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ON THE EXISTENCE OF SOLUTIONS OF QUASILINEAR WAVE EQUATIONS WITH VISCOSITY

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ABSTRACT. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In this paper, we consider the existence of solutions of the following problem:

(1.1)

$$\begin{aligned}
&u_{tt}(t,x) - \operatorname{div}\{\sigma(|\nabla u(t,x)|^2)\nabla u(t,x)\} - \Delta u(t,x) - \Delta u_t(t,x) \\
&\quad + \delta|u_t(t,x)|^{p-1}u_t(t,x) = \mu|u(t,x)|^{q-1}u(t,x), \\
&\qquad x \in \Omega, \quad t \in [0,T], \\
&u(t,x)|_{\partial\Omega} = 0, \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega,
\end{aligned}$$

where $q > 1$, $p \geq 1$, $\delta > 0$, $\mu \in \mathbb{R}$, Δ the Laplacian in $\mathbb{R}^N$ and $\sigma = \sigma(v^2)$ is a positive function like as $\frac{1}{(1+v^2)^{1/2}}$.

1. Introduction

Many authors have been studied about the existence and uniqueness of solutions of (1.1) by using various methods. Equation (1.1) with $\delta = \mu = 0$ was introduced by Greenberg and Maccamy ([5]) as a model of a modified quasilinear wave equation which has global smooth solutions for large data. Since then, many authors have investigated the global existence as well as the asymptotic behavior of solutions to this and related equations (see Alikakos and Rostamin [2], Engler [4], Kawashima and Shibata [7], Mizohata and Ukai [9] and Nakao [10] and the references cited in these papers). When $\sigma = \mu = 0$ and $\delta = 1$, Nakao ([12]) has considered the existence and decay properties

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of solution of (1.1). On the other hand, Matsuyama and Ikehata ([8]) have considered the following quasilinear wave equation

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2\, dx\Big)\,\Delta u + \delta|u_t|^{p-1}u_t = \mu|u|^{q-1}u, \tag{1.2}$$

where $M(s)$ is a positive C^1 class function for $s \geq 0$ satisfying $M(s) \geq m_0 > 0$ with a constant m_0. In fact, Equation (1.2) has its origin in the mathematical description of small amplitude vibrations of an elastic string whose ends are held a fixed distance apart, hinged or clamped and is either elastic or compressed by an axial force. In fact, a mathematical model for (1.2) is an initial boundary value problem for the nonlinear hyperbolic equation

(1.3)

$$\rho h\frac{\partial^2 u}{\partial t^2} = \left\{p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} + f \quad \text{for } 0 < x < L,\ t \geq 0,$$

where u is the lateral deflection, x the space coordinate, t the time, E the Young modulus, ρ the mass density, h the cross section area, L the length, p_0 the initial axial tension and f the external force. See Narasimha ([13]).

Kirchhoff first introduced (1.3) in the study of oscillations of elastic strings and plates, so that (1.3) is called the wave equation of Kirchhoff type after his name.

In the present paper, we will study the existence and uniqueness of solutions of (1.1) with $\delta > 0$ and $\mu \neq 0$ by using Galerkin method. Our paper is organized as follows: In section 2, we give lemmas and state the main result. In section 3, we study the existence and uniqueness of weak solution of (1.1).

2. Preliminaries

In this section we present some lemmas that will be necessary throughout this paper.

LEMMA 2.1 (Sobolev-Poincaré [1]). *If either* $1 \leq q < +\infty \quad (N = 1, 2)$ *or* $1 \leq q \leq \frac{N+2}{N-2} \quad (N \geq 3)$ *is satisfied, then there is a constant* C_* *such that*

$$\|u\|_{q+1} \leq C_*\|\nabla u\|_2 \quad \textit{for} \quad u \in H_0^1(\Omega).$$

LEMMA 2.2 (Gagliardo-Nirenberg [1]). *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then the inequality*

$$\|u\|_{W^{k,q}} \leq C\|u\|_{W^{m,p}}^{\theta}\|u\|_r^{1-\theta} \quad \textit{for} \quad W^{m,p}(\Omega) \cap L^r(\Omega)$$

holds with some $C > 0$ and $\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$ provided that $0 < \theta \leq 1$ (we assume $0 < \theta < 1$ if $q = +\infty$).

For further a priori estimates, we need the generalized Gronwall's inequality which is due to Bihari and Langenhop.

LEMMA 2.3. ([3]). *If $k > 0$ and $c \geq 0$ are constants and $g(s)$ is positive, nondecreasing for $s \geq 0$, then the inequality*

$$\phi(t) \leq k + c\int_0^t \psi(s)g(\phi(s))\,ds \tag{2.1}$$

implies that $\phi(t) \leq G^{-1}(c\int_0^t \psi(s)\,ds)$, where $G(\eta) = \int_k^\eta \frac{1}{g(s)}\,ds,\ \ \eta > k > 0$. If $g(s) = s$, then the inequality (2.1) is the usual Gronwall's inequality and Lemma 2.3 *reads as follows:*

$$\phi(t) \leq k + c\int_0^t \psi(s)\phi(s)\,ds$$

implies that

$$\phi(t) \leq k\ \exp\ \left(c\int_0^t \psi(s)\,ds\right), \quad t \geq 0.$$

Lastly, we indicate the following two propositions which are necessary to obtain convergence results.

PROPOSITION 2.4. (Temam [14]). *Let X and Y be two Banach spaces such that $X \subset Y$ with a continuous injection. If a function ϕ belongs to $L^\infty(0,T;X)$ and is weakly continuous with values in Y, then ϕ is weakly continuous with values in X.*

PROPOSITION 2.5 (Lions [6]). *Let X be a Banach space. If $f \in L^p(0,T;X)$ and $f' \in L^p(0,T;X)$ $(1 \le p \le \infty)$, then f, possibly after redefinition on a set of measure zero, is continuous from $[0,T]$ to X.*

Now we consider the following boundary value problem:

$$\begin{aligned} &u_{tt}(t,x) - \operatorname{div}\{\sigma(|\nabla u(t,x)|^2)\nabla u(t,x)\} - \Delta u(t,x) - \Delta u_t(t,x) \\ &\quad + \delta|u_t(t,x)|^{p-1}u_t(t,x) = \mu|u(t,x)|^{q-1}u(t,x), \\ &\qquad x \in \Omega, \quad t \in [0,T], \\ &u(t,x)|_{\partial\Omega} = 0, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega. \end{aligned} \tag{2.2}$$

We define the potential $J(u(t))$, energy $E(u(t))$ and I-positive set $\mathcal{W}$ associated with equation (2.2) by

$$J(u(t)) = \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{\mu}{q+1}\|u(t)\|_{q+1}^{q+1},$$
$$E(u(t)) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\int_\Omega\int_0^{|\nabla u(t)|^2}\sigma(\eta)\,d\eta\,dx + J(u(t))$$
and $\mathcal{W} = \{u(t) \in H_0^1(\Omega) \cap H^2(\Omega) \mid I(u(t)) > 0\} \cup \{0\}$,

where $I(u(t)) = \|\nabla u(t)\|_2^2 - \mu\|u(t)\|_{q+1}^{q+1}$.

To obtain our theorem, we shall make the following assumptions on σ:

$\sigma(\cdot)$ belongs to $C^2([0,\infty))$ and satisfies

(H_1) $0 \le \sigma(v^2) \le k_1$ and $|\sigma'(v^2)| \le k_1 < \infty$,

(H_2) there exists $r > 0$ such that $\sigma(v^2) - 2|\sigma'(v^2)|v^2 \ge k_2\{\sigma(v^2) + 2|\sigma'(v^2)|v^2\}^r$ for some $k_1, k_2 > 0$.

REMARK. Without loss of generality, we may assume $r > 2$ in (H_2) of the above hypothesis. When $\sigma(v^2) = \frac{1}{\sqrt{1+v^2}}$, we see $\sigma(v^2) - 2|\sigma'(v^2)|v^2 = (1+v^2)^{-\frac{3}{2}}$ and $\sigma(v^2) + 2|\sigma'(v^2)|v^2 \le 2(1+v^2)^{-\frac{1}{2}}$ and hence we can take $r = 3$.

Our results read as follows.

THEOREM 2.6. *Let N be a positive integer. Under assumptions (H_1) and (H_2), suppose that $\delta > 0$, $\mu > 0$ and $p < \min\{q, \frac{N+4q-Nq}{2}\}$ is such that*

$$(i)\ 1 \le p < \infty\ (N = 1, 2),$$
$$(ii)\ 1 \le p \le 3,\ 1 < q \le 5\ (N = 3),$$
$$(iii)\ 1 \le p \le \frac{N}{N-2},$$
$$\frac{N}{N-2} \le q \le \min\left\{\frac{N+2}{N-2}, \frac{N-2}{[N-4]^+}\right\}\ (N \ge 4).$$

If $u_0 \in \mathcal{W} \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ and

$$\mu C_*^{q+1}\left(\frac{2(q+1)}{q-1}\right)^{\frac{q-1}{2}} E(u_0)^{\frac{q-1}{2}} < 1,$$

then the problem (2.2) *has a solution $u = u(t,x)$ satisfying $u \in L^\infty\ (0,T;\ H_0^1\ (\Omega) \cap H^2(\Omega))$, $u' \in L^\infty\ (0,T; H_0^1(\Omega))$ and $u'' \in L^\infty\ (0,T; L^2(\Omega))$.*

3. Proof of Theorem 2.6

We now consider the problem (2.2). Throughout this section, we always assume that $u_0 \in \mathcal{W} \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$. We employ the Galerkin method to construct a solution. We present by $(w_j)_{j\in\mathbb{N}}$ be a basis in $H_0^1(\Omega) \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$ and by V_m the subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ generated by the m-first vectors $w_1, \cdots, w_m$ and define $u_m(t) = \sum_{j=1}^m g_{jm}(t)w_j$, where $u_m(t)$ is the solution of the following problem:

$$\begin{aligned}
&(u''_m(t) - \operatorname{div}\{\sigma(|\nabla u_m(t)|^2)\nabla u_m(t)\} - \Delta u_m(t) - \Delta u'_m(t), w) \\
&\quad + \delta|u'_m(t)|^{p-1}(u'_m(t), w) = \mu|u_m(t)|^{q-1}(u_m(t), w), w \in V_m,
\end{aligned}$$

$$(3.1)\quad u_m(0) = u_{0m} = \sum_{j=1}^m (u_0, w_j)w_j \to u_0 \quad \text{in } H_0^1(\Omega) \cap H^2(\Omega),$$

$$u'_m(0) = u_{1m} = \sum_{j=1}^m (u_1, w_j)w_j \to u_1 \quad \text{in } H_0^1(\Omega).$$

Note that we can solve the system (3.1) by a Picard's iteration method. Hence the system (3.1) has a local solution in $[0, T_m)$ with $0 < T_m \le T$. The extension of the solution to the whole interval $[0, T]$ is a consequence of priori estimates we are going to obtain below.

A Priori Estimates I

Multiplying the equation (3.1) by $u'_m(t)$, then we have

(3.2)
$$\frac{d}{dt}\left(\frac{1}{2}\|u'_m(t)\|_2^2 + \frac{1}{2}\|\nabla u_m(t)\|_2^2 + \frac{1}{2}\int_\Omega \Gamma(t)\,dx - \frac{\mu}{q+1}\|u_m(t)\|_{q+1}^{q+1}\right) + \|\nabla u'_m(t)\|_2^2 + \delta\|u'_m(t)\|_{p+1}^{p+1} = 0,$$

where $\Gamma(t) = \int_0^{|\nabla u_m(t)|^2} \sigma(\eta)\,d\eta$.

Integrating (3.2) from 0 to t, then we have the energy identity

$$E(u_m(t)) + \int_0^t \|\nabla u'_m(s)\|_2^2\,ds + \delta\int_0^t \|u'_m(s)\|_{p+1}^{p+1}\,ds = E(u_{0m}), \tag{3.3}$$

where $E(u_m(t)) = \frac{1}{2}\|u'_m(t)\|_2^2 + \frac{1}{2}\|\nabla u_m(t)\|_2^2 + \frac{1}{2}\int_\Omega \Gamma(t)\,dx - \frac{\mu}{q+1}\|u_m(t)\|_{q+1}^{q+1}$.

Next to obtain a priori bound, we need the following results.

Lemma 3.1. *Assume that either $1 \le q < +\infty$ $(N = 1, 2)$ or $1 \le q \le \frac{N+2}{N-2}$ $(N \ge 3)$ is satisfied. Let $u_m(t)$ be the solution of (3.1) with $u_{0m} \in \mathcal{W} \cap H^2(\Omega)$ and $u_{1m} \in H_0^1(\Omega)$. If*

$$\mu C_*^{q+1}\left(\frac{2(q+1)}{q-1}\right)^{\frac{q-1}{2}} E(u_{0m})^{\frac{q-1}{2}} < 1, \tag{3.4}$$

then $u_m(t) \in \mathcal{W}$ on $[0, T]$, that is, $\|\nabla u_m\|_2^2 - \mu\|u_m\|_{q+1}^{q+1} > 0$ on $[0, T]$.

Proof. Since $I(u_{0m}) > 0$, it follows from the continuity of $u_m(t)$ that

$$I(u_m(t)) \ge 0 \quad \text{for some interval near} \quad t = 0. \tag{3.5}$$

Let $t_{\max}$ be a maximal time (possibly $t_{\max} = T_m$) when (3.5) holds on $[0, t_{\max})$. Note that

$$
\begin{aligned}
J(u_m(t)) &= \frac{1}{q+1} I(u_m(t)) + \frac{q-1}{2(q+1)} \|\nabla u_m(t)\|_2^2 \\
&\geq \frac{q-1}{2(q+1)} \|\nabla u_m(t)\|_2^2 \quad \text{on} \quad [0, t_{\max}).
\end{aligned} \tag{3.6}
$$

By the energy identity (3.3) and (3.6), we have

$$
\begin{aligned}
\|\nabla u_m(t)\|_2^2 &\leq \frac{2(q+1)}{q-1} J(u_m(t)) \\
&\leq \frac{2(q+1)}{q-1} E(u_m(t)) \\
&\leq \frac{2(q+1)}{q-1} E(u_{0m}) \quad \text{on} \quad [0, t_{\max}).
\end{aligned} \tag{3.7}
$$

It follows from the Sobolev-Poincaré's inequality, (3.4) and (3.7) that

$$
\begin{aligned}
\mu \|u_m(t)\|_{q+1}^{q+1} &\leq \mu C_*^{q+1} \|\nabla u_m(t)\|_2^{q+1} \\
&\leq \mu C_*^{q+1} \left(\frac{2(q+1)}{q-1} E(u_{0m}) \right)^{\frac{q-1}{2}} \|\nabla u_m(t)\|_2^2 \\
&< \|\nabla u_m(t)\|_2^2 \quad \text{on} \quad [0, t_{\max}).
\end{aligned}
$$

Therefore, we get $I(u_m(t)) > 0$ on $[0, t_{\max})$. This implies that we can take $t_{\max} = T$. This completes the proof of Lemma 3.1. □

Note that Lemma 3.1 implies that

$$
\begin{aligned}
E(u_m(t)) &= \frac{1}{2} \|u_m'(t)\|_2^2 + \frac{1}{q+1} I(u_m(t)) + \frac{q-1}{2(q+1)} \|\nabla u_m(t)\|_2^2 \\
&\quad + \frac{1}{2} \int_\Omega \Gamma(t)\, dx \\
&\geq \frac{1}{2} \|u_m'(t)\|_2^2 + \frac{q-1}{2(q+1)} \|\nabla u_m(t)\|_2^2 + \frac{1}{2} \int_\Omega \Gamma(t)\, dx.
\end{aligned} \tag{3.8}
$$

Thus, (3.3) and (3.8) imply

$$
\begin{aligned}
&\frac{1}{2}\|u_m'(t)\|_2^2 + \frac{q-1}{2(q+1)}\|\nabla u_m(t)\|_2^2 + \frac{1}{2}\int_\Omega \Gamma(t)\,dx \\
&\quad + \int_0^t \delta\|u_m'(s)\|_{p+1}^{p+1}\,ds + \int_0^t \|\nabla u_m'(s)\|_2^2\,ds \\
&\le E(u_{0m}).
\end{aligned} \tag{3.9}
$$

A Priori Estimates II

Multiplying the equation (3.1) by $-\Delta u_m(t)$, then we have

$$
\begin{aligned}
&\frac{d}{dt}(u_m'(t), \Delta u_m(t)) - \|\nabla u_m'(t)\|_2^2 + \|\Delta u_m(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u_m(t)\|_2^2 \\
&\quad + \int_\Omega \operatorname{div}\{\sigma(|\nabla u_m(t)|^2)\nabla u_m(t)\}\Delta u_m(t)\,dx \\
&\quad - \delta\int_\Omega |u_m'(t)|^{p-1}u_m'(t)\Delta u_m(t)\,dx \\
&= -\mu\int_\Omega |u_m(t)|^{q-1}u_m(t)\Delta u_m(t)\,dx.
\end{aligned} \tag{3.10}
$$

Since $p \ge 1$, we can use the fact that $L^{2p}(\Omega) \hookrightarrow L^2(\Omega)$ and so (3.9) implies

$$
\begin{aligned}
&\left|\delta\int_\Omega |u_m'(t)|^{p-1}u_m'(t)\Delta u_m(t)\,dx\right| \\
&\le \delta\|u_m'(t)\|_{2p}^p\|\Delta u_m(t)\|_2 \\
&\le \delta C_*^p\|u_m'(t)\|_2^p\|\Delta u_m(t)\|_2 \\
&\le \delta C_*^p(2E(u_{0m}))^{\frac{p}{2}}\|\Delta u_m(t)\|_2 \\
&\le C_1 + \frac{1}{2}\|\Delta u_m(t)\|_2^2, \\
&\quad \text{where} \quad C_1 = \frac{1}{2}\delta^2C_*^{2p}(2E(u_{0m}))^p.
\end{aligned} \tag{3.11}
$$

On the other hand, Schwarz's inequality and (3.9) imply

$$
\begin{aligned}
&\left|\int_\Omega |u_m(t)|^{q-1}u_m(t)\Delta u_m(t)\,dx\right| \\
&\le q\|u_m(t)\|_{q+1}^{q-1}\|\nabla u_m(t)\|_{q+1}^2 \\
&\le qC_*^{q+1}\|\nabla u_m(t)\|_2^{q-1}\|\Delta u_m(t)\|_2^2 \\
&\le qC_*^{q+1}\left(\frac{2(q+1)}{q-1}E(u_{0m})\right)^{\frac{q-1}{2}}\|\Delta u_m(t)\|_2^2.
\end{aligned}
\tag{3.12}
$$

From (3.10)-(3.12), we get

(3.13)

$$
\begin{aligned}
&\frac{d}{dt}(u_m'(t),\Delta u_m(t)) + \frac{1}{2}\|\Delta u_m(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u_m(t)\|_2^2 \\
&\quad + \int_\Omega \operatorname{div}\{\sigma(|\nabla u_m(t)|^2)\nabla u_m(t)\}\Delta u_m(t)\,dx \\
&\le C_1 + \|\nabla u_m'(t)\|_2^2 + q\mu C_*^{q+1}\left(\frac{2(q+1)}{q-1}E(u_{0m})\right)^{\frac{q-1}{2}}\|\Delta u_m(t)\|_2^2.
\end{aligned}
$$

On the other hand, integration by parts gives

(3.14)

$$
\begin{aligned}
&\int_\Omega \operatorname{div}\{\sigma(|\nabla u_m|^2)\nabla u_m\}\Delta u_m\,dx \\
&= \sum_{i,j=1}^N\sum_{k=1}^N\int_\Omega\left\{\sigma(|\nabla u_m|^2)\frac{\partial^2 u_m}{\partial x_i\partial x_j} + 2\sigma'(|\nabla u_m|^2)\frac{\partial u_m}{\partial x_k}\frac{\partial^2 u_m}{\partial x_i\partial x_k}\frac{\partial u_m}{\partial x_i}\right\} \\
&\qquad \cdot\frac{\partial^2 u_m}{\partial x_i\partial x_j}\,dx \\
&\quad + (N-1)\int_{\partial\Omega}(\sigma(|\nabla u_m|^2) + 2\sigma'(|\nabla u_m|^2)|\nabla u_m|^2)\left|\frac{\partial u_m}{\partial n}\right|^2 H(x)\,dS, \\
&\ge \int_\Omega(\sigma(|\nabla u_m|^2) - 2|\sigma'(|\nabla u_m|^2)|\,|\nabla u_m|^2)|D^2u_m(t)|^2\,dx \\
&\quad + (N-1)\int_{\partial\Omega}(\sigma(|\nabla u_m|^2) + 2\sigma'(|\nabla u_m|^2)|\nabla u_m|^2)\left|\frac{\partial u_m}{\partial n}\right|^2 H(x)\,dS,
\end{aligned}
$$

where $H(x)$ denotes the mean curvature of $\partial\Omega$ at x with respect to the outward normal and $|D^2u_m|^2 = \sum_{i,j}|\frac{\partial^2 u_m}{\partial x_i\partial x_j}|^2$. By assumption (H_1) of Hypothesis and a standard trace theorem, we see

(3.15)

$$\left|(N-1)\int_{\partial\Omega}(\sigma(|\nabla u_m|^2)+2\sigma'(|\nabla u_m|^2)|\nabla u_m|^2)|\frac{\partial u_m}{\partial n}|^2 H(x)\,dS\right|$$
$$\le C_2\int_{\partial\Omega}\left|\frac{\partial u_m}{\partial n}\right|^2 dS$$
$$\le C_3\|\nabla u_m\|_2^{2(1-\theta_2)}\|\Delta u_m\|_2^{2\theta_2}$$
$$\le (1-\theta_2)C_3^{\frac{1}{1-\theta_2}}\|\nabla u_m\|_2^2+\theta_2\|\Delta u_m\|_2^2$$
$$\le C_3^{\frac{1}{1-\theta_2}}\frac{2(q+1)(1-\theta_2)}{q-1}E(u_{0m})+\theta_2\|\Delta u_m\|_2^2$$
$$\le C_4+\theta_2\|\Delta u_m\|_2^2,$$
$$\text{where}\quad C_4=C_3^{\frac{1}{1-\theta_2}}\frac{2(q+1)(1-\theta_2)}{q-1}E(u_{0m}).$$

Thus (3.13)-(3.15) give

(3.16)

$$\frac{d}{dt}(u_m'(t),\Delta u_m(t))+\frac{1}{2}\|\Delta u_m(t)\|_2^2+\frac{1}{2}\frac{d}{dt}\|\Delta u_m(t)\|_2^2$$
$$+\int_\Omega(\sigma(|\nabla u_m(t)|^2)-2|\sigma'(|\nabla u_m(t)|^2)|\;|\nabla u_m(t)|^2)|D^2u_m(t)|^2\,dx$$
$$\le C_1+\|\nabla u_m'(t)\|_2^2+q\mu C_*^{q+1}\left(\frac{2(q+1)}{q-1}E(u_{0m})\right)^{\frac{q-1}{2}}\|\Delta u_m(t)\|_2^2$$
$$+|(N-1)\int_{\partial\Omega}(\sigma(|\nabla u_m|^2)+2\sigma'(|\nabla u_m|^2)|\nabla u_m|^2)|\frac{\partial u_m}{\partial n}|^2H(x)\,dS|$$
$$\le C_1+C_4+\|\nabla u_m'(t)\|_2^2+C_6\|\Delta u_m(t)\|_2^2,$$
$$\text{where}\quad C_6=\theta_2+q\mu C_*^{q+1}\left(\frac{2(q+1)}{q-1}E(u_{0m})\right)^{\frac{q-1}{2}}.$$

Therefore,

(3.17)

$$\frac{d}{dt}(u_m'(t),\Delta u_m(t))+\frac{1}{2}\frac{d}{dt}\|\Delta u_m(t)\|_2^2$$
$$+\int_\Omega(\sigma(|\nabla u_m(t)|^2)-2|\sigma'(|\nabla u_m(t)|^2)|\;|\nabla u_m(t)|^2)|D^2u_m(t)|^2\,dx$$

$$\leq C_7 + \|\nabla u_m'(t)\|_2^2 + C_8\|\Delta u_m(t)\|_2^2,$$

where $C_7 = C_1 + C_4$ and $C_8 = |C_6 - \frac{1}{2}|$.

Integrating (3.17), from (3.9) we get

$$\begin{aligned}
&\int_0^t \int_\Omega (\sigma(|\nabla u_m(s)|^2) - 2|\sigma'(|\nabla u_m(s)|^2)|\ |\nabla u_m(s)|^2)|D^2 u_m(s)|^2\,dxds \\
&\quad + \frac{1}{2}\|\Delta u_m(t)\|_2^2 \\
&\leq \|u_m'(t)\|_2\|\Delta u_m(t)\|_2 + \|\nabla u_1\|_2\|\nabla u_0\|_2 + \frac{1}{2}\|\Delta u_0\|_2^2 \\
&\quad + \int_0^t (C_7 + \|\nabla u_m'(s)\|_2^2 + C_8\|\Delta u_m(s)\|_2^2)\,ds \\
&\leq C_9\|u_m'(t)\|_2^2 + \epsilon\|\Delta u_m(t)\|_2 + \|\nabla u_1\|_2\|\nabla u_0\|_2 + \frac{1}{2}\|\Delta u_0\|_2^2 + E(u_{0m}) \\
&\quad + \int_0^t (C_7 + C_8\|\Delta u_m(s)\|_2^2)\,ds \\
&\leq \epsilon\|\Delta u_m(t)\|_2 + C_{10} + C_{11}\int_0^t \|\Delta u_m(s)\|_2^2\,ds,
\end{aligned}$$

where $0 \leq \epsilon \leq \frac{1}{2}$. Thus, we have

(3.18)
$$\begin{aligned}
&\int_0^t \int_\Omega (\sigma(|\nabla u_m(s)|^2) - 2|\sigma'(|\nabla u_m(s)|^2)|\ |\nabla u_m(s)|^2)|D^2 u_m(s)|^2\,dxds \\
&\quad + (\frac{1}{2} - \epsilon)\|\Delta u_m(t)\|_2^2 \leq C_{10} + C_{11}\int_0^t \|\Delta u_m(s)\|_2^2\,ds.
\end{aligned}$$

Next, multiplying the equation (3.1) by $-\Delta u_m'(t)$, then we have

(3.19)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u_m'(t)\|_2^2 + \|\Delta u_m(t)\|_2^2\Big) \\
&\quad + p\delta\int_\Omega |u_m'(t)|^{p-1}|\nabla u_m'(t)|^2\,dx \\
&\quad + \|\Delta u_m'(s)\|_2^2 + \int_\Omega \operatorname{div}\{\sigma(|\nabla u_m(t)|^2)\nabla u_m(t)\}\Delta u_m'(t)\,dx \\
&= -\mu\int_\Omega |u_m(t)|^{q-1}u_m(t)\Delta u_m'(t)\,dx.
\end{aligned}$$

Integrating (3.19) from 0 to t, then we have

$$\begin{aligned}(3.20)\qquad &\frac{1}{2}\|\nabla u_m'(t)\|_2^2+\frac{1}{2}\|\Delta u_m(t)\|_2^2+\int_0^t\|\Delta u_m'(s)\|_2^2\,ds\\ &\quad+\int_0^t\int_\Omega \operatorname{div}\{\sigma(|\nabla u_m(s)|^2)\nabla u_m(s)\}\Delta u_m'(s)\,dxds\\ &\le\frac{1}{2}\|\nabla u_1\|_2^2+\frac{1}{2}\|\Delta u_0\|_2^2\\ &\quad+\mu\Big|\int_0^t\int_\Omega |u_m(s)|^{q-1}u_m(s)\Delta u_m'(s)\,dxds\Big|,\end{aligned}$$

where we have used the fact that $p\delta\int_0^t\int_\Omega |u_m'(s)|^{p-1}|\nabla u_m'(s)|^2\,dxds\ge 0$.

Note that

$$\begin{aligned}(3.21)\qquad &\mu\int_0^t\int_\Omega |u_m(s)|^{q-1}u_m(s)\Delta u_m'(s)\,dxds\\ &\le q\mu\int_0^t\int_\Omega |u_m(s)|^{q-1}|\nabla u_m(s)|\,|\nabla u_m'(s)|\,dxds\\ &\le q\mu\int_0^t\||u_m(s)|^{q-1}\nabla u_m(s)\|_2\|\nabla u_m'(s)\|_2\,ds.\end{aligned}$$

Sobolev-Poincaré's inequality implies

$$\begin{aligned}(3.22)\qquad \||u_m(s)|^{q-1}\nabla u_m(s)\|_2 &\le C_{12}\|u_m(s)\|_{(q-1)N}^{q-1}\|\nabla u_m(s)\|_{\frac{2N}{N-2}}\\ &\le C_{13}\|u_m(s)\|_{(q-1)N}^{q-1}\|\Delta u_m(s)\|_2.\end{aligned}$$

Now, in the case $\frac{N}{N-2}\le q\le\min\{\frac{N+2}{N-2},\frac{N-2}{[N-4]^+}\}$ $(N\ge 4)$, we observe from Gagliardo-Nirenberg's inequality that

$$\begin{aligned}(3.23)\qquad &\|u_m(s)\|_{(q-1)N}^{q-1}\\ &\le C_{14}\|u_m(s)\|_{\frac{2N}{N-2}}^{(q-1)(1-\theta_3)}\|\Delta u_m(s)\|_2^{(q-1)\theta_3}\\ &\le C_{15}\|\nabla u_m(s)\|_2^{(q-1)(1-\theta_3)}\|\Delta u_m(s)\|_2^{(q-1)\theta_3}\end{aligned}$$

$$\le (1-(q-1)\theta_3)C_{15}^{\frac{1}{1-(q-1)\theta_3}}\|\nabla u_m(s)\|_2^{\frac{(q-1)(1-\theta_3)}{1-(q-1)\theta_3}}$$
$$+ (q-1)\theta_3\|\Delta u_m(s)\|_2$$
$$\le (1-(q-1)\theta_3)C_{15}^{\frac{1}{1-(q-1)\theta_3}}\left(\frac{2(q+1)}{q-1}E(u_{0m})\right)^{\frac{(q-1)(1-\theta_3)}{2(1-(q-1)\theta_3)}}$$
$$+ (q-1)\theta_3\|\Delta u_m(s)\|_2$$
$$\text{with} \quad \theta_3 = \frac{N-2}{2} - \frac{1}{q-1} (< 1).$$

Thus, (3.22) and (3.23) imply

(3.24)
$$\||u_m(s)|^{q-1}\nabla u_m(s)\|_2$$
$$\le C_{13}(C_{16} + (q-1)\theta_3\|\Delta u_m(s)\|_2)\|\Delta u_m(s)\|_2$$
$$\le C_{17}(\|\Delta u_m(s)\|_2 + \|\Delta u_m(s)\|_2^2),$$

where $C_{16} = (1-(q-1)\theta_3)C_{15}^{\frac{1}{1-(q-1)\theta_3}}\left(\frac{2(q+1)}{q-1}E(u_{0m})\right)^{\frac{(q-1)(1-\theta_3)}{2(1-(q-1)\theta_3)}}$.

Therefore, from (3.21) and (3.24), we have

$$\mu\int_0^t\int_\Omega |u_m(s)|^{q-1}u_m(s)\Delta u'_m(s)\,dxds$$
$$\le q\mu C_{17}\int_0^t (\|\Delta u_m(s)\|_2 + \|\Delta u_m(s)\|_2^2)\|\nabla u'_m(s)\|_2\,ds. \tag{3.25}$$

On the other hand, since

(3.26)
$$\operatorname{div}\{\sigma(|\nabla u_m|^2)\nabla u_m\}$$
$$= \sigma(|\nabla u_m|^2)\Delta u_m + 2\sigma'(|\nabla u_m|^2)\sum_{i=1}^N \frac{\partial u_m}{\partial x_i}\cdot\sum_{i,j=1}^N \frac{\partial u_m}{\partial x_j}\frac{\partial^2 u_m}{\partial x_i\partial x_j}$$
$$\le \sigma(|\nabla u_m|^2)\Delta u_m + 2\sigma'(|\nabla u_m|^2)\sum_{i=1}^N\left|\frac{\partial u_m}{\partial x_i}\right|^2\cdot\left(\sum_{i,j=1}^N \left|\frac{\partial^2 u_m}{\partial x_i\partial x_j}\right|^2\right)^{\frac{1}{2}}$$
$$= \sigma(|\nabla u_m|^2)\Delta u_m + 2\sigma'(|\nabla u_m|^2)|\nabla u_m|^2|D^2u_m|$$
$$\le (\sigma(|\nabla u_m|^2) + 2\sigma'(|\nabla u_m|^2)|\nabla u_m|^2)|D^2 u_m|,$$

we get

$$
\begin{aligned}
&\left|\int_0^t \int_\Omega \operatorname{div}\{\sigma(|\nabla u_m(s)|^2)\nabla u_m(s)\}\Delta u_m'(s)\,dxds\right| \\
(3.27)\qquad &\le \frac{1}{2}\int_0^t \int_\Omega \Big(\sigma(|\nabla u_m(s)|^2) + 2\sigma'(|\nabla u_m(s)|^2)|\nabla u_m(s)|^2\Big)^2 \\
&\qquad |D^2 u_m(s)|^2\,dxds \\
&\quad + \frac{1}{2}\int_0^t \int_\Omega |\Delta u_m'(s)|^2\,dxds.
\end{aligned}
$$

Now, assumption (H_2) implies that
(3.28)

$$
\begin{aligned}
&\frac{1}{2}\int_0^t \int_\Omega \{\sigma(|\nabla u_m(s)|^2) + 2\sigma'(|\nabla u_m(s)|^2)|\nabla u_m(s)|^2\}^2 |D^2 u_m(s)|^2\,dxds \\
&= \frac{1}{2}\int_0^t \int_\Omega \{\sigma(|\nabla u_m(s)|^2) + 2\sigma'(|\nabla u_m(s)|^2)|\nabla u_m(s)|^2\}^2 \\
&\qquad |D^2 u_m(s)|^{\frac{4}{r}} |D^2 u_m(s)|^{\frac{2(r-2)}{r}}\,dxds \\
&\le \frac{1}{2}\Big(\int_0^t \int_\Omega \{\sigma(|\nabla u_m(s)|^2) + 2\sigma'(|\nabla u_m(s)|^2)|\nabla u_m(s)|^2\}^r \\
&\qquad |D^2 u_m(s)|^2 dxds\Big)^{\frac{2}{r}} \Big(\int_0^t \int_\Omega |D^2 u_m(s)|^2 dxds\Big)^{\frac{r-2}{r}} \\
&\le \frac{C_{18}}{2}\Big(\int_0^t \int_\Omega \{\sigma(|\nabla u_m(s)|^2) - 2|\sigma'(|\nabla u_m(s)|^2)|\ |\nabla u_m(s)|^2\} \\
&\qquad |D^2 u_m(s)|^2 dxds\Big)^{\frac{2}{r}} \Big(\int_0^t \int_\Omega |D^2 u_m(s)|^2 dxds\Big)^{\frac{r-2}{r}} \\
&\le \frac{C_{18}}{2}\Big(C_{10} + C_{11}\int_0^t \|\Delta u_m(s)\|_2^2 ds\Big)^{\frac{2}{r}} \Big(\sup_t \|\Delta u_m(t)\|_2^2\Big)^{\frac{r-2}{r}} \\
&\le C_{19} + C_{20}\int_0^t \|\Delta u_m(s)\|_2^2\,ds + \epsilon\|\Delta u_m(t)\|_2^2,
\end{aligned}
$$

where we have used (3.17) and the Young's inequality at the last stage.

Thus, from (3.27) and (3.28), we have

(3.29)

$$\left|\int_0^t\int_\Omega \operatorname{div}\{\sigma(|\nabla u_m(s)|^2)\nabla u_m(s)\}\Delta u_m'(s)dxds\right|$$
$$\le C_{19}+\epsilon\|\Delta u_m(t)\|_2^2+C_{20}\int_0^t\|\Delta u_m(s)\|_2^2\,ds+\frac{1}{2}\int_0^t\|\Delta u_m'(s)\|_2^2\,ds.$$

Hence (3.20), (3.25) and (3.29) imply

$$\begin{aligned}&\frac{1}{2}\|\nabla u_m'(t)\|_2^2+(\frac{1}{2}-\epsilon)\|\Delta u_m(t)\|_2^2+\frac{1}{2}\int_0^t\|\Delta u_m'(s)\|_2^2\,ds\\&\le\frac{1}{2}\|\nabla u_1\|_2^2+\frac{1}{2}\|\Delta u_0\|_2^2\\&\quad+q\mu C_{17}\int_0^t(\|\Delta u_m(s)\|_2+\|\Delta u_m(s)\|_2^2)\|\nabla u_m'(s)\|_2\,ds\\&\quad+C_{19}+C_{20}\int_0^t\|\Delta u_m(s)\|_2^2\,ds.\end{aligned}\tag{3.30}$$

Therefore we have

$$\begin{aligned}&\|\nabla u_m'(t)\|_2^2+\|\Delta u_m(t)\|_2^2+\int_0^t\|\Delta u_m'(s)\|_2^2\,ds\\&\le C_{21}+C_{22}\int_0^t\Big((\|\Delta u_m(s)\|_2+\|\Delta u_m(s)\|_2^2)\|\nabla u_m'(s)\|_2\\&\qquad+\|\Delta u_m(s)\|_2^2\Big)\,ds,\end{aligned}\tag{3.31}$$

where C_{21} and C_{22} are some constants. Therefore, we obtain

$$E_1(t)\le C_{21}+C_{23}\int_0^t E_1(s)+E_1(s)^{\frac{3}{2}}\,ds,$$

where $E_1(t)=\|\nabla u_m'(t)\|_2^2+\|\Delta u_m(t)\|_2^2$.

Now we set $g(s)=s+s^{\frac{3}{2}}$ on $s\ge 0$. Then we have

$$E_1(t)\le C_{21}+C_{23}\int_0^t g(E(s))\,ds.$$

Note that $g(s)$ is continuous and nondecreasing on $s \geq 0$. By applying Bihari-Langenhop's inequality, we get

$$E_1(t) \leq G^{-1}(C_{23}t), \quad \text{where} \quad G(s) = \int_{C_{21}}^{s} \frac{1}{g(r)}\, dr, \ s > C_{21} > 0$$

and this estimate imply

$$E_1(t) = \|\nabla u_m'(t)\|_2^2 + \|\Delta u_m(t)\|_2^2 \leq M_1(T) \tag{3.32}$$

for some constant $M_1 > 0$ independent of m.

In particular, (3.31) and (3.32) imply

$$\int_0^t \|\Delta u_m'(s)\|_2^2\, ds \leq M_2(T). \tag{3.33}$$

A Priori Estimates III

Finally, multiplying the equation (3.1) by $u_m''(t)$, then we have

(3.34)

$$\begin{aligned}
&\|u_m''(t)\|_2^2 - \int_\Omega \operatorname{div}\{\sigma(|\nabla u_m(t)|^2)\nabla u_m(t)\} u_m''(t)\, dx \\
&\quad - (\Delta u_m(t), u_m''(t)) - (\Delta u_m'(t), u_m''(t)) + \delta |u_m'(t)|^{p-1}(u_m'(t), u_m''(t)) \\
&= \mu |u_m(t)|^{q-1}(u_m(t), u_m''(t)).
\end{aligned}$$

Note that from $L^{2p}(\Omega) \hookrightarrow L^2(\Omega)$ and (3.9),

$$\begin{aligned}
\delta |u_m'(t)|^{p-1}(u_m'(t), u_m''(t)) &\leq \delta \int_\Omega |u_m'(t)|^p |u_m''(t)|\, dx \\
&\leq \delta \|u_m'(t)\|_{2p}^p \|u_m''(t)\|_2 \\
&\leq \delta C_*^p \|u_m'(t)\|_2^p \|u_m''(t)\|_2 \\
&\leq C_{27} \|u_m''(t)\|_2.
\end{aligned} \tag{3.35}$$

Now, it follows from (3.9) the Gagliardo-Nirenberg's inequality and our assumptions on q that

(3.36)

$$\begin{aligned}
\mu |u_m(t)|^{q-1}(u_m(t), u_m''(t)) &\leq \mu \|u_m(t)\|_{2q}^q \|u_m''(t)\|_2 \\
&\leq C_{28} \|\nabla u_m(t)\|_2^{\theta_4} \|u_m(t)\|_2^{1-\theta_4} \|u_m''(t)\|_2 \\
&\leq C_{29} \|u_m''(t)\|_2^2 \quad \text{with} \quad \theta_4 = \frac{(q-1)N}{2q}
\end{aligned}$$

for some constants C_{28} and C_{29}. Now, we have

$$\begin{aligned}
&\operatorname{div}\{\sigma(|\nabla u_m|^2)\nabla u_m\} \\
&= \nabla \cdot \{\sigma(|\nabla u_m|^2)\nabla u_m\} \\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_j}\{\sigma(|\nabla u_m|^2)\frac{\partial}{\partial x_j}\} \qquad (3.37)\\
&= \sigma(|\nabla u_m|^2)\Delta u_m + 2\sigma'(|\nabla u_m|^2)\sum_{i,j=1}^{N} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\frac{\partial^2 u_m}{\partial x_i \partial x_j} \\
&\le \sigma(|\nabla u_m|^2)|\Delta u_m| + 2|\sigma'(|\nabla u_m|^2)|\ |\nabla u_m|^2 |D^2 u_m|.
\end{aligned}$$

Thus, our assumption (H_1) implies that

(3.38)

$$\left|\int_\Omega \operatorname{div}\{\sigma(|\nabla u_m(t)|^2)\nabla u_m(t)\} u_m''(t)\,dx\right| \le C_{30}\|\Delta u_m(t)\|_2 \|u_m''(t)\|_2.$$

Thus (3.34)-(3.38) imply

$$\|u_m''(t)\|_2^2 \le C_{31}\Big(1 + \|\Delta u_m(t)\|_2 + \|\Delta u_m'(t)\|_2\Big)\|u_m''(t)\|_2 \tag{3.39}$$

Thus, from (3.32), (3.33) and (3.39) we have

(3.40)

$\|u_m''(t)\|_2 \le M_3(T)$ for some constant $M_3 > 0$ independent of m.

Limiting process

By above estimates, $\{u_m\}$ has a subsequence still denoted by $\{u_m\}$ such that

$$u_m \to u \quad \text{in} \quad L^\infty(0,T; H_0^1(\Omega)\cap H^2(\Omega)) \quad \text{weak}^*, \tag{3.41}$$

$$u_m' \to u' \quad \text{in} \quad L^\infty(0,T; H_0^1(\Omega)) \quad \text{weak}^*, \tag{3.42}$$

$$u_m'' \to u'' \quad \text{in} \quad L^\infty(0,T; L^2(\Omega)) \quad \text{weak}^*, \tag{3.43}$$

$$u'_m \to u' \quad \text{in} \quad L^2(0,T;H_0^1(\Omega)) \quad \text{weak}, \tag{3.44}$$

$$u'_m \to u' \quad \text{in} \quad L^{p+1}(0,T;L^{p+1}(\Omega)) \quad \text{weak}, \tag{3.45}$$

$$\text{div}\{\sigma(|\nabla u_m|^2)\nabla u_m\} \to \xi \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{weak}^*, \tag{3.46}$$

$$\begin{aligned} &|u'_m|^{p-1}u'_m \to \phi \quad \text{in} \quad L^{\frac{p+1}{p}}((0,T)\times\Omega) \quad \text{weak}, \\ &|u_m|^{q-1}u_m \to \psi \quad \text{in} \quad L^{\frac{q+1}{q}}((0,T)\times\Omega) \quad \text{weak}. \end{aligned} \tag{3.47}$$

It follows from a classical compactness argument (cf. Lions [6]) that as $m \to \infty$

$$|u'_m|^{p-1}u'_m \to |u'|^{p-1}u' \text{ in } L^{\frac{p+1}{p}}((0,T)\times\Omega) \text{ weak}, \tag{3.48}$$

$$|u_m|^{q-1}u_m \to |u|^{q-1}u \text{ in } L^{\frac{q+1}{q}}((0,T)\times\Omega) \text{ weak}. \tag{3.49}$$

We shall show that $\xi = \text{div}\{\sigma(|\nabla u|^2)\nabla u\}$ is satisfied. For any $w \in C_0(0,\infty;H_0^1(\Omega))$, we have

$$\begin{aligned} &\int_0^T (\xi - \text{div}\{\sigma(|\nabla u|^2)\nabla u\}, w)\,dt \\ &= \int_0^T (\xi - \text{div}\{\sigma(|\nabla u_m|^2)\nabla u_m\}, w)\,dt \\ &\quad + \int_0^T (\text{div}\{\sigma(|\nabla u_m|^2)\nabla u_m\} - \text{div}\{\sigma(|\nabla u|^2)\nabla u\}, w)\,dt. \end{aligned} \tag{3.50}$$

It follows from (3.46) that the first term of the right hand sides of (3.50) tends to zero. Also, (3.41) and mean value theorem imply

$$\begin{aligned} &\int_0^T (\text{div}\{\sigma(|\nabla u_m|^2)\nabla u_m\} - \text{div}\{\sigma(|\nabla u|^2)\nabla u\}, w)\,dt \\ &= \int_0^T (\text{div}\{\sigma(|\nabla u_m|^2)(\nabla u_m - \nabla u)\}, w)\,dt \\ &\quad - \text{div}\{(\sigma(|\nabla u_m|^2) - \sigma(|\nabla u|^2))\nabla u\}, w)\,dt \\ &\le C_{24}\int_0^T \|\nabla u_m - \nabla u\|_2\|\nabla w\|_2 dt \\ &\quad + C_{25}\max_s\{\sigma'(s)\}\int_0^T \|\nabla u_m - \nabla u\|_2\|\nabla w\|_2\,dt \\ &\quad \to 0 \text{ as } m \to \infty \end{aligned} \tag{3.51}$$

and hence we conclude $\xi = \text{div } \{\sigma(|\nabla u|^2)\nabla u\}$. On the other hand, we can use Aubin-Lions' compactness lemma (cf. proposition 2.4 and 2.5) and so we can extract from $\{u_m\}$ subsequence still denoted by $\{u_m\}$ such that for each $t \in [0, T]$

$$u_m(t) \to u(t) \quad \text{strongly in} \quad H_0^1(\Omega). \tag{3.52}$$

By letting $m \to \infty$ in (3.1), we can find that u satisfies the equation:

$$\begin{aligned} &(u''(t) - \text{div}\{\sigma(|\nabla u(t)|^2)\nabla u(t)\} - \Delta u(t) - \Delta u'(t), w) \\ &\quad + \delta|u'(t)|^{p-1}(u'(t), w) = \mu|u(t)|^{q-1}(u(t), w) \\ &\quad \text{for all} \quad w \in H_0^1(\Omega). \end{aligned} \tag{3.53}$$

Now, the above result (3.52) implies

$$u_m(0) = u_{0m} \to u(0) \quad \text{strongly in} \quad H_0^1(\Omega). \tag{3.54}$$

Thus, from (3.1) and (3.54), $u(0) = u_0$. Also, from (3.42) we obtain

$$(u_m'(0) - u'(0), w) \to 0 \quad \text{as} \quad m \to \infty \quad \text{for each} \quad w \in H_0^1(\Omega). \tag{3.55}$$

Thus, (3.1) and (3.55) imply $u'(0) = u_1$. This completes the proof of Theorem 2.6. □

References

[1] R. A. Adams, *Sobolev spaces*, Academic press, New York, San Francisco, London, 1975.

[2] N. D. Alikakos and R. Rostamian, *Gradient estimates for degenerate diffusion equations*, Math. Ann. **259** (1982), 53-70.

[3] R. Bellman, *Inequalities*, Springer-Verlag Berlin, Heidelberg, New York, 1971.

[4] H. Engler, *Gradient estimates for solutions of parabolic equations and systems*, J. Math. Anal. Appl. **47** (1990), 309-329.

[5] J. Greenberg, *On the existence, uniqueness and stability of the equation* $\rho x_{tt} = E(X_x)X_{xx} + X_{xxt}$, J. Math. Anal. Appl. **25** (1969), 575-591.

[6] J. L. Lions, *Quelques méthode de Résolution des probléme aux limites Non-linéaire*, Dunod Gauthier-Villars, Paris, 1969.

[7] S. Kawashima and Y. Shibata, *Global existence and exponential stability of small solutions to nonlinear viscoelasticity*, Comm. Math. Phys. **148** (1992), 189-208.

[8] T. Matsuyama and R. Ikehata, *On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms*, J. Math. Anal. Appl. **204** (1996), 729-753.
[9] K. Mizohata and S. Ukai, *The global existence, uniqueness of small amplitude solutions to the nonlinear acoustic wave equations*, J. Math. Kyoto Univ. **33** (1993), 505-522.
[10] M. Nakao, *Energy decay for the quasilinear wave equation with viscosity*, Math. Z. **219** (1995), 289-299.
[11] ———, *Existence of an anti-periodic solution for the quasilinear wave equation with viscosity*, J. Math. Anal. Appl. **204** (1996), 754-764.
[12] ———, *On strong solutions of the quasilinear wave equation with viscosity*, Adv. Math. Sci. Appl. **6** (1996), 267-278.
[13] K. Narasimha, *Nonlinear vibration of an elastic string*, J. Sound Vib. **8** (1968), 134-146.
[14] R. Temam, *Infinite dimensional dynamical systems in mechanics and physics*, vol. 68, Applied Math. Sci., Springer-Verlag, 1988.

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