|x+y|_p \leq \max(|x|_p, |y|_p)$. If $a,b \in \mathbb{Q}_p$, we write $a \equiv b \pmod{p^N}$ if $|a-b|_p \leq p^{-N}$, or equivalently, $(a-b)/p^N \in \mathbb{Z}_p$, that is, if the first nonzero digit in the p-adic expansion of $a-b$ occurs no sooner than the p^N-place. In $\mathbb{Q}_p$, it is not hard to see that all discs of finite radius are compact. Each compact neighborhood $a+p^N\mathbb{Z}_p$ in $\mathbb{Q}_p$ is both open and closed, where $a + p^N\mathbb{Z}_p = \{x \in \mathbb{Z}_p \mid x \equiv a \pmod{p^N}\}$, $0 \leq a \leq p^N - 1$. Let d be a fixed positive integer and p be a fixed odd prime number. We set

$$\begin{aligned} X &= \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned} \tag{1.2}$$

where $0 \leq a \leq dp^N - 1$. In special case if $d = 1$, then $X = \mathbb{Z}_p$ and $X^* = \mathbb{Z}_p^*$. The set of invertible elements in the ring $\mathbb{Z}_p$ is $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$.

We shall now introduce definitions of the p-adic distribution and the p-adic measure (see [3], [5]).

A p-adic *distribution* μ on X means that if $U \subset X$ is the disjoint union of compact-open sets $U_1, U_2, \cdots, U_n$, then $\mu(U) = \mu(U_1) + \mu(U_2) + \cdots + \mu(U_n)$. An $\mathbb{C}_p$-valued *measure* μ on X is a finitely additive bounded map from the set of compact-open $U \subset X$ to $\mathbb{C}_p$.

Now, we give the key example of the p-adic distribution. The p-adic Haar distribution μ_0 is defined by

$$\mu_0(a+p^N\mathbb{Z}_p) := \frac{1}{p^N}. \tag{1.3}$$

This extends to the unique measure (up to a constant multiple) on $\mathbb{Z}_p$. It suffices to check that

$$\sum_{b=0}^{p-1} \mu_0(a+bp^N+p^{N+1}\mathbb{Z}_p) = \mu_0(a+p^N\mathbb{Z}_p).$$

We denote by $UD(\mathbb{Z}_p, \mathbb{C}_p)$ the $\mathbb{C}_p$-Banach algebra of all *uniformly differentiable* functions $f : \mathbb{Z}_p \to \mathbb{C}_p$ under the usual pointwise operations and valuation V where $V(f) = \min\{v(f), R(f)\}$ with

$$R(f) = \inf\left\{ v_p\left(\frac{f(x)-f(y)}{x-y}\right) \;\middle|\; x, y \in \mathbb{Z}_p,\ x \neq y \in \mathbb{Z}_p \right\},$$

where $v(f) = \inf_{x\in\mathbb{Z}_p} v_p(f(x))$ (see [10]).

For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, we have an integral $I_0(f)$ with respect to the invariant measure μ_0 :

$$I_0(f) := \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{N\to\infty} \sum_{a=0}^{p^N-1} f(a)\mu_0(a+p^N\mathbb{Z}_p).$$

LEMMA 1. *For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, we have*

$$I_0(f_{n+1}) = I_0(f_n) + f'(n)$$

where $f_n(x) = f(x+n)$, $n \in \mathbb{Z}_{\geq 0}$. In particular,

$$I_0(f_1) = I_0(f) + f'(0).$$

Proof. The proof is clear. □

If $f' \equiv 0$ on $\mathbb{Z}_p$, then the integral $I_0(f)$ is invariant with respect to shifts, i.e.,

$$\int_{\mathbb{Z}_p} f(x+n)\, d\mu_0(x) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x),$$

where $n \in \mathbb{Z}_{\geq 0}$.

LEMMA 2. (Witt's formula) *For $n \in \mathbb{Z}_{\geq 0}$, we have*

$$B_n = \int_{\mathbb{Z}_p} x^n \, d\mu_0(x),$$

where $\mu_0(x + p^N\mathbb{Z}_p) = \frac{1}{p^N}$.

Proof. This follows easily from Lemma 1. □

Let $\mathbb{Z}_p^*$ be the group of p-adic units, and let $1 + p\mathbb{Z}_p$ is the subgroup of $\mathbb{Z}_p^*$ consisting of all elements of the form $1+pa$, $a \in \mathbb{Z}_p$. Let C be the cyclic group of order $p-1$ consisting of $(p-1)$-th roots of unity in $\mathbb{Q}_p$. Each x in $\mathbb{Z}_p^*$ can be uniquely written in the form $x = \omega(x)\langle x\rangle$, where $\omega(x)$ and $\langle x\rangle$ denote the projections of x on C and $1+p\mathbb{Z}_p$, respectively.

We see easily that if $p > 2$, then

$$(1.4) \qquad \omega(x) = \lim_{n\to\infty} x^{p^n} \quad \text{and} \quad \langle x\rangle^{p-1} = 1 + pq_x, \qquad \forall\, q_x \in \mathbb{Z}_p.$$

Hence, we can deduce from (1.4) that for any $x \in \mathbb{Z}_p^*$,

$$\omega(x) = x(1+pq_x)^{\frac{1}{1-p}}, \qquad \forall\, q_x \in \mathbb{Z}_p.$$

In particular,

$$\sum_{x=1}^{p^n}{}^{*} x^m (1+pq_x)^{\frac{m}{1-p}} = \begin{cases} 0, & \text{if } p-1 \nmid m; \\ p^{n-1}(p-1), & \text{if } p-1 | m, \end{cases}$$

where $\sum^*$ means to take the sum over all integers prime to p in given ranges.

We now prove the following general congruence:

PROPOSITION 1. *For any prime $p > 5$ and $i \geq 1$, we have*

$$B_{i(p-1)} \equiv 1-\frac{1}{p}-i(1-\alpha_p)+\frac{i(i-1)}{2}\left(B_{2(p-1)}+\frac{1}{p}+1-2\alpha_p\right)(\text{mod } p^3),$$

where $\alpha_p = (1+pB_{p-1})/p$.

Proof. From Lemma 2 with $n = i(p-1), i \geq 1$ and $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$, we obtain

$$\begin{aligned}(1-p^{i(p-1)-1})B_{i(p-1)} &= \lim_{N\to\infty}\frac{1}{p^N}\sum_{i=1}^{p^N}{}^{*}\, x^{i(p-1)} \\ &= \int_{\mathbb{Z}_p^*} x^{i(p-1)}d\mu_0(x).\end{aligned}$$

Let p be a prime with $p > 5, i \geq 1$. Then by the von Staudt-Clausen theorem, $pB_{i(p-1)} \in \mathbb{Z}_p$, and we find clearly that

$$B_{i(p-1)} \equiv \int_{\mathbb{Z}_p^*} x^{i(p-1)}d\mu_0(x) \quad (\text{mod } p^3).$$

Now, we put the Fermat quotient q_x by $x^{p-1} = 1 + pq_x$ for any integer $x \in \mathbb{Z}_p^*$. Then for any $x \in \mathbb{Z}_p^*$, we have

$$x^{i(p-1)} \equiv 1 + ipq_x + \frac{i(i-1)}{2}p^2q_x^2 \quad (\text{mod } p^3).$$

By using this congruence, we show that

$$\begin{aligned}&\int_{\mathbb{Z}_p^*} x^{i(p-1)}d\mu_0(x) \\ &\equiv \int_{\mathbb{Z}_p^*} d\mu_0(x) + ip\int_{\mathbb{Z}_p^*} q_x d\mu_0(x) + \frac{i(i-1)}{2}p^2\int_{\mathbb{Z}_p^*} q_x^2 d\mu_0(x) \ (\text{mod } p^3) \\ &\equiv 1-\frac{1}{p}+ip\int_{\mathbb{Z}_p^*}\frac{x^{p-1}-1}{p}d\mu_0(x) \\ &\quad + \frac{i(i-1)}{2}p^2\int_{\mathbb{Z}_p^*}\frac{(x^{p-1}-1)^2}{p^2}d\mu_0(x) \ (\text{mod } p^3) \\ &\equiv 1-\frac{1}{p}-i(1-\alpha_p)+\frac{i(i-1)}{2}\left(B_{2(p-1)}+\frac{1}{p}+1-2\alpha_p\right) \ (\text{mod } p^3),\end{aligned}$$

where $\alpha_p = (1 + pB_{p-1})/p$. □

On the other hand, let $B_k(x)$ be the Bernoulli polynomials defined by the power series $\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}$. Then it is clear that the Bernoulli numbers are constant term of the Bernoulli polynomials, that is, $B_k = B_k(0)$. We can easily find that the relation $B_k(x) = \sum_{i=0}^{k}\binom{k}{i}B_i x^{k-i} = \sum_{i=0}^{k}\binom{k}{i}B_{k-i}x^i$. It is known that for any rational integer $n \geq 1$ and $k \geq 0$,

$$B_k(x) = n^{k-1}\sum_{i=0}^{n-1} B_k\left(\frac{x+i}{n}\right). \tag{1.5}$$

The above Bernoulli polynomials are closely related to the p-adic distributions.

Let $f : \mathbb{Z} \to \mathbb{C}$ be a function with period d, the positive integer, i.e., $f(j) = f(k)$ for $j \equiv k \pmod{d}$. The generalized Bernoulli numbers $B_{m,f}$, $m \geq 0$ is defined by

$$\sum_{0\leq a<d} \frac{f(a)e^{at}t}{e^{dt}-1} = \sum_{m=0}^{\infty} B_{m,f}\frac{t^m}{m!}. \tag{1.6}$$

Then $B_{m,f} = d^{m-1}\sum_{a=0}^{d-1} f(a)B_m\left(\frac{a}{d}\right)$, where $B_m(x)$ is the Bernoulli polynomials.

For $k \in \mathbb{Z}_{\geq}0$, $\mu_{B,k}$ (cf. [5]) is defined by

$$\mu_{B,k}(a + dp^N\mathbb{Z}_p) := (dp^N)^{k-1}B_k\left(\frac{a}{dp^N}\right). \tag{1.7}$$

Then we can show that $\mu_{B,k}$ extends uniquely to the distribution on X.

Let $f = \chi$ be a primitive Dirichlet character with the conductor d. In the p-adic case, the generalized Bernoulli numbers $B_{m,\chi}$ can be represented by the integral form as follow:

LEMMA 3. *For a Dirichlet character χ of conductor d, we have the integral representations*

(1) $\int_X \chi(x)\,d\mu_{B,k}(x) = B_{k,\chi}$;
(2) $\int_{pX} \chi(x)\,d\mu_{B,k}(x) = \chi(p)p^{k-1}B_{k,\chi}$.

In particular,

$$B_{k,\chi} = \lim_{N\to\infty} \frac{1}{dp^N} \sum_{a=0}^{dp^N-1} \chi(a)a^k.$$

Proof. The proof is clear. □

We rewrite the generalized Bernoulli numbers as integral forms

$$B_{k,\chi} = \lim_{N\to\infty} \frac{1}{dp^N} \sum_{a=0}^{dp^N-1} \chi(a)a^k = \int_X \chi(x)x^k \, d\mu_0(x),$$

where $\mu_0(a + dp^N\mathbb{Z}_p) = \frac{1}{dp^N}$.

Therefore we obtain the following:

LEMMA 4. *For $k \geq 0$, we have*

$$\int_X \chi(x)\, d\mu_{B,k}(x) = \int_X \chi(x)x^k \, d\mu_0(x).$$

This can plainly be rewritten as $d\mu_{B,k}(x) = x^k d\mu_0(x)$.

Now, we can show that

$$\begin{aligned} B_{k,\chi} &= \lim_{N\to\infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N}{}^{*} \chi(x)x^k + \lim_{N\to\infty} \frac{1}{dp^N} \sum_{y=1}^{dp^{N-1}} \chi(py)(py)^k \\ &= \lim_{N\to\infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N}{}^{*} \chi(x)x^k + p^{k-1}\chi(p)B_{k,\chi}. \end{aligned}$$

We thus define

$$L_p(r,\chi) := \frac{1}{r-1} \int_{X^*} \chi(x)x^{1-r} \, d\mu_0(x), \tag{1.8}$$

where $r \in \mathbb{Z},\ r \neq 1$.

From (1.8) we see that

$$L_p(1-k,\chi) = -\frac{1}{k}\lim_{N\to\infty}\frac{1}{dp^N}\sum_{x=1}^{dp^N}{}^{*}\chi(x)x^k$$
$$= -\frac{1}{k}(1-p^{k-1}\chi(p))B_{k,\chi}.$$

If χ is the constant function with a period $d=1$, then we have

$$L_p(1-k,1) = -(1-p^{k-1})\frac{B_k}{k} = \zeta_p(1-k),$$

where ζ_p is defined in [5, p. 44].

Note that N. Koblitz (see [5]) considered the p-adic ζ-function having the value $-(1-p^{k-1})\frac{B_k}{k}$ at the positive integer k, i.e.,

$$\zeta_p(1-k) = \frac{1}{\alpha^{-k}-1}\int_{\mathbb{Z}_p^*} x^{k-1}\,\mu_{1,\alpha}, \tag{1.9}$$

where α is any rational integer not equal to 1, not divisible by p, and

$$\mu_{1,\alpha}(a+p^N\mathbb{Z}_p) = \frac{1}{\alpha}\left[\frac{\alpha a}{p^N}\right]_g + \frac{1/\alpha - 1}{2}$$

for the greatest integer function $[\cdot]_g$.

2. q-Analogue of the Bernoulli numbers and the Dirichlet L-function

The q-analogue Bernoulli numbers $\mathcal{B}_n(q)$ and the q-analogue Bernoulli polynomials $\mathcal{B}_n(x;q)$ may be defined by means of the generating functions

$$\frac{t}{qe^t-1} = \sum_{n=0}^{\infty}\mathcal{B}_n(q)\frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{qe^t-1} = \sum_{n=0}^{\infty}\mathcal{B}_n(x;q)\frac{t^n}{n!}, \tag{2.1}$$

respectively. The q-analogue Bernoulli numbers $\mathcal{B}_n(q)$ also can be written as $qe^{(\mathcal{B}(q)+1)t} - e^{\mathcal{B}(q)t} = t$, or if we equate the powers of t, then

$$\mathcal{B}_0 = 0, \quad q(\mathcal{B}+1)^n - \mathcal{B}_n = \begin{cases} 1, & \text{if } n=1; \\ 0, & \text{if } n>1, \end{cases} \tag{2.2}$$

with the usual convention of replacing $\mathcal{B}^n(q)$ by $\mathcal{B}_n$. We can easily find the following relation:

$$\mathcal{B}_n(x;q) = \sum_{i=0}^{n} \binom{n}{i} \mathcal{B}_i x^{n-i}, \quad \mathcal{B}_n(0;q) = \mathcal{B}_n. \tag{2.3}$$

If $q \neq 1$, then for $n \geq 1$

$$\frac{\mathcal{B}_n(q)}{n} = \frac{q^{-1}}{1-q^{-1}} H_{n-1}(q^{-1}), \tag{2.4}$$

where $H_{n-1}(q^{-1})$ means the $(n-1)$-th Euler numbers (cf. [7], [8]). If $q = 1$, then $\mathcal{B}_n(q) = B_n$, where B_n is the ordinary Bernoulli numbers.

We can define a q-analogue zeta function $\zeta_q(s)$: For $s \in \mathbb{C}$,

$$\zeta_q(s) := \sum_{n=1}^{\infty} \frac{q^n}{n^s}, \tag{2.5}$$

which converges for all s if $|q| < 1$, for $\mathrm{Re}(s) > 0$ if $|q| = 1, q \neq 1$, and for $\mathrm{Re}(s) > 1$ if $q = 1$. We easily see that $\zeta_q(s)$ can be extended to the whole s-plane by the contour integral.

The values of $\zeta_q(s)$ at non-positive integers are obtained by the following proposition:

PROPOSITION 2. *For any positive integer k, we have*

$$\zeta_q(1-k) = \begin{cases} -q\mathcal{B}_1(q), & \text{if } k = 1; \\ -\frac{\mathcal{B}_k(q)}{k}, & \text{if } k > 1. \end{cases}$$

Proof. It is clear from (2.5). □

COROLLARY 1. *For any positive integer $k > 1$,*

$$\zeta_q(1-k) = \frac{1}{1-q} H_{k-1}(q^{-1}),$$

where $H_{k-1}(q^{-1})$ means the $(n-1)$-th Euler numbers.

Proof. It is clear from (2.4). □

COROLLARY 2. *For* $n \geq 1$, $\zeta_{\frac{1}{2}}(-n) = -\frac{\mathcal{B}_{n+1}(\frac{1}{2})}{n+1} = \sum_{k=1}^{\infty} \frac{k^n}{2^k}$ *which satisfy the recurrence relation*

$$\zeta_{\frac{1}{2}}(-n) = 1 + \sum_{j=0}^{n-1} \binom{n}{j} \zeta_{\frac{1}{2}}(-j).$$

REMARK. The q-analogue ζ-function $\zeta_q(s)$ is related to the polylogarithm $\mathrm{Li}_n(z)$, e.g.,

$$\zeta_{\frac{1}{2}}(-n) = \mathrm{Li}_{-n}\left(\frac{1}{2}\right),$$

where $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$, $|z| < 1$.

Let χ be a primitive Dirichlet character of conductor d. We define the generalized q-analogue Bernoulli numbers by

$$\sum_{a=0}^{d-1} \frac{\chi(a) q^a t e^{at}}{q^d e^{dt} - 1} = \sum_{k=0}^{\infty} \mathcal{B}_{k,\chi}(q) \frac{t^n}{k!}. \tag{2.6}$$

By using the definition of $\zeta_q(s)$, we can define a q-L-function $L_q(s,\chi)$: For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$,

$$L_q(s,\chi) := \sum_{n=1}^{\infty} \frac{q^n \chi(n)}{n^s}, \qquad |q| \leq 1. \tag{2.7}$$

We easily see that $L_q(s,\chi)$ can be extended to the whole s-plane by the contour integral.

The values of $L_q(s,\chi)$ at non-positive integers can be obtained similarly as in the q-analogue zeta function.

PROPOSITION 3. *For any positive integer* k, *we have*

$$L_q(1-k,\chi) = -\frac{\mathcal{B}_{k,\chi}(q)}{k}.$$

Proof. From the definition of generalized q-analogue Bernoulli numbers $\mathcal{B}_{k,\chi}(q)$

$$\mathcal{B}_{k,\chi}(q) = \sum_{a=0}^{d-1} q^a \chi(a) d^{k-1} \mathcal{B}_k\Big(\frac{a}{d}; q^d\Big).$$

Therefore

$$\begin{aligned}
\mathcal{B}_{k,\chi}(q) &= k \sum_{a=0}^{d-1} q^a \chi(a) \left(\frac{d}{dt}\right)^{k-1} \frac{e^{at}}{q^d e^{dt} - 1}\bigg|_{t=0} \\
&= -k \left(\frac{d}{dt}\right)^{k-1} \sum_{l=0}^{\infty} \sum_{a=0}^{d-1} q^{ld+a} \chi(ld+a) e^{(ld+a)t}\bigg|_{t=0} \\
&= -k \left(\frac{d}{dt}\right)^{k-1} \sum_{n=1}^{\infty} q^n \chi(n) e^{nt}\bigg|_{t=0} \\
&= -k L_q(1-k, \chi),
\end{aligned}$$

since $\left(\frac{d}{dt}\right)^{k-1} e^{nt}\Big|_{t=0} = n^{k-1}$. This implies that for $k \geq 1$,

$$L_q(1-k, \chi) = -\frac{\mathcal{B}_{k,\chi}(q)}{k}.$$

□

3. q-Analogue of the p-adic L-functions and congruences

In this section, we have some congruences for the generalized q-analogue Bernoulli numbers in a method similar to [4], [5].

Let C_{p^n} be the cyclic group consisting of all p^n-th roots of unity in $\mathbb{C}_p$ for each $n \geq 0$ and $\mathbf{T}_p$ the direct limit of C_{p^n} with respect to the natural homomorphisms, i.e.,

$$\mathbf{T}_p = \{\omega \in \mathbb{C}_p \mid \omega^{p^n} = 1 \text{ for some } n \geq 0\}.$$

Hence $\mathbf{T}_p$ is the union of all C_{p^n} with discrete topology (see [10]).

By Lemma 1, if $f(x) = q^x e^{xt} \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, then

$$(qe^t - 1) I_0(q^x e^{xt}) = \ln q + t.$$

For $q \in \mathbf{T}_p$, we have

$$(qe^t - 1)I_0(q^x e^{xt}) = t.$$

Hence

$$I_0(q^x e^{xt}) = \frac{t}{qe^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(q)\frac{t^n}{n!}.$$

Therefore we obtain the following:

PROPOSITION 4. *We have the Witt's formula*

$$\mathcal{B}_n(q) = \int_{\mathbb{Z}_p} q^x x^n \, d\mu_0(x)$$

for $q \in \mathbf{T}_p$ with $n \geq 0$.

For $q \in \mathbf{T}_p$ and $n \geq 0$, the q-analogue Bernoulli numbers $\mathcal{B}_n(q)$ and the q-analogue Bernoulli polynomials are represented by

$$\mathcal{B}_n(q) = \int_{\mathbb{Z}_p} q^x x^n \, d\mu_0(x) \tag{3.1}$$

and

$$\mathcal{B}_n(x; q) = \int_{\mathbb{Z}_p} q^t (x + t)^n \, d\mu_0(t), \tag{3.2}$$

respectively. Let $\mathcal{B}_{n,\chi}(q)$ denote the n-th generalized q-analogue Bernoulli numbers belonging to the Dirichlet character χ with the conductor d. Then we have a q-analogue of Witt's formula in the p-adic cyclotomic field $\mathbb{Q}_p(\chi)$ as follow:

$$\mathcal{B}_{n,\chi}(q) = \lim_{N\to\infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N} \chi(x) x^n q^x \qquad n \geq 0. \tag{3.3}$$

Hence the above expression of $\mathcal{B}_{n,\chi}(q)$ is equal to

$$\lim_{N\to\infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N}{}^{*} \chi(x) x^n q^x + \lim_{N\to\infty} \frac{1}{dp^N} \sum_{y=1}^{dp^{N-1}} \chi(py)(py)^n q^{py}.$$

Using (3.3), we deduce that

$$\mathcal{B}_{n,\chi}(q) = \lim_{N\to\infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N}{}^{*} \chi(x)x^n q^x + p^{n-1}\chi(p)\mathcal{B}_{n,\chi}(q^p).$$

We plainly have

$$\mathcal{B}_{n,\chi}(q) - p^{n-1}\chi(p)\mathcal{B}_{n,\chi}(q^p) = \lim_{N\to\infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N}{}^{*} \chi(x)x^n q^x. \tag{3.4}$$

For $r \in \mathbb{Z}$, $r \neq 1$, let us define

$$\zeta_{p,q}(r) := \frac{1}{r-1} \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=1}^{p^N}{}^{*} \frac{q^x}{x^{r-1}}.$$

Then by Proposition 4, we have

$$\zeta_{p,q}(1-k) = -\frac{1}{k}(\mathcal{B}_k(q) - p^{k-1}\mathcal{B}_k(q^p)).$$

If $q = 1$, then $\zeta_{p,q}(1-k)$ is the p-adic ζ-function $\zeta_p(1-k)$ in (1.9). Let us also define

$$L_p(r, \chi; q) := \frac{1}{r-1} \lim_{N\to\infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N}{}^{*} \chi(x)\langle x\rangle^{1-r} q^x, \tag{3.5}$$

where $\langle x\rangle = \frac{x}{\omega(x)}$.

We have the following:

Proposition 5. *For $k \geq 1$ and $q \in \mathbf{T}_p$,*

$$L_p(1-k, \chi\omega^k; q) = -\frac{1}{k}(\mathcal{B}_{k,\chi}(q) - p^{k-1}\chi(p)\mathcal{B}_{k,\chi}(q^p)).$$

REMARK. Put

$$\mathbf{U}_p := \left\{ q \in \mathbb{C}_p \,\middle|\, |q-1|_p < p^{-\frac{1}{p-1}} \right\}.$$

For a p-adic number $q \in \mathbf{U}_p$, $\mathcal{B}_n(q)$ can be related to the another type of Bernoulli numbers $B_n(q)$, which satisfy the recursive relations

$$B_0(q) = 1, \quad q(B(q)+1)^n - B_n(q) = \begin{cases} \frac{q-1}{\log q}, & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases}$$

Our aim is to construct an analogue of the distribution $\mu_{B,k}$ in (1.7). The desired distribution of q-analogue Bernoulli numbers is given by next lemma:

LEMMA 5. (1) *For any rational integer $m \geq 1$ and $k \geq 0$,*

$$\mathcal{B}_k(x;q) = m^{k-1} \sum_{i=0}^{m-1} q^i \mathcal{B}_k\!\left(\frac{x+i}{m}; q^m\right).$$

(2) *Let $q \in \mathbb{C}_p$. For any positive integer N, k and d, let $\mu_{\mathcal{B},k;q}$ be defined by*

$$\mu_{\mathcal{B},k;q}(a + dp^N\mathbb{Z}_p) = (dp^N)^{k-1} q^a \mathcal{B}_k\!\left(\frac{a}{dp^N}; q^{dp^N}\right).$$

Then $\mu_{\mathcal{B},k;q}$ extends uniquely to a distribution on X.

Proof. The proof is clear. □

Hereafter, we assume that for $q \in \mathbb{C}_p$

$$\left|1 - q^{dp^N}\right|_p \geq 1 \quad \text{and} \quad |q|_p \leq 1$$

for $N \geq 0$ (cf. [2, p. 459 Proposition 2]).

PROPOSITION 6. *$|\mu_{\mathcal{B},1;q}(U)|_p \leq M$ for all compact-open $U \subset X$, where M is some constant.*

Proof. Applying Lemma 5 (2) with $k=1$, we obtain

$$\mu_{\mathcal{B},1;q}(a+dp^N\mathbb{Z}_p)=q^a\frac{1}{q^{dp^N}-1}.$$

On the other hand, since every compact-open U is a finite disjoint union of intervals $a+dp^N\mathbb{Z}_p$ and $|1-q^{dp^N}|_p\geq 1$, we may conclude that $|\mu_{\mathcal{B},1;q}(U)|_p\leq \max|\mu_{\mathcal{B},1;q}(a+dp^N\mathbb{Z}_p)|_p\leq M$ for some constant M. □

Now, we will give a relation between $\mu_{\mathcal{B},k;q}$ and $\mu_{\mathcal{B},1;q}$.

It is not hard to show that any open subset which is compact is a finite union of compact-open sets of the form $a+dp^N\mathbb{Z}_p$. Therefore, we obtain the following:

THEOREM 1. (1) For all $k=1,2,\cdots$

$$\mu_{\mathcal{B},k;q}(a+dp^N\mathbb{Z}_p)\equiv ka^{k-1}\mu_{\mathcal{B},1;q}(a+dp^N\mathbb{Z}_p)\pmod{p^N},$$

where both sides of this congruence lie in $\mathbb{Z}_p$.
(2) $\mu_{\mathcal{B},k;q}$ is a measure for all $k=1,2,\cdots$.

Proof. (1) By using Lemma 5 (2) and the equation (2.3), we obtain

$$\begin{aligned}
&\mu_{\mathcal{B},k;q}(a+dp^N\mathbb{Z}_p)\\
&=(dp^N)^{k-1}q^a\mathcal{B}_k\left(\frac{a}{dp^N};q^{dp^N}\right)\\
&=(dp^N)^{k-1}q^a\sum_{i=0}^{k}\binom{k}{i}\mathcal{B}_i\left(q^{dp^N}\right)\left(\frac{a}{dp^N}\right)^{k-i}\\
&=(dp^N)^{k-1}q^a\left(k\mathcal{B}_1\left(q^{dp^N}\right)\left(\frac{a}{dp^N}\right)^{k-1}+\binom{k}{2}\mathcal{B}_2\left(q^{dp^N}\right)\left(\frac{a}{dp^N}\right)^{k-2}+\cdots\right)\\
&\equiv q^aka^{k-1}\frac{1}{q^{dp^N}-1}\pmod{p^N}.
\end{aligned}$$

This completes the proof of our assertion (1).

For the proof of (2), we have to show that $\mu_{\mathcal{B},k;q}(a + dp^N\mathbb{Z}_p)$ is bounded. By the above assertion (1), we have

$$\begin{aligned}|\mu_{\mathcal{B},k;q}(a+dp^N\mathbb{Z}_p)|_p &= |xp^N + ka^{k-1}\mu_{\mathcal{B},1;q}(a+dp^N\mathbb{Z}_p)|_p \\ &\quad (\text{ for some } x \in \mathbb{Z}_p) \\ &\le \max\{|xp^N|_p, |ka^{k-1}\mu_{\mathcal{B},1;q}(a+dp^N\mathbb{Z}_p)|_p\} \\ &\le \max\{|p^N|_p, |ka^{k-1}\mu_{\mathcal{B},1;q}(a+dp^N\mathbb{Z}_p)|_p\} \\ &< \infty.\end{aligned}$$

□

Note that $\mu_{\mathcal{B},1;q}(a + dp^N\mathbb{Z}_p) = \frac{q^a}{q^{dp^N}-1}$ is the same as Koblitz measure (see [2]).

Corollary 3. *Let $f : X \to X$ be the function given by $f(x) = x^{k-1}$ for a fixed positive integer k. Then for all compact-open $U \subset X$,*

$$\int_U 1\, \mu_{\mathcal{B},k;q}(x) = k \int_U f\, \mu_{\mathcal{B},1;q}(x).$$

Proof. It follows from Theorem 1. □

Define the n-th generalized q-analogue Bernoulli numbers belonging to the character χ by

$$\mathcal{B}_{n,\chi}(q) = \sum_{a=0}^{d-1} q^a\chi(a)d^{n-1}\mathcal{B}_n\Big(\frac{a}{d}; q^d\Big)\,. \tag{3.6}$$

We express the the generalized q-analogue Bernoulli numbers as integral forms over X, by using the measure $\mu_{\mathcal{B},k;q}(x)$.

Proposition 7. *Let χ be a primitive Dirichlet character of conductor d. Then*

(1) $\int_X \chi(x)\, \mu_{\mathcal{B},k;q}(x) = \mathcal{B}_{k,\chi}(q)$.
(2) $\int_{pX} \chi(x)\, \mu_{\mathcal{B},k;q}(x) = \chi(p)p^{k-1}\mathcal{B}_{k,\chi}(q^p)$.
(3) $\int_X \chi(x)\, \mu_{\mathcal{B},k;q^{\frac{1}{\alpha}}}(\alpha x) = \chi\big(\frac{1}{\alpha}\big)\, \mathcal{B}_{k,\chi}\Big(q^{\frac{1}{\alpha}}\Big)$.
(4) $\int_{pX} \chi(x)\, \mu_{\mathcal{B},k;q^{\frac{1}{\alpha}}}(\alpha x) = \chi\big(\frac{p}{\alpha}\big)\, p^{k-1}\mathcal{B}_{k,\chi}\big(q^{\frac{p}{\alpha}}\big)$.

Proof. It follows immediately from (3.6) and Lemma 5 (2). □

Using Proposition 7, we have

$$\begin{aligned}\int_{X^*} \chi(x)\, \mu_{\mathcal{B},k;q}(x) &= \int_X \chi(x)\, \mu_{\mathcal{B},k;q}(x) - \int_{pX} \chi(x)\, \mu_{\mathcal{B},k;q}(x) \\ &= \mathcal{B}_{k,\chi}(q) - \chi(p)p^{k-1}\mathcal{B}_{k,\chi}(q^p).\end{aligned} \tag{3.7}$$

For the simplicity, we now set the operator $\chi^a = \chi^{a,k;q}$ on $f(q)$ by $\chi^a f(q) = a^{k-1}\chi(a)f(q^a)$ for some positive integer a (see [4], [7]). Then

$$\int_{X^*} \chi(x)\, \mu_{\mathcal{B},k;q}(x) = (1-\chi^p)\mathcal{B}_{k,\chi}(q).$$

Finally, we set $\langle x\rangle := \frac{x}{\omega(x)}$, where ω is the first kind Teichmüller character and $\langle x\rangle^{p^N} \equiv 1 \pmod{p^N}$. Put $\chi_k = \chi\omega^{-k}$. By Corollary 3, we have

$$\begin{aligned}\int_{X^*} \chi_k(x)\, \mu_{\mathcal{B},k;q}(x) &= \int_{X^*} \chi_k(x) k x^{k-1}\, \mu_{\mathcal{B},1;q}(x) \\ &= \int_{X^*} \chi_1(x)\langle x\rangle^{k-1} k\, \mu_{\mathcal{B},1;q}(x).\end{aligned}$$

If $k_1 \equiv k_2 \pmod{(p-1)p^N}$, then (cf. [5, §II. 6])

$$\begin{aligned}(1-\chi_{k_1}^p)\mathcal{B}_{k_1,\chi_{k_1}}(q) &= \int_{X^*} \chi_{k_1}(x)\, \mu_{\mathcal{B},k_1;q}(x) \\ &= \int_{X^*} \chi_1(x)\langle x\rangle^{k_1-1} k_1\, \mu_{\mathcal{B},1;q}(x) \\ &\equiv \int_{X^*} \chi_1(x)\langle x\rangle^{k_2-1} k_2\, \mu_{\mathcal{B},1;q}(x) \pmod{p^N} \\ &= \int_{X^*} \chi_{k_2}(x)\, \mu_{\mathcal{B},k_2;q}(x) \\ &= (1-\chi_{k_2}^p)\mathcal{B}_{k_2,\chi_{k_2}}(q).\end{aligned}$$

Therefore, we obtain the following theorems:

THEOREM 2. (Kummer type Congruences for the q-analogue Bernoulli numbers) *If* $k_1 \equiv k_2 \pmod{(p-1)p^N}$, *then*

$$(1-\chi_{k_1}^p)\mathcal{B}_{k_1,\chi_{k_1}}(q) \equiv (1-\chi_{k_2}^p)\mathcal{B}_{k_2,\chi_{k_2}}(q) \qquad (\text{mod } p^N).$$

THEOREM 3. (p-adic q-L-function) *The q-analogue of the p-adic L-function*

$$L_p(s,\chi;q) \stackrel{\text{def}}{=} \frac{1}{s-1}\int_{X^*} \langle x\rangle^{-s}\chi_1(x)(1-s)\,\mu_{\mathcal{B},1;q}(x), \qquad s \in \mathbb{Z}_p,$$

interpolates the values

$$-\frac{1}{k}(1-\chi_k^p)\mathcal{B}_{k,\chi_k}(q)$$

when $s = 1-k$ with the positive integer k.

References

[1] Z. I. Borevich and I. R. Shafarevich, *Number Theory*, Academic Press, 1996.

[2] N. Kobilitz, *A new proof of certain formulas for p-adic L-functions*, Duke Math. J. **46** (1979), 455-468.

[3] ———, *p-adic Analysis: a Short Course on Recent Work*, Mathematical Society Lecture Notes, Series 46, London: Cambridge Univ. Press, 1980.

[4] ———, *On Carlitz's q-Bernoulli numbers*, J. Number Theory **14** (1982), 332-339.

[5] ———, *p-adic Number, p-adic Analysis, and Zeta-Functions*, 2nd, Springer-Verlag, New York, 1984.

[6] T. Kim, *On explicit formulas of p-adic q-L-function*, Kyushu J. Math. **48** (1994), 73-86.

[7] ———, *An analogue of Bernoulli numbers and their congruences*, Rep. Fac. Sci. Engrg. Saga Univ. Math. **22** (1994), 21-26.

[8] J.-W. Son and M.-S. Kim, *On poly-Eulerian numbers*, Bull. Korean Math. Soc. **36** (1999), 47-61.

[9] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd, Springer-Verlag, New York, 1997.

[10] C. F. Woodcock, *An invariant p-adic integral on $\mathbb{Z}_p$*, J. London Math. Soc. (2), **8** (1974), 731-734.

Min-Soo Kim
Department of Mathematics
Kyungnam University
Masan 631-701, Korea
E-mail: mskim@nobel.kyungnam.ac.kr

Jin-Woo Son
Department of Mathematics
Kyungnam University
Masan 631-701, Korea
E-mail: sonjin@hanma.kyungnam.ac.kr