J. Korean Math. Soc. **37** (2000), No. 2, pp. 245–251

ONE REMARK FOR CR EQUIVALENCE PROBLEM

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ABSTRACT. Assume that two boundaries of worm domains, which are parametrized by harmonic functions, are CR equivalent. Then we determine the Taylor expansion of CR equivalence mapping and get a relation of harmonic functions.

1. Introduction

In this article, we study the CR equivalence problem for the boundaries of certain weakly pseudoconvex domains. Originary, E. Cartan studied this problem for hypersurfaces of nondegenerate Levi forms, and S. S. Chern, J. K. Moser ([2]) and N. Tanaka succeeded him. Their basic idea is the following ([1]). Let M be a $2n+1$ dimensional real hypersurface and E a line bundle generated by an annihilator of complexified holomorphic tangent bundle of M. Denote by Y its principal fiber bundle. Then in Y, we can find one forms so that the total number equals the dimension of Y and they are everywhere linearly independent. Denote by W a set of such one forms. Let M' be another $2n+1$ dimensional real hypersurface and suppose that its corresponding concepts are denoted by dashes. Then finding a local biholomorphic mapping $f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ with $f(M) \subset M'$ is reduced to finding a real analytic diffeomorphism $\tilde{f} : Y \to Y'$ under which the forms in W are respectively equal to the forms in W'.

Recently two new ideas have been studied. One idea is studied by J. M. Lee and D. Burns. They studied the CR equivalence problem in terms of CR automorphisms of hypersurfaces. J. M. Lee ([6]) proved the following theorem by the use of Webster metric ([7]).

Received May 30, 1999.
1991 Mathematics Subject Classification: 32C16, 32F40.
Key words and phrases: worm domain, CR mappings, CR equivalence.

THEOREM. *Let M be a compact, connected, strictly pseudoconvex CR manifold of dimension $2n+1 \geq 3$. Then the identity component of CR automorphisms of M is compact, unless M is globally CR equivalent to the $(2n+1)$-sphere with its standard CR structure.*

There are several theorems on equivalence problem studied from view point of automorphisms of manifolds in the conformal geometry. But, as far as the author knows, there are only a few in the CR geometry.

Another idea is related to the extension problem of proper holomorphic mappings. Making use of Fefferman's Theorem ([4]), if there exists a biholomorphic mapping between strictly pseudoconvex domains with C^∞ boundaries, then these boundaries are CR equivalent. In case of weakly pseudoconvex domains, A. Kodama ([5]) considers the domain, called a generalized complex ellipsoid,

$$E(k,\alpha) = \left\{ (z_1,\dots,z_n) \in \mathbb{C}^n : \sum_{j=1}^{k} |z_j|^2 + \left(\sum_{j=k+1}^{n} |z_j|^2 \right)^{\alpha} < 1 \right\}$$

and he gets conditions on k, α, l, β for the boundaries of $E(k,\alpha)$ and $E(l,\beta)$ being CR equivalent by the use of Webster metric.

To state a main result of this article, we shall prepare some notation. We consider the hypersurface in $\mathbb{C}^2$ defined as

$$M_h = \{(z,w) \in \mathbb{C}^2 \mid r_h := |z - \exp\{ih(w,\bar{w})\}|^2 - 1 = 0\},$$

which is a boundary of a worm domain ([3]). Here $h(w,\bar{w})$ is a harmonic function defined on a neighborhood of the origin with $h(0,0) = 0$. Since this hypersurface contains a complex line $\{z = 0\}$, it is of infinite type, and M_h fails to be strictly pseudoconvex precisely on this line. Let h' be another harmonic function. Assume that there exists a real analytic CR equivalence mapping $F : M_h \to M_{h'}$ defined on a neighborhood of the origin with $F(0) = 0$. Then, by definition, we have $F(\{z = 0\}) \subset \{z = 0\}$. The main result says that if there exists such a mapping, then each component is a "variable split" function.

THEOREM. *Let $F = (f,g) : M_h \to M_{h'}$ be as above. Then*

$$f = z, \quad g = \sum_{q \geq 1} b_q w^q, \quad b_1 \neq 0,$$

where (z, w) *is a standard coordinate of* $\mathbb{C}^2$, *and the relation* $h = h'(g, \bar{g})$ *holds.*

This theorem treats very simple infinite type hypersurfaces. But since the CR equivalence problem of such hypersurfaces has not studied well yet and the determining the forms of CR equivalence mappings has not studied, this theorem is still interesting. In the proof of Theorem, the property that "M_h fails to be strictly pseudoconvex precisely on the line" is not used effectively, so we may say more about a form of CR equivalence mapping by using this property.

2. The uniqueness theorem for CR functions and their Taylor expansions

Let M_h be a hypersurface as in §1. Since $(\partial r_h/\partial z)(0) \neq 0$, $r_h = 0$ can be rewritten as

$$z + \sum_{j\geq 1} \bar{z}^j \exp\{(j+1)ih(w, \bar{w})\} = 0.$$

Therefore M_h is parameterized by three independent variables $(\bar{z}, w, \bar{w})$. We shall use these variables, unless otherwise specified. Let

$$\begin{aligned}\bar{L} = &\sum_{j\geq 1} j\bar{z}^{j-1} \exp\{(j+1)ih(w, \bar{w})\} \frac{\partial}{\partial \bar{w}} \\ &\qquad - i\frac{\partial h}{\partial \bar{w}} \sum_{j\geq 1} (j+1)\bar{z}^j \exp\{(j+1)ih(w, \bar{w})\} \frac{\partial}{\partial \bar{z}}\end{aligned}$$

be a tangential Cauchy–Riemann vector field defined on a neighborhood of the origin.

LEMMA 1. *If* f *is a real analytic CR function on* M_h *with* $f(\bar{z}, w, \bar{w})|_{\bar{w}=0} = 0$, *then* f *is identically zero.*

Proof. Expand $f(\bar{z}, w, \bar{w})$ near the origin as

$$f(\bar{z}, w, \bar{w}) = \sum_{\alpha\geq 0} f_\alpha(\bar{z}, w)\bar{w}^\alpha.$$

Then $f(\bar{z}, w, \bar{w})|_{\bar{w}=0} = 0$ implies $f_0(\bar{z}, w) = 0$. Observing coefficients of $\bar{w}^0, \bar{w}^1, \ldots$ in $\bar{L}f = 0$, we have $f_1(\bar{z}, w) = f_2(\bar{z}, w) = \cdots = 0$. This completes the proof. □

By Lemma 1, we obtain the following expansion of a CR function.

LEMMA 2. *Any real analytic CR function defined on M_h is expanded as*

$$f(\bar{z}, w, \bar{w}) = \sum_{\alpha, p \geq 0} a_{\alpha,p} \left(- \sum_{j \geq 1} \bar{z}^j \exp\{(j+1) i h(w, \bar{w})\} \right)^{\alpha} w^p.$$

Proof. Expand $f(\bar{z}, w, \bar{w})$ on $M_h \cap \{\bar{w} = 0\}$ as

$$f(\bar{z}, w, \bar{w})|_{\bar{w}=0} = \sum_{\alpha, p \geq 0} \tilde{a}_{\alpha,p} \bar{z}^{\alpha} w^p.$$

Then define $a_{\alpha,p}$ inductively by

$$\sum_{\alpha, p \geq 0} \tilde{a}_{\alpha,p} \bar{z}^{\alpha} w^p = \sum_{\alpha, p \geq 0} a_{\alpha,p} \left(- \sum_{j \geq 1} \bar{z}^j \exp\{(j+1) i h(w, 0)\} \right)^{\alpha} w^p,$$

and put

$$F(\bar{z}, w, \bar{w}) = \sum_{\alpha, p \geq 0} a_{\alpha,p} \left(- \sum_{j \geq 1} \bar{z}^j \exp\{(j+1) i h(w, \bar{w})\} \right)^{\alpha} w^p.$$

Since we have $(f - F)|_{\bar{w}=0} = 0$ and $\bar{L}(f - F) = 0$, we get the conclusion by Lemma 1. □

Let F be a CR equivalence mapping as in Theorem. Then it follows from $F(\{z = 0\}) \subset \{z = 0\}$ that each component f, g are expanded as

$$f(\bar{z}, w, \bar{w}) = \sum_{\substack{\alpha \geq 1, \\ p \geq 0}} a_{\alpha,p} \left(- \sum_{j \geq 1} \bar{z}^j \exp\{(j+1) i h(w, \bar{w})\} \right)^{\alpha} w^p,$$

$$g(\bar{z}, w, \bar{w}) = \sum_{\beta + q \geq 1} b_{\beta,q} \left(- \sum_{j \geq 1} \bar{z}^j \exp\{(j+1) i h(w, \bar{w})\} \right)^{\beta} w^q.$$

3. Proof of Theorem

Let $(\mathcal{F}, \mathcal{G}) : M_{h'} \to M_h$ be an inverse mapping of (f, g), and suppose that $\mathcal{G}$ is expanded with coefficients $B_{\nu,m}$ as in the last part of §2. If there exist ν (≥ 1) and m such that $B_{\nu,m} \neq 0$, then, by $f = \mathcal{O}(\bar{z})$,

$$\begin{aligned}\mathcal{G}(\bar{f}, g, \bar{g}) &= \sum_{m\geq 1} B_{0,m} g^m + \mathcal{O}(\bar{z}) \\ &= w.\end{aligned}$$

Therefore such ν, m do not exist. Since the same is true for the coefficients of g, we have $b_{\beta,q} = 0$ for $\beta \geq 1, q \geq 0$. Put $b_{0,q} = b_q$. Now we get the expansion of g. Since (f, g) is a CR equivalence mapping, we have $a_{1,0}, b_1 \neq 0$.

As $(f, g)(M_h) \subset M_{h'}$ near the origin, we get

$$|f|^2 - f\exp\{-ih'(g, \bar{g})\} - \bar{f}\exp\{ih'(g, \bar{g})\} = 0. \tag{1}$$

Substituting the expansions of f and g into (1) and observing the term $\bar{z}$, we have the relation

$$\begin{aligned}&\sum_{p\geq 1} a_{1,p} w^p \exp\{ih(w, \bar{w}) - ih'(g, \bar{g})(w, \bar{w})\} \\ &\qquad = \sum_{p\geq 1} \overline{a_{1,p}} \bar{w}^p \exp\{-ih(w, \bar{w}) + ih'(g, \bar{g})(w, \bar{w})\}.\end{aligned} \tag{2}$$

Let $H(w)$ and $H'(w)$ be holomorphic functions with $2\mathrm{Re}H(w) = h(w, \bar{w})$ and $2\mathrm{Re}H'(w) = h'(g, \bar{g})(w, \bar{w})$ near the origin. Then above relation implies that $a_{1,0} \in \mathbb{R}$ and that $a_{1,0}\exp\{i(\overline{H(w)} - \overline{H'(w)})\}$ is a real valued holomorphic function. It follows from $a_{1,0} \neq 0$ that we conclude that $\overline{H(w)} = \overline{H'(w)}$, namely, $h(w, \bar{w}) = h'(g, \bar{g})(w, \bar{w})$, which is the desired relation of h and h'. This relation together with (2) implies $a_{1,p} = 0$ for $p \geq 1$. Substituting these into the expansion of f, and picking the coefficients of $\bar{z}^2$ in (1), we get

$$\begin{aligned}&\sum_{p\geq 0} a_{2,p} w^p \exp\{ih(w, \bar{w})\} + \frac{1}{2}(a_{1,0}^2 - a_{1,0}) \\ &\qquad = -\sum_{p\geq 0} \overline{a_{2,p}} \bar{w}^p \exp\{-ih(w, \bar{w})\} - \frac{1}{2}(a_{1,0}^2 - a_{1,0}).\end{aligned} \tag{3}$$

This implies that the left hand side is a purely imaginary valued function for any independent variables $w, \bar{w}$. Putting $w = 0$, we get a purely imaginary valued holomorphic function $a_{2,0}\exp\{ih(0,w)\}+(1/2)(a_{1,0}^2-a_{1,0})$, which implies $a_{2,0} = 0$ and $a_{1,0} = 1$. Then substituting these into (3) and putting $\bar{w} = 0$, we get a purely imaginary valued holomorphic function $\exp\{ih(w,0)\}\sum_{p\geq 1} a_{2,p}w^p$. This means $a_{2,p} = 0$ for $p \geq 0$. We shall prove $a_{\alpha,p} = 0$ for $p \geq 0, \alpha \geq 3$ by induction. Assume that $a_{2,p} = \cdots = a_{\lambda-1,p} = 0$ for $p \geq 0$. Substituting these assumptions and $a_{1,0} = 0$ into the expansion of f, from (1) we have

$$(-1)^\lambda \sum_{p\geq 0} a_{\lambda,p}w^p\exp\{-ih(w,\bar{w}) - 2i\lambda h(w,\bar{w})\} - \sum_{p\geq 0} \overline{a_{\lambda,p}}\bar{w}^p\exp\{ih(w,\bar{w})\} = 0$$

by picking up the coefficient of $\bar{z}^\lambda$ from the resulting equality. By the same argument as above, we get $a_{\lambda,p} = 0$ for any $p \geq 0$. As a result, we conclude that $a_{\alpha,p} = 0$ for $(\alpha,p) \neq (1,0)$ and therefore the expansion of f is

$$f(\bar{z},w,\bar{w}) = -\sum_{j\geq 1} \bar{z}^j\exp\{(j+1)ih(w,\bar{w})\}.$$

Now changing the variables into the standard coordinates in $\mathbb{C}^2$, we conclude that $f = z$. This completes the proof. □

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