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BRILL-NOETHER THEORY FOR RANK 1 TORSION FREE SHEAVES ON SINGULAR PROJECTIVE CURVES

E. Ballico

Abstract. Let X be an integral Gorenstein projective curve with $g := p_a(X) \geq 3$. Call $G^r_d(X, **)$ the set of all pairs (L, V) with $L \in \mathrm{Pic}(X)$, $\deg(L) = d$, $V \subseteq H^0(X, L)$, $\dim(V) = r+1$ and V spanning L. Assume the existence of integers d, r with $1 \leq r \leq d \leq g-1$ such that there exists an irreducible component, Γ, of $G^r_d(X, **)$ with $\dim(\Gamma) \geq d - 2r$ and such that the general $L \in \Gamma$ is spanned at every point of $\mathrm{Sing}(X)$. Here we prove that $\dim(\Gamma) = d - 2r$ and X is hyperelliptic.

Let X be an integral projective curve with $g := p_a(X) \geq 2$ defined over an algebraically closed field $\mathbf{K}$. We want to study the Brill - Noether theory of special line bundles on X. However, if $L \in \mathrm{Pic}(X)$, $h^0(X, L) \geq 2$ but L is not spanned by its global sections at some point of $\mathrm{Sing}(X)$ the subsheaf L' of L generated by $H^0(X, L)$ may be not locally free but only torsion free. This observation explains why in the theory of special line bundles on singular curves one has to consider also torsion free sheaves and motivates the introduction of the following notations. For all positive integers d, r set $W^r_d(X) :=\{$rank 1 torsion free sheaves L on X with $\deg(L) = d$ and $h^0(X, L) \geq r+1\}$, $W^r_d(X, *) := \{L \in W^r_d(X) :$ L is a flat limit of a family of line bundles on $X\}$, $W^r_d(X, **) := W^r_d(X) \cap \mathrm{Pic}(X)$, $W^r_d(X, ***) := \{L \in W^r_d(X, **) :$ L is spanned by its global sections at every point of $\mathrm{Sing}(X)\}$ and $\rho(g, r, d) := g - (r+1)(g+r-d)$. Every rank 1 torsion free sheaf on X is the flat limit of a family of line bundles if and only if X has only planar singularities (see [18] or [1] and references

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therein). Similarly, one defines the sets $G^r_d(X)$, $G^r_d(X,*), G^r_d(X,**)$ and $G^r_d(X,***)$. Most of our results are effective only for Gorenstein curves. In section 1 we fix the notation and give a few remarks. At the beginning of section two we introduce four notions of gonality; $\mathrm{gon}(X,2)$ seems to be the more important one. In the same section we will prove the next two theorems which are a partial extension to singular curves of the so-called Martens - Mumford's theory.

THEOREM 0.1. *Let X be an integral Gorenstein projective curve with $g := p_a(X) \geq 3$. Assume the existence of integers d, r with $1 \leq r \leq d \leq g-1$ such that there exists an irreducible component, Γ, of $G^r_d(X,**)$ with $\dim(\Gamma) \geq d-2r$ and such that the general $L \in \Gamma$ is spanned at every point of $\mathrm{Sing}(X)$. Then $\dim(\Gamma) = d-2r$ and X is hyperelliptic.*

THEOREM 0.2. *Let X be an integral Gorenstein projective curve with $g := p_a(X) \geq 3$. Assume that X is not hyperelliptic. Assume the existence of integers d, r with $1 \leq r \leq d \leq g-1$ such that there exists an irreducible component, Γ, of $G^r_d(X,**)$ with $\dim(\Gamma) \geq d-2r-1$ and such that the general $L \in \Gamma$ is spanned at every point of $\mathrm{Sing}(X)$. Then $\dim(\Gamma) = d-2r-1$ and either $\mathrm{gon}(X,2) = 3$ or X is a double covering of an integral curve C with $p_a(C) = 1$ (i.e., X is a generalized bielliptic curve). Viceversa, every triple covering $f : X \to \mathbf{P}^1$ has $G^1_3(X,**) \neq \emptyset$ and for every double covering of an integral curve C with $p_a(C) = 1$ there exists an irreducible component Γ of $G^1_4(X,**)$ with $\dim(\Gamma) = 1$ formed by pull-backs of ${g^1_2}'s$ on C.*

In section 2 we will apply Theorems 0.1 and 0.2 to the existence of base point free line bundles on X with degree g or $g-1$ (see Theorem 2.5 and 2.6). In the same section we will classify all integral curves X with arithmetic genus $g \geq 2$ such that every $L \in \mathrm{Pic}^g(X)$ has $h^1(X,L) = 0$: such curves do exist (see Proposition 2.7, Theorem 2.12 and Corollary 2.13).

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1. We will use the following notations, conventions and definitions. Let X be an integral projective curve and $\pi : Y \to X$ its normalization. For every $P \in \mathrm{Sing}(X)$, let $\mathbf{O}'_P := \Pi_{Q\in\pi^{-1}(P)}\mathbf{O}_{Y,Q}$ be the integral closure of $\mathbf{O}_{X,P}$ in its total ring of fractions and C_P the conductor of $\mathbf{O}_{X,P}$ in $\mathbf{O}'_P$. We will add a superscript ^ to denote the corresponding objects for the formal completion of $\mathbf{O}_{X,P}$. Set $\mathbf{I}_P := C_P$ if X is either Gorenstein or unibranch at P and $\mathbf{I}_P := C_{P^2}$ otherwise; this choice is motivated by [7], Remark 2 at p. 21, and (in the unibrach case) the key result [11], 1.4; the sheaf $\mathbf{I}_P$ defines the ideal sheaf of an effective divisor, D_P, of Y supported on $\pi^{-1}(P)$ and with $\deg(D_P) = \dim_{\mathbf{K}}(\mathbf{O}_{X,P}/\mathbf{I}_P) = \dim_{\mathbf{K}}(\mathbf{O}_{X,P}\hat{}/\mathbf{I}_P\hat{})$; let $\delta''(Q)$ be the degree of the connected component of D_P supported by Q; hence $\delta''(Q) > 0$ for every $Q \in \pi^{-1}(P)$ and $\Sigma_{Q\in\pi^{-1}(P)}\delta''(Q) = \deg(D_P)$. Set $\delta(P) := \dim_K(\mathbf{O}'_P/\mathbf{O}_{X,P})$, $\delta'(P) := \dim_{\mathbf{K}}(\mathbf{O}'_P/C_P)$, $\delta''(P) := \dim_{\mathbf{K}}(\mathbf{O}_{X,P}/\mathbf{I}_P) = \Sigma_{Q\in\pi^{-1}(P)}\delta''(Q)$, $\delta(X) := \Sigma_{P\in\mathrm{Sing}(X)}\delta(P)$, $\delta'(X) := \Sigma_{P\in\mathrm{Sing}(X)}\delta'(P)$ and $\delta''(X) := \Sigma_{P\in\mathrm{Sing}(X)}\delta''(P)$. Hence if X is Gorenstein we have $\delta''(X) = \delta'(X) = 2\delta(X)$. Set $g'' := g - \delta(X) = p_a(Y)$ (the geometric genus of X).

REMARK 1.1. Let X be a Gorenstein irreducible projective curve. Since ω_X and all elements of $W^r_d(X, **)$ are locally free, we may give as in [3], Ch. IV, a scheme structure to $G^r_d(X, **)$. Fix $L \in W^r_d(X, **)$. With this scheme structure the tangent space of $G^r_d(X, **)$ at L is given by the cokernel of the cup-product map $\mu_0 : H^0(X, L) \otimes H^0(X, \omega_X \otimes L^{-1}) \to H^0(X, \omega_X)$ (see [3], Ch. IV, §4, or [2], Proof of Th. 3.3).

REMARK 1.2. Fix integers q, q', t, t', g with $q \geq 0$, $q' \geq 0$, $t > 0$, $t' > 0$ and smooth curves C, C' with $p_a(C) = q$ and $p_a(C') = q'$. Let X be an integral projective curve with $p_a(X) = g$ and such that there exist morphisms $f : X \to C$ and $f' : X \to C'$ with $\deg(f) = t$ and $\deg(f') = t'$. Assume that the induced map $(f, f') : X \to C \times C'$ is birational. Then, exactly as in the case in which X is smooth, we have Castelnuovo - Severi inequality $g \leq tq + t'q' + (t-1)(t'-1)$ ([16], Cor. to Th. 1), because this inequality is just an inequality for the arithmetic genus of suitable divisors on the smooth surface $C \times C'$. Furthermore, if $g = tq + t'q' + (t-1)(t'-1)$, then the morphism (f, f') must be an isomorphism and hence X must have only planar singularities. Assume that there is no morphism $f : X \to C$ with $\deg(f) \cdot p_a(C)$ low and that $\mathrm{gon}(X, 2)$ (see Definition 2.1) is much lower than g; as in the smooth case Castelnuovo - Severi inequality implies that $\mathrm{gon}(X, 2)$ is computed by a unique pencil and that there is a lower bound for the first integer $d > \mathrm{gon}(X, 2)$ such that there exists a spanned $L \in \text{Pic}^d(X)$with $h^0(X,L) \geq 2$.

REMARK 1.3. Assume X Gorenstein and $g := p_a(X) \geq 2$. The line bundle ω_X is spanned, i.e., the canonical map $u_\omega : X \to \mathbf{P}^{g-1}$ is a morphism ([6], Th. D, or [R]). The canonical map u_ω is not birational if and only if it is a two - to - one morphism and in this case X is "hyperelliptic" ([6], Prop. 3.10, or [R]). If X is not hyperelliptic, then u_ω is an embedding ([R], Th. 15, or [13], Th. 1.6). For a discussion of this topic, even in the non-Gorenstein case, see [20]. Fix positive integers d, r with $\rho(g,r,d) \geq 0$ and assume X not hyperelliptic. The standard proof of the inequality $\dim(G^r_d(X,**)) \geq \rho(g,r,d)$ given in [12], p. 260, just uses $u_\omega(X)$ and the Grassmannian $G(d-g-r,g)$ of all projective subspaces of dimension $d-r-1$ of $\mathbf{P}^{g-1}$; since u_ω is an embedding, this proof shows that for every $L \in W^r_d(X,**)$ every irreducible component of $W^r_d(X,**)$ containing L has dimension at least $\rho(g,r,d)$.

REMARK 1.4. Let X be an integral projective hyperelliptic curve and $L \in \text{Pic}^2(X)$ its hyperelliptic spanned line bundle. In particular X is Gorenstein ([10], part (b) of Th. A). Set $g := p_a(X)$. Since $L^{\otimes(g-1)} \cong \omega_X$ and $S^{g-1}(H^0(X,L)) = H^0(X,\omega_X)$, every spanned line bundle, M, on X with $h^1(X,M) \neq 0$ is isomorphic to a line bundle $L^{\otimes t}$ for some integer t with $0 \leq t \leq g-1$. In particular $\deg(M)$ is even. Hence by Riemann - Roch the line bundles $L^{\otimes t}$ with $0 \leq t \leq [g/2]$ are the only spanned line bundles on X with degree at most g. Furthermore, for every integer d with $2 \leq d \leq g$ there exists an irreducible component, Γ, of $W^r_d(X,**)$ with $\dim(\Gamma) = d-2r$ and whose general member is a line bundle spanned at every point of $\text{Sing}(X)$. Compare this classical example with the partial extension of Martens - Mumford's theory stated in the introduction and proved in section two.

Using the existence theorem for special divisors on smooth projective curves ([3], Ch. IV) it is very easy to prove the following result.

PROPOSITION 1.5. *Fix integers g,r,d with $g \geq 2$, $d > 0$, $r > 0$ and $\rho(g,d,r) \geq 0$. Let X be an integral projective curve with $p_a(X) = g$ and whose only singularities are smoothable singularities. Then $W^r_d(X) \neq \emptyset$ and $\dim(W^r_d(X)) \geq \rho(g,r,d)$.*

Proof. Since X has only smoothable singularities, X is a flat limit of an integral family of connected smooth curves of genus g. Hence it is sufficient to use the existence of the relative generalized Jacobian for flat families of integral projective curves ([1]), the semicontinuity of cohomology and the semicontinuity of the fiber dimension for proper maps. □

2. First we will give 4 notions of gonality and 8 notions of Clifford index for singular curves.

DEFINITION 2.1. Let X be an integral projective curve. Let $\mathrm{gon}(X,1)$ be the first integer t such that there exists $L \in \mathrm{Pic}^t(X)$ with $h^0(X,L) \geq 2$; taking $L(-P)$ instead of L for a general $P \in X$ we see that any such L has $h^0(X,L) = 2$ and it is spanned at each point of X_{reg}. Let $\mathrm{gon}(X,2)$ be the first integer t such that there exists $L \in \mathrm{Pic}^t(X)$ with $h^0(X,L) \geq 2$ and L base point free; taking $L(-P)$ instead of L for a general $P \in X$ we see easily that any such L has $h^0(X,L) = 2$. Let $\mathrm{gon}(X,3)$ be the first integer t such that there exists a rank 1 torsion free sheaf L with $\deg(L) \geq 2$ and $\deg(L) = t$; notice that any such L is spanned by its global sections and $h^0(X,L) = 2$. Let $\mathrm{gon}(X,4)$ be the first integer t such that there exists a rank 1 torsion free sheaf L , L flat limit of a family of lines bundles, with $\deg(L) \geq 2$ and $\deg(L) = t$; notice that any such L has $h^0(X,L) = 2$ and it is spanned at each point of X_{reg}.

DEFINITION 2.2. Let X be an integral projective curve with $g := p_a(X) \geq 2$. For any rank 1 torsion free sheaf L on X let $\mathrm{Cliff}(L) := \deg(L) - 2(h^0(X,L)) + 2$ be its Clifford dimension. Set $\mathrm{Cliff}(X,1) := \max\{\mathrm{Cliff}(L)$ for L line bundle on X with $h^0(X,L) \geq 2$ and $0 < \deg(L) \leq g-1\}$. Set $\mathrm{Cliff}(X,2) := \max\{\mathrm{Cliff}(L)$ for L spanned line bundle on X with $h^0(X,L) \geq 2$ and $0 < \deg(L) \leq g-1\}$. Set $\mathrm{Cliff}(X,3) := \max\{\mathrm{Cliff}(L)$ for L rank 1 torsion free sheaf on X with $h^0(X,L) \geq 2$ and $0 < \deg(L) \leq g-1\}$. Set $\mathrm{Cliff}(X,4) := \max\{\mathrm{Cliff}(L)$ for L rank 1 torsion free sheaf on X with L flat limit of a family of line bundles, $h^0(X,L) \geq 2$ and $0 < \deg(L) \leq g-1\}$. Set $\mathrm{Cliff}(X,5) := \max\{\mathrm{Cliff}(L)$ for L line bundle on X with $h^0(X,L) \geq 2$ and $h^1(X,L) \geq 2\}$. Set $\mathrm{Cliff}(X,6) := \max\{\mathrm{Cliff}(L)$ for L spanned line bundle on X with $h^0(X,L) \geq 2$ and $h^1(X,L) \geq 2\}$. Set $\mathrm{Cliff}(X,7) := \max\{\mathrm{Cliff}(L)$ for L rank 1 torsion free sheaf on X with $h^0(X,L) \geq 2$ and

$h^1(X, L) \geq 2\}$. Set Cliff$(X, 8) :=$ max{Cliff(L) for L rank 1 torsion free sheaf on X with L flat limit of a family of line bundles, $h^0(X, L) \geq 2$ and $h^1(X, L) \geq 2\}$.

Every rank 1 torsion free sheaf on an integral projective curve X is a flat limit of a family of line bundles if and only if X has only planar singularities ([1] or [18]). Hence if X has only planar singularities we have gon$(X, 3)$ =gon$(X, 4)$, Cliff$(X, 3)$ =Cliff$(X, 4)$ and Cliff$(X, 7)$ =Cliff$(X, 8)$.

Now we will prove Theorems 0.1 and 0.2.

Proof of Theorem 0.1. We may repeat the proof of the smooth case given in [3], p. 192, for the following reasons. We may reduce to the case in which a general $L \in \Gamma$ is spanned (lowering if necessary d) without loosing the condition that a general element of Γ is locally free because a general $L \in \Gamma$ is spanned at every point of Sing(X). We may apply the infinitesimal theory of $G^r_d(X, **)$ by Remark 1.1. We may apply Clifford's inequality by [10], Th. A. We may apply the base point free pencil trick, because it holds with the same proof for Gorenstein singular curves; notice that here we need that we use a spanned line bundle; we could not use a line bundle spanned at each point of X_{reg} and this is the only point of the proof in which this assumption is used. □

Proof of Theorem 0.2. By Theorem 0.1 we have $\dim(\Gamma) = d - 2r - 1$. As in the first few lines of the proof of [3], Th. IV.5.2, we may reduce to the case $r = 1$, without loosing the key condition "a general $L \in \Gamma$ is spanned at every point of Sing(X)". Again, as in the proof of [3], Th. IV.5.2, we obtain $d \leq 5$ and that if $d = 5$ then g must be 7. If $d = 3$ we obtain gon$(X, 2) = 2$. Now assume $d = 4$. As in [3] we have $g \geq 6$. Since $\dim(\Gamma) = 1$ we may repeat the proof of [3], pp. 194-195, and obtain that for $L, L' \in \Gamma$ with $L \neq L'$ we have $h^0(X, L \otimes L') = 4$. As in [3], p. 195, this gives a morphism $\phi : X \to \mathbf{P}^2$ with $\deg(\phi)\deg(f(X)) = 6$ and $\phi(X)$ spanning P^2. If $\deg(\phi(X)) = 2$, we obtain gon$(X, 2) \leq 3$. If $\deg(\phi(X)) = 3$ we obtain that X is a double covering of an integral curve $C := \phi(X)$ with $p_a(C) = 1$. If $\deg(\phi(X)) = 6$ the map ϕ must be an embedding because $g \geq 6$. It remains to exclude the case $d = 5$ and $g = 7$. By Riemann - Roch for a general $L \in \Gamma$ we have $\omega_X \otimes L^{-2} \cong \mathbf{O}_X(D)$ for some effective

Cartier divisor on X. By Serre duality and Riemann - Roch we have $h^0(X, L(D)) = h^1(X, \omega_X \otimes L^{-1}) \geq 3$. Since X is not hyperelliptic, by [10] $\omega_X \otimes L^{-1}$ cannot have a base point and must send birationally X onto a plane curve of degree 5 and hence with arithmetic genus at most 6, contradiction. □

The following lemma was proved in [13], first paragraph on page 379.

LEMMA 2.3. *Let X be an integral projective curve and $L \in \mathrm{Pic}^t(X)$ such that $h^0(X, L) \geq 2$ and L is spanned at each point of* Sing(X). *Then gon$(X, 2) \leq t$.*

Proof. By assumption the scheme-theoretic base locus of L is contained in X_{reg} and hence either it is empty or it is an effective Cartier divisor, B. Since $L(-B) \in \mathrm{Pic}^z(X)$ with $z = t - \deg(D) \leq t$, we conclude. □

Let X be an integral projective curve. The singularities of X give a lower bound on gon$(X, 2)$ as shown by the following example.

EXAMPLE 2.4. Let X be an integral projective curve with Sing$(X) \neq \emptyset$ and $\pi : Y \to X$ the normalization. For every $P \in \mathrm{Sing}(X)$ let $k(X, P)$ be the degree of the effective divisor $\pi^{-1}(P)$ of Y; here $\pi^{-1}(P)$ denotes the scheme-theoretic fiber of π at P. Set $k(X) := \max_{P \in \mathrm{Sing}(X)}\{k(X, P)\}$. Notice that $k(X) \geq 2$ because Sing$(X) \neq \emptyset$ by assumption. We claim that gon$(X, 2) \geq k(X)$. Take any $L \in \mathrm{Pic}^{\mathrm{gon}(X,2)}(X)$ computing gon$(X, 2)$. Thus L is spanned. Since $\dim(X) = 1$ it is easy to check the existence of a linear subspace V of $H^0(X, L)$ with $\dim(V) = 2$ and V spanning L. Hence $\pi^*(V)$ is a subspace of $H^0(Y, \pi^*(L))$ spanning $\pi^*(L)$. Thus $\pi^*(V)$ induces a morphism $f : Y \to \mathbf{P}^1$ with $\deg(f)$ =gon$(X, 2)$. By construction f factors through π. Since every scheme-theoretic fiber of f is an effective divisor on Y with degree $\deg(f)$, we have $\deg(f) \geq k(X)$, as claimed.

It is known that a smooth curve X of genus $g \geq 2$ has a base point free line bundle of degree g, unless g is odd and X is hyperelliptic; furthermore, a smooth curve X of genus $g \geq 7$ has a base point free line bundle of degree $g - 1$, unless X is either hyperelliptic or bielliptic (see [15] or [8], Lemma 2.1.1, or [5], Th. 0.1, for references and stronger statements).

THEOREM 2.5. *Let X be an integral Gorenstein projective curve. Then there exists $L \in \mathrm{Pic}^g(X)$ spanned by its global sections, unless g is odd and X is hyperelliptic.*

Proof. By Remark 1.4 everything is obvious in the hyperelliptic case. Hence we may assume X not hyperelliptic. By [14] we have $\mathrm{gon}(X,2) \leq g$. By definition of $\mathrm{gon}(X,2)$ the result is obvious if $\mathrm{gon}(X,2) = g$. Hence we may assume $\mathrm{gon}(X,2) < g$. Adding as base points only points of X_{reg} and using Remark 1.3 we obtain that for every integer d with $\max\{\mathrm{gon}(X,2), [(g+3)/2]\} \leq d \leq g$ there exists an irreducible component $G(d)$ of $G^1_d(X,**)$ such that $\dim(G(d)) \geq \rho(g,1,d) = 2d - g - 2$ and such that a general pair $(L,V) \in G(d)$ is spanned at every point of $\mathrm{Sing}(X)$. In particular we have $\dim(G(g)) \geq g-2$. In order to obtain a contradiction we assume that a general element of $G(g)$ is not spanned. Since a general element of $G(g)$ is spanned at every point of $\mathrm{Sing}(X)$, this implies the existence of an component T of $G^1_{g-1}(X,**)$ such that for a general $L \in G(g)$ there exists $M \in T$ and $P \in X_{\mathrm{reg}}$ such that $L \cong M(P)$, $h^0(X,L) = h^0(X,M)$ and M is spanned at each point of $\mathrm{Sing}(X)$. Hence $\dim(T) \geq g-3$. By Theorem 0.1 we obtain that X is hyperelliptic, contradiction. □

THEOREM 2.6. *Let X be an integral Gorenstein projective curve with $g := p_a(X) \geq 5$ and $\mathrm{gon}(X,2) \leq g-1$. Then there exists $L \in \mathrm{Pic}^{g-1}(X)$ spanned by its global sections, unless either $g(X,2) \leq 3$ or X is bielliptic.*

Proof. By the definition of $\mathrm{gon}(X,2)$ the result is obvious if $\mathrm{gon}(X,2) = g-1$. Hence we may assume $\mathrm{gon}(X,2) < g-1$ and repeat verbatim the proof of Theorem 2.5 with the pair of integers $(g-1, g-2)$ instead of the pair of integers $(g, g-1)$ and quoting Theorem 0.2 instead of Theorem 0.1. For non-Gorenstein curves the numerical behaviour of the Brill - Noether theory of special line bundles may be much worst. Indeed, the theory may be empty as shown by the following key example; for the definition of seminormality and its geometric interpretation for curves, see [21] and [9]. □

PROPOSITION 2.7. *Fix an even integer $g \geq 2$. Let X be an integral projective curve with a unique singular point, P, which is seminormal* *and with $g+1$ branches. Assume that the normalization of X is rational, i.e., assume $p_a(X) = g$. Then for every line bundle L on X with* $\deg(L) \leq g$ *we have* $h^0(X, L) \leq 1$.

Proof. Fix $L \in \mathrm{Pic}(X)$ with $\deg(L) \leq g$ and $d := \deg(L) \leq g$. Fix $P \in X_{\mathrm{reg}}$. Assume $h^0(X, L) \geq 2$. Thus $h^0(X, L((g-d)P)) \geq 2$. Hence by Riemann - Roch we have $h^1(X, L((g-d)P)) \neq 0$ and thus by Serre duality $h^0(X, \mathrm{Hom}(L((g-d)_P), \omega_X)) \neq 0$. Take $f \in H^0(X, \mathrm{Hom}(L((g-d)_P), \omega_X))$, $f \neq 0$. By [7], example after Def. 2.2.3 at line 11 on p. 18, $\mathrm{Coker}(f)$ is a skyscraper sheaf whose connected component supported by P is a vector space of dimension at least $2g-(g+1)$. Since $\deg(\omega_X) - \deg(L((g-P))) = g-2$, this is absurd.□

Definition 2.9. Let X be an integral projective curve and R a rank 1 torsion free sheaf on X. For every $P \in \mathrm{Sing}(X)$ let $l(R, P)$ be the minimal integer t such that there exists a rank 1 trivial $O_{X,P}$-module A with $A \subseteq R_P$ and $\dim_{\mathbf{K}}(R_P/A) = t$; here R_P denotes the stalk of R at P as $O_{X,P}$-module. The same definition applies to a rank 1 torsion free $O_{X,P}\hat{}$-module $R\hat{}$; this is [7], Def. 2.2.3.

We need the following well-known lemma (see e.g. [4], Remark 1.14).

Lemma 2.10. *Let X be an integral projective curve and R a rank 1 torsion free sheaf on X. Then* $\deg(R) - \Sigma_{P \in \mathrm{Sing}(X)} l(R, P)$ *is the maximal degree of a line bundle L on X with $L \subseteq R$.*

Lemma 2.11. *Let X be an integral projective curve. Then $2p_a(X) - 2 - 2\delta(X) + \delta'(X)$ is the maximal degree of a line bundle L on X with $L \subseteq \omega_X$.*

Proof. Fix $P \in \mathrm{Sing}(X)$. By [7], example after Def. 2.2.3 at line 11 on p. 18, we have $l(\omega_X, P) = 2\delta(X, P) - \delta'(X, P)$. Hence we conclude by Lemma 2.10. As an immediate corollary of Lemma 2.11 we obtain the following result. □

Theorem 2.12. *Let X be an integral projective curve. Assume $g := p_a(X) \geq 2$. There is no line bundle $L \in \mathrm{Pic}^g(X)$ with $h^1(X, L) \neq 0$ if and only if the normalization of X is rational,* $\mathrm{card}(\mathrm{Sing}(X)) = 1$ *and the unique singular point, P, of X is seminormal, i.e., it has g+1 branches.*

COROLLARY 2.13. *The curve X described in* Proposition 2.7 *is the "only" curve with $g := p_a(X) \geq 2$ such that there is no line bundle L on X with* $\deg(L) \leq g$ *and with* $h^0(X, L) \geq 2$.

Here we explain the means of the word "only" in the statement of Corollary 2.13. We start with any $g+1$ distinct points $Q_1, \cdots, Q_{g+1}$ of $\mathbf{P}^1$ and call $X(Q_1, \cdots, Q_{g+1})$ the unique curve obtained from $\mathbf{P}^1$ gluing together the points $Q_1, \cdots, Q_{g+1}$ in the sense of [21] or [9]. Any curve X in Proposition 2.7 and Corollary 2.13 arises in this way for a unique choice (up to the action of $\mathrm{Aut}(\mathbf{P}^1)$) of the points $Q_1, \cdots, Q_{g+1}$. Hence if $g = 2$ the curve X is unique, up to isomorphisms; if $g \geq 3$ there is an action of $\mathrm{Aut}(\mathbf{P}^1)$ on the set of all subsets of $\mathbf{P}^1$ with cardinality $g+1$ and this action is studied in [17], §2; roughly speaking, for $g \geq 3$ the set of all such X depends on $g-2$ parameters and there is a smooth, irreducible variety of dimension $g+1$ parametrizing all such X in such a way that every isomorphism class of one X corresponds to a 3-dimensional subscheme of the parameter space.

References

[1] A. Altman, S. L. Kleiman and A. Iarrobino, *Irreducibility of the compactified Jacobian*, in: Singularities of Real and Complex Maps, Proceedings of the Nordic Summer School, Oslo, 1976, 1-12, P. Holm Editor, Sijthoff and Noordhof, 1977.

[2] E. Arbarello and M. Cornalba, *Su una congettura di Petri*, Comment. Math. Helvetici **56**(1981), 1-38.

[3] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of Algebraic Curves,*, Springer-Verlag **I**(1985).

[4] E. Ballico, *Stable sheaves on reduced projective curves*, Ann. Mat. pura appl. (to appear).

[5] E. Ballico and Ch. Keem, *On multiple coverings of irrational curves*, Arch. Math. **65**(1995), 151-160.

[6] F. Catanese, *Pluricanonical Gorenstein curves, in: Enumerative Geometry and Classical Algebraic Geometry*, Progress in Math. 24, Birkhäuser, 1982, 51-95.

[7] P. Cook, *Local and global aspects of the module theory of singular curves*, Ph. D. Thesis, 1993.

[8] M. Coppens, Ch. Keem and G. Martens, *Primitive linear series on curves*, Manuscripta Math. **77**(1992), 237-264.

[9] E. D. Davis, *On the geometric interpretation of seminormality*, Proc. Amer. Math. Soc. **68**(1978), 1-5.

[10] D. Eisenbud, J. Harris, J. Koh and M. Stillman, *Appendix to: Determinantal equations for curves of high degree, by D. Eisenbud, J. Koh, M. Stillman*, Amer. J. Math. **110** (1988), 513-539.

[11] G.-M. Greuel and G. Pfister, *Moduli spaces for torsion free modules on curves singularities. I*, J. Alg. Geom. **2** (1993), 81-135.

[12] P. A. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, 1978.

[13] R. Hartshorne, *Generalized divisors on Gorenstein curves and a theorem of Noether*, J. Math. Kyoto Univ. **26** (1986), 375-386.

[14] M. Homma, *Separable gonality of a Gorenstein curve*, Matematica contemporâ-nea (to appear).

[15] R. Horiuchi, *Gap orders of meromorphic functions on Riemann surfaces*, J. reine angew. Math. **336** (1982), 213-220.

[16] E. Kani, *On Castelnuovo's inequality*, J. reine angew. Math. **352** (1984), 24-70.

[17] D. Mumford and K. Suominen, *Introduction to the theory of moduli*, in: Algebraic Geometry, Oslo 1970, pp. 171-222, Wolters - Noordhoff Publishing, 1972.

[18] C. J. Rego, *The compactified Jacobian*, Ann. Sci. Ec. Norm. Sup. 4e serie **13** (1980), 211-223.

[19] M. Rosenlicht, *Equivalence relations on algebraic curves*, Ann. Math. **56** (1952), 169-191.

[20] K.-O. Stöhr, *On the poles of regular differentials of singular curves*, Bol. Soc. Bras. Mat. **24** (1993), 103-136.

[21] C. Traverso, *Seminormality and Picard group*, Ann. Scuola Norm. Sup. Pisa **24** (1970), 585-595.

Department of Mathematics
University of Trento
38050 Povo (TN) - Italy
fax: italy + 0461881624
E-mail: ballico@science.unitn.it