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FATOU THEOREMS OLD AND NEW: AN OVERVIEW OF THE BOUNDARY BEHAVIOR OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. We consider the boundary behavior of a Hardy class holomorphic function, either on the disc D in the complex plane or on a domain in multi-dimensional complex space. Although the two theories are formally different, we postulate some unifying features, and we suggest some future directions for research.

0. Introduction

In 1906, Pierre Fatou [9] proved the following remarkable theorem:

THEOREM 0.1. *Let f be a bounded holomorphic function on the unit disc D in $\mathbb{C}$. Then, for almost every $\theta \in [0, 2\pi)$, the limit*

$$f^*(e^{i\theta}) \equiv \lim_{r \to 1^-} f(re^{i\theta})$$

exists.

It is of interest to say a few words about the proof. If the holomorphic function has power series expansion

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

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then in polar coordinates we may write

$$f(re^{i\theta}) = \sum_{j=0}^{\infty} a_j r^j e^{ij\theta}.$$

Thus seeking the limit $\lim_{r\to 1^-} f(re^{i\theta})$ is the same as calculating the limit of the Abel mean of the Fourier series

$$S \sim \sum_{j=0}^{\infty} a_j e^{ij\theta}.$$

At the time that Fatou did his work, Fejer's theorem about the convergence of the Cesàro means of the Fourier series of a piecewise continuous function was a matter of great interest, so it stands to reason that Fatou would have had Cesàro convergence on his mind. And it was known that Cesàro convergence implies Abel convergence. So in fact Fatou proved that the Cesàro means of the Fourier series whose coefficients come from a bounded holomorphic function converge.

About a dozen years later, Privalov and Plessner (see [38]) made a seminal contribution to the theory by noticing that the radial convergence used in Fatou's theorem is far too restrictive. Let $\alpha > 1$. For $\theta \in [0, 2\pi)$, define

$$\Gamma_\alpha(e^{i\theta}) = \{z \in D : |z - e^{i\theta}| < \alpha(1 - |z|)\}.$$

The set Γ_α is known as a *Stolz region* or non-tangential approach region. See Figure 1. Then the result is this:

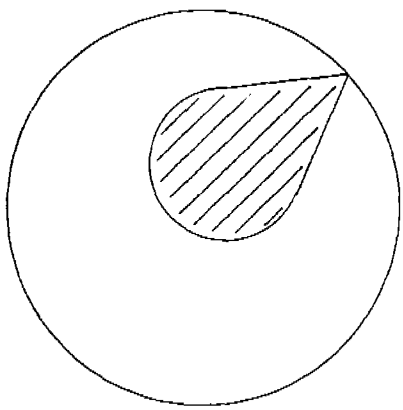

Figure 1

THEOREM 0.2. *Let f be a bounded holomorphic function on D. Fix a real number $\alpha > 1$. Then, for almost every $\theta \in [0, 2\pi)$, the limit*

$$f^*(e^{i\theta}) \equiv \lim_{\Gamma_\alpha(e^{i\theta}) \ni z \to e^{i\theta}} f(z)$$

exists.

Of course we have incorporated into the statement of this result the obvious fact that the limit over the larger non-tangential approach regions will be the same as the limit taken over the radii (which they contain).

1. The Existence of the Boundary Function

First let us see why f^* exists in any sense whatever. Let f be a bounded harmonic (there is no need to assume that the function is holomorphic) function on the unit disc in $\mathbb{C}$. For $0 < r < 1$, define

$$f_r(e^{i\theta}) = f(re^{i\theta}).$$

Then $\{f_r\}$ is a bounded set in $L^\infty(\partial D)$. And notice that L^∞ is the dual of L^1. So $\{f_r\}$ is a bounded set in $(L^1)^*$. But the Banach-Alaoglu theorem tells us that the unit ball in the dual of a Banach space is weak-$*$ compact (see [39]). Thus there is a subsequence f_{r_j} that converges weak-$*$ to some limit function $f^* \in L^\infty$.

Now fix $0 < r < 1$ and $0 \le \theta < 2\pi$ and set $\phi(t) = P_r(e^{i(\theta - t)})$. Here P_r is the usual Poisson kernel for the unit disc (see formula (4.0)). The definition of the weak-$*$ topology tells us that

$$\int f_{r_j}(e^{it})\phi(t)\, dt \to \int f^*(e^{it})\phi(t)\, dt$$

as $j \to \infty$. Writing out the definitions of the components of this formula then yields

$$\int f(r_j e^{it}) P_r(e^{i(\theta - t)})\, dt \to \int f^*(e^{it}) P_r(e^{i(\theta - t)})\, dt.$$

The reproducing property of the Poisson kernel allows us to rewrite the lefthand side to obtain

$$f(r \cdot r_j e^{i\theta}) \to \int f^*(e^{it}) P_r(e^{i(\theta - t)})\, dt.$$

Finally, we may evaluate the limit of the left-hand side and obtain

$$f(re^{i\theta}) = \int f^*(e^{it})P_r(e^{i(\theta-t)})\,dt.$$

Thus our original bounded holomorphic function f is the Poisson integral of the (abstractly obtained) boundary function f^*. It is no accident that we have chosen to denote this function by f^*, for our next goal is to show that f converges *pointwise* to f^* in the sense of either Theorem 0.1 or Theorem 0.2. Our main tool for doing so will be maximal functions.

2. A Digression on Maximal Functions

The modern method for studying the Privalov/Plessner theorem is by way of maximal functions. We now give a brief description of this methodology.

In fact we shall endeavor to paint our problem on a larger canvass. Our ultimate goal in this paper is to look at analogs of Theorems 0.1 and 0.2 in higher dimensions. Maximal functions will also be key to understanding that situation correctly, and they will be a unifying theme for all the various theories. And there will be several different maximal functions at play. So we may as well, in this section, present things from a "higher dimensional" point of view. Let us begin with $\mathbb{R}^N$.

Let $g \in L^1_{\text{loc}}(\mathbb{R}^N)$. We define

$$Mg(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |g(t)|\,dm(t).$$

[Here we denote Lebesgue measure by m.] This is the *Hardy-Littlewood maximal function*. The operator M is not linear, but it is *sublinear*; that is, it satisfies the inequality

$$M(f+g)(x) \leq Mf(x) + Mg(x).$$

It will be useful to know how M acts on the Lebesgue spaces L^p.

The key to such boundedness issues is to establish a *covering lemma*.

PROPOSITION 2.1. *Let $K \subseteq \mathbb{R}^N$ be a compact set that is covered by the open balls $\{B_\alpha\}_{\alpha\in A}, B_\alpha = B(c_\alpha, r_\alpha)$. There is a subcollection*

$B_{\alpha_1}, B_{\alpha_2}, \ldots, B_{\alpha_p}$, *consisting of pairwise disjoint balls, such that*

$$\bigcup_{j=1}^{p} 3B_{\alpha_j} \supseteq K.$$

[*Here* $3B_{\alpha_j} \equiv B(c_{\alpha_j}, 3r_{\alpha_j})$.]

Proof. Since K is compact, we may immediately assume that there are only finitely many B_α. Let B_{α_1} be the ball in this collection that has the greatest radius (this ball may not be unique). Let B_{α_2} be the ball that has greatest radius and is also disjoint from B_{α_1}. Assume now that balls $B_{\alpha_1}, \ldots, B_{\alpha_{j-1}}$ have been selected. At the j^{th} step choose the (not necessarily unique) ball of greatest radius that is disjoint from $B_{\alpha_1}, \ldots, B_{\alpha_{j-1}}$. Continue. The process ends in finitely many steps. We claim that the B_{α_j} chosen in this fashion do the job.

It is enough to show that $B_\alpha \subsetneq \cup_j B(c_{\alpha_j}, 3r_{\alpha_j})$ for every α. Fix an α. If $\alpha = \alpha_j$ for some j then we are done. If $\alpha \notin \{\alpha_j\}$, then let j_0 be the first index j with $B_{\alpha_j} \cap B_\alpha \neq \emptyset$ (there must be one, otherwise the process would not have stopped). Then $r_{\alpha_{j_0}} \geq r_\alpha$; otherwise we selected $B_{\alpha_{j_0}}$ incorrectly. But then clearly $B(c_{\alpha_{j_0}}, 3r_{\alpha_{j_0}}) \supseteq B(c_\alpha, r_\alpha)$ as desired. □

The next result that we prove is called a *weak type* $(1,1)$ *estimate.* It is a substitute for the more familiar *strong type* $(1,1)$ *estimate* $\|Mf\|_{L^1} \leq C \cdot \|f\|_{L^1}$ (which turns out to be false).

Lemma 2.2. *If* $f \in L^1(\mathbb{R}^N)$, *then*

$$m\{x \in \mathbb{R}^N : Mf(x) > \lambda\} \leq C\frac{\|f\|_{L^1}}{\lambda},$$

all $\lambda > 0$.

Proof. Let $S_\lambda = \{x \in \mathbb{R}^N : Mf(x) > \lambda\}$. Let K be a compact subset of S_λ. It suffices to estimate $m(K)$. Now, for each $x \in K$, there is a ball B_x centered at x such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(t)| dm(t) > \lambda. \tag{2.2.1}$$

The balls $\{B_x\}_{x\in K}$ cover K. We may choose, by Proposition 2.1, disjoint balls $B_{x_1}, B_{x_2}, \ldots, B_{x_p}$ such that $\{3B_{x_j}\}$ cover K. Then

$$\begin{aligned} m(K) &\leq \sum_{j=1}^{p} m(3B_{x_j}) \\ &= 3^N \sum_{j=1}^{p} m(B_{x_j}). \end{aligned}$$

But (2.2.1) implies that the last line is majorized by

$$3^N \sum_j \frac{\int_{B_{x_j}} |f(t)|dm(t)}{\lambda} \leq 3^N \frac{\|f\|_{L^1}}{\lambda}.$$

This completes the proof of the theorem. □

COROLLARY 2.3. *Let $1 < p \leq \infty$. There is a constant C_p such that, for $f \in L^p(\mathbb{R}^N)$,*

$$\|Mf\|_{L^p} \leq C_p \cdot \|f\|_{L^p}.$$

Proof. The assertion for $p = \infty$ is true simply by inspection of the definition of M. We also know, by the lemma, that M is weakly bounded on L^1. Now the Marcinkiewicz interpolation theorem (see [42]) implies that M is bounded on L^p. That is the assertion of the corollary. □

Now we have formulated and proved Lemma 2.2 on the Euclidean space $\mathbb{R}^N$. But in fact it is easy to see that variants are true in other settings. If $\Omega \subseteq \mathbb{R}^N$ is a smoothly bounded domain (C^2 boundary suffices for most purposes) then we may define a ball with center $x \in \partial\Omega$ and radius $r > 0$ by

$$\beta_1(x,r) = \{t \in \partial\Omega : |x - t| < r\}.$$

Then the covering lemma applies *grosso modo* to the balls $\beta_1(x,r)$, just because $\beta_1(x,r) = B(x,r) \cap \partial\Omega$. The proof of Lemma 2.2 does not literally apply to the balls $\beta_1(x,r)$ because (denoting surface measure on $\partial\Omega$ by σ) it is no longer the case that there is a universal constant C such that $\sigma(\beta_1(x,3r)) = C \cdot \sigma(\beta_1(x,r))$. But there *will* be a constant C' (depending on the curvatures of $\partial\Omega$) such that $\sigma(\beta_1(x,3r)) \leq C' \cdot \sigma(\beta_1(x,r))$. That suffices to prove an analog of Lemma 2.2 in the setting of $\partial\Omega$.

There are even more general contexts in which we shall need analogs of Theorems 2.1 and 2.2, so we now formulate a set of axioms (which havc evolved through work of Hörmander [11], K. T. Smith [40], and Coifman/Weiss [7]) that are a natural setting for the type of analysis we have been describing.

DEFINITION 2.4. Suppose that we are given a set X that is equipped with a function $\rho : X \times X \to \mathbb{R}^+$. We assume that ρ satisfies the three conditions

(2.4.1): $\rho(x, y) = 0$ if and only if $x = y$;
(2.4.2): $\rho(x, y) = \rho(y, x)$;
(2.4.3): There is a constant $C_2 > 0$ such that, if $x, y, z \in X$, then

$$\rho(x, z) \leq C_2\bigl[\rho(x, y) + \rho(y, z)\bigr].$$

[We call this displayed relationship the *quasi-triangle inequality.*]

Then ρ is called a *quasi-metric* for the space X.

We take the balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$ to be the sub-basis for a topology on X.

DEFINITION 2.5 (Axioms for a Space of Homogeneous Type). Assume that the space X is equipped with a quasi-metric ρ. Also assume that there is given a measure μ on X. For convenience, we assume that μ is a regular Borel measure. We say that (X, ρ, μ) is a *space of homogeneous type* if the following axioms are satisfied:

(2.5.1): For each $x \in X$ and $r > 0$, $0 < \mu[B(x, r)] < \infty$;
(2.5.2) [The Doubling Property]: There is a constant $C_1 > 0$ such that, for any $x \in X$ and $r > 0$, we have

$$\mu[B(x, 2r)] \leq C_1 \cdot \mu[B(x, r)];$$

REMARKS.

(1) There are many different definitions of "space of homogeneous type" (see [7], [4], [24], [2] for example). In some applications, it is convenient to replace the quasi-triangle inequality by the so-called "enveloping property": There is a constant $K > 0$ such that if $B(x, r) \cap B(y, s) \neq \emptyset$ and $s \geq r$ then $B(y, Ks) \supseteq B(x, r)$.

(2) It is not difficult to see that Euclidean space, equipped with the standard Euclidean metric and with Lebesgue measure, is a space of homogeneous type. Also the boundary of the unit disc (i.e., the unit circle) is a space of homogeneous type when equipped with the usual arc-length measure and with Euclidean distance. We shall encounter more profound examples of spaces of homogeneous type later in the paper. See [15] for a more detailed discussion of this concept.

We define the Hardy-Littlewood maximal function on a space X of homogeneous type by

$$Mf(P) = \sup_{r>0} \frac{1}{\mu(B(P,r))} \int_{B(P,r)} |f(x)| \, d\mu(x).$$

The reader may check that, on any space X of homogeneous type equipped with quasi-metric ρ and measure μ, and with balls defined accordingly, analogs of both Proposition 2.1 and Lemma 2.2 will be true. It is for this reason that we have introduced the abstraction of spaces of homogeneous type. It will turn out that, in the setting of several complex variables that we study in detail later, spaces of homogeneous type are the correct framework in which to understand the boundary behavior of holomorphic functions.

3. Return to Holomorphic Functions on the Unit Disc

Now we use the Hardy-Littlewood maximal function on the unit circle to gain control of the boundary behavior of holomorphic functions on the disc. We record here, for the record, that the maximal function is

$$Mf(e^{i\theta}) = \sup_{R>0} \frac{1}{2R} \int_{(\theta-R,\theta+R)} |f(e^{it})| \, dt.$$

PROPOSITION 3.1. *If $e^{i\theta} \in \partial D, 1 < \alpha < \infty$, then there is a constant $C_\alpha > 0$ such that if $f \in L^1(\partial D)$, then*

$$\sup_{re^{i\phi} \in \Gamma_\alpha(e^{i\theta})} |P_r f(e^{i\phi})| \leq C_\alpha M f(e^{i\theta}).$$

Proof. For $re^{i\phi} \in \Gamma_\alpha(e^{i\theta})$, we have

$$|\theta - \phi| \leq 2\alpha(1-r).$$

Therefore, for $1/\alpha \le r < 1$, we obtain

$$\begin{aligned}
&|P_r f(e^{i\phi})| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i(\phi-\psi)})\frac{1-r^2}{1-2r\cos\psi+r^2}\,d\psi\right| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i(\phi-\psi)})\frac{1-r^2}{(1-r)^2+2r(1-\cos\psi)}\,d\psi\right| \\
&\le \frac{4}{2\pi}\sum_{j=0}^{[\log_2(\pi/[\alpha(1-r)])]+1}\int_{S_j}|f(e^{i(\phi-\psi)})|\frac{1-r^2}{(1-r)^2+2r(2^{j-1}\alpha(1-r))^2}\,d\psi \\
&+\frac{1}{2\pi}\int_{|\psi|<\alpha(1-r)}|f(e^{i(\phi-\psi)})|\frac{1-r^2}{(1-r)^2}\,d\psi,
\end{aligned}$$

where $S_j = \{\psi : 2^j\alpha(1-r) \le |\psi| < 2^{j+1}\alpha(1-r)\}$. Now this is

$$\begin{aligned}
\le\;& \frac{8}{4\pi\alpha^2}\sum_{j=0}^{\infty}\frac{1}{2^{2j-2}(1-r)}\int_{|\psi|<(2+2^{j+1})\alpha(1-r)}|f(e^{i(\theta-\psi)})|\,d\psi \\
&+\frac{2}{2\pi}\frac{1}{1-r}\int_{|\psi|<3\alpha(1-r)}|f(e^{i(\theta-\psi)})|\,d\psi \\
\le\;& \frac{256}{\pi}\sum_{j=0}^{\infty}2^{-j}\left[\frac{1}{2\alpha(2+2^{j+1})(1-r)}\int_{|\psi|<(2+2^{j+1})\alpha(1-r)}|f(e^{i(\theta-\psi)})|\,d\psi\right] \\
&\quad+\frac{6\alpha}{\pi}\frac{1}{2\cdot 3\alpha(1-r)}\int_{|\psi|<3\alpha(1-r)}|f(e^{i(\theta-\psi)})|\,d\psi \\
\le\;& \frac{256}{\pi}\cdot\sum_{j=0}^{\infty}2^{-j}Mf(e^{i\theta})+\frac{6\alpha}{\pi}Mf(e^{i\theta}) \\
\le\;& \frac{512}{\pi}Mf(e^{i\theta})+\frac{6\alpha}{\pi}Mf(e^{i\theta}) \\
\equiv\;& C\cdot Mf(e^{i\theta}).
\end{aligned}$$

If $0 < r < 1/\alpha$ then

$$\begin{aligned}
|P_r f(e^{i\phi})| \;&\le\; \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{i(\phi-\psi)})|(2\alpha/(\alpha-1))d\psi \\
&\le\; \frac{2\alpha}{\alpha-1}Mf(e^{i\theta}).
\end{aligned}$$

Thus we have estimated $P_r f$ in terms of Mf. The proof is complete. □

Now we will examine Theorems 0.1 and 0.2 from a broader perspective. Rather than consider only *bounded functions*, we now define a growth condition. If f is a harmonic function on the unit disc and $0 < p < \infty$ then we say that $f \in \mathbf{h}^p(D)$ if

$$\|f\|_{\mathbf{h}^p} \equiv \sup_{0<r<1} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right]^{1/p} < \infty.$$

Observe that $\| \ \|_{\mathbf{h}^p}$ is a true norm only when $p \geq 1$. We say that $f \in \mathbf{h}^\infty$ if f is bounded. The corresponding norm is $\|f\|_{\mathbf{h}^\infty} \equiv \sup |f|$. If $f \in \mathbf{h}^p$ and f is, in addition, holomorphic, then we write $f \in H^p(D)$. The space H^p is the classical Hardy space.

Now our first complete result about boundary behavior of functions is as follows:

THEOREM 3.2. *Let $f \in \mathbf{h}^p(D)$ and $1 < p \leq \infty$. Let f^* be as in Theorem 0.2 and $1 < \alpha$. Then*

$$\lim_{\Gamma_\alpha(e^{i\theta}) \ni z \to e^{i\theta}} f(z) = f^*(e^{i\theta}), \quad a.e.\ e^{i\theta} \in \partial D.$$

Proof. It suffices to handle the case $p < \infty$ and f real-valued. If $\epsilon > 0$ then choose $g \in C(\partial D)$ real-valued so that $\|f^* - g\|_{L^p(\partial D)} < \epsilon^2$. We know by the theory of the Dirichlet problem that

$$\lim_{\Gamma_\alpha(e^{i\theta}) \ni z \to e^{i\theta}} g(z) = g(e^{i\theta}), \qquad \text{all} \quad e^{i\theta} \in \partial D, \tag{3.2.1}$$

where $g(re^{i\theta}) \equiv P_r g(e^{i\theta})$. Therefore

$$\begin{aligned} m\{e^{i\theta} : &\limsup_{\Gamma_\alpha(e^{i\theta}) \ni z \to e^{i\theta}} |f(z) - f^*(e^{i\theta})| > \epsilon\} \\ \leq\ & m\{e^{i\theta} : \limsup_{\Gamma_\alpha(e^{i\theta}) \ni z \to e^{i\theta}} |f(z) - g(z)| > \epsilon/3\} \\ & +m\{e^{i\theta} : \limsup_{\Gamma_\alpha(e^{i\theta}) \ni z \to e^{i\theta}} |g(z) - g(e^{i\theta})| > \epsilon/3\} \\ & +m\{e^{i\theta} : |g(e^{i\theta}) - f^*(e^{i\theta})| > \epsilon/3\} \\ \leq\ & m\{e^{i\theta} : C_\alpha M(f^* - g) > \epsilon/3\} + 0 + (\|g - f^*\|_{L^p}/(\epsilon/3))^p. \end{aligned}$$

In the last estimate we used Proposition 3.1 and Chebycheff's inequality. Now the last line is majorized by

$$C'_\alpha \left[\|f^* - g\|_{L^p}/(\epsilon/3)\right]^p + 3^p \epsilon^p \leq C''_\alpha \epsilon^p.$$

It follows that

$$\square \qquad \lim_{\Gamma_\alpha(e^{i\theta})\ni z\to e^{i\theta}} f(z) = f^*(e^{i\theta}), \qquad a.e. \quad e^{i\theta}\in\partial D.$$

The informal statement of Theorem 3.2 is that f has non-tangential boundary limits almost everywhere.

Note once again the restriction in Theorem 3.2 to $1 < p \le \infty$. Is it possible to push the value of p down below 1? In case the function f is holomorphic then the answer is "yes". There are several ways to see why this is so. For the moment, we explain the point by way of Blaschke products.

If a is a complex number such that $|a| < 1$ then we define

$$\phi_a(\zeta) = \frac{\zeta - a}{1 - \overline{a}\zeta}$$

for $\zeta \in D$. If $\sum_j (1 - |a_j|) < \infty$ then it is a classical fact that

$$B(\zeta) = \prod_j \frac{-\overline{a_j}}{|a_j|}\phi_{a_j}(\zeta)$$

converges, uniformly on compact subsets of D (see [14], §§8.1 for details).

Further, if $f \in H^p(D)$, $0 < p \le \infty$ and if $\{a_j\}$ are the zeros of f, listed with multiplicities, then $\sum_j (1 - |a_j|) < \infty$. Thus the product defining $B(\zeta)$ converges. So we can consider

$$F(\zeta) = \frac{f(\zeta)}{B(\zeta)}.$$

It can be shown that $F \in H^p$ and has the same norm as f. And of course F is zero free. Then the function $G = F^{p/2}$ (when $p < \infty$) is holomorphic and $G \in H^2$. By Theorem 3.2, G has non-tangential boundary limits almost everywhere. But then, untangling what we have said, F has non-tangential boundary limits almost everywhere. Hence so does f.

That completes the proof. $\square$

4. The Case of Several Real Variables

We have gone into almost painful detail with the disc (i.e., the one-dimensional) case because the arguments that are used there are fundamental to all that follows. Because of the care we have taken thus far, we may present the next arguments in a rather flowing fashion.

Suppose now that f is a harmonic function on the unit ball B in $\mathbb{R}^N$. Further assume that $f \in \mathbf{h}^p$, $1 < p \leq \infty$, where

$$\|f\|_{\mathbf{h}^p(B)} \equiv \sup_{0<r<1} \left\{ \int_{\partial B} |f(r\xi)|^p \, d\sigma(\xi) \right\}^{1/p} \qquad \text{when } 1 < p < \infty$$

and $\mathbf{h}^\infty$ is just the bounded harmonic functions with the obvious norm.

A functional analysis argument, exactly like that in §§1, shows that a boundary function f^* exists. Moreover, we have an explicit formula for the Poisson kernel for the ball:

$$P(x, y) = c_N \cdot \frac{1 - |x|^2}{|x - y|^N},$$

where c_N is $1/\sigma(\partial B)$. If $f \in L^1(\partial B)$ then we write

$$Pf(x) \equiv \int_{\partial B} P(x, t) f(t) \, d\sigma(t).$$

The reader may consult [14] for details about P. The kernel is derived in that source, and the exact value of c_N (which is of no interest for us here) is determined. In point of fact the reader may check that in dimension two, using polar coordinates, the Poisson kernel assumes the familiar form

$$P_r(e^{i\theta}) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \tag{4.0}$$

Now we are going to need some non-tangential approach regions. Let $\alpha > 1$ and $P \in \partial B$. Then

$$\Gamma_\alpha(P) \equiv \{x \in B : |x - P| \leq \alpha(1 - |x|)\}.$$

We define balls $\beta_1(P, r)$ in the boundary of B, just as we did in §§2. The corresponding maximal function is defined as usual. And it will be bounded on the L^p spaces in the expected way because ∂B, equipped with these balls and with surface area measure and the Euclidean metric, is a space of homogeneous type.

The proof of Proposition 3.1 may now be imitated, step by step, to show that if $f \in L^1(\partial B)$ then Pf satisfies

$$\sup_{t \in \Gamma_\alpha(x)} |Pf(t)| \leq C \cdot Mf(x).$$

Here we are using a suitable version of the Hardy-Littlewood maximal function on ∂B, as discussed at the end of §§2.

Finally, the proof of Theorem 3.2 may be imitated verbatim to establish the following result:

THEOREM 4.1. *Let* $1 < p \leq \infty$. *Let* $f \in \mathbf{h}^p(B)$. *Fix* $\alpha > 1$. *Then*

$$\lim_{\Gamma_\alpha(P)\ni x\to P} f(x) = f^*(P) \qquad \text{a.e. } P \in \partial B.$$

It should be stressed that a crucial aspect of the proof of the result of Theorem 4.1 is that we have an explicit formula for the Poisson kernel, and therefore may estimate it just as we did in the proof of Proposition 3.1. What would happen if the ball B were replaced by a more general smoothly bounded domain Ω?

First, what does $\mathbf{h}^p(\Omega)$ *mean*? A careful answer to this question is developed in [14, Chapter 8]. A correct, but more informal, answer is that we replace the spherical surfaces that we have been considering so far by hypersurfaces that are parallel to the boundary of Ω. See Figure 2.

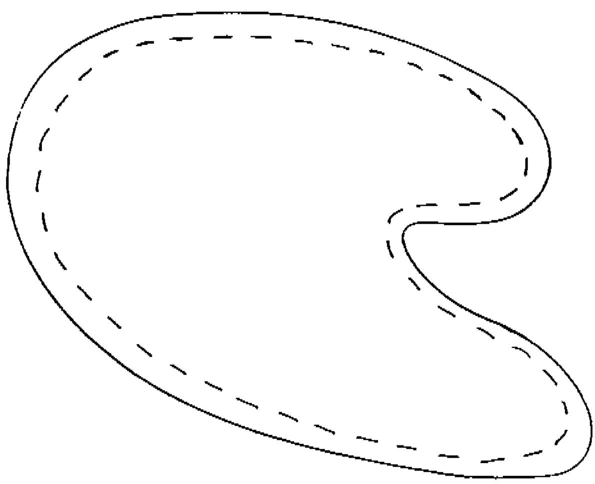

Figure 2

In this circumstance there is no hope of obtaining an explicit formula for the Poisson kernel of Ω. Nonetheless, by using delicate comparison domain arguments (see [14, Chapter 8]), it is possible to show that

$$P(x,y) \approx C \cdot \frac{\delta(x)}{|x-y|^N} \qquad x \in \Omega,\ y \in \partial\Omega\,.$$

Here $\delta(x)$ is the distance of x to $\partial\Omega$. As a result, $P(x, y)$ may be estimated in much the same way that we estimated the explicitly given Poisson kernel for the ball.

The other aspect of the "round analysis" that we did for the disc and the ball that will not work on an arbitrary domain is the dilation argument used to define the functions f_r in the functional analysis proof of the existence of f^* (§§1). In fact a different type of analysis (due to E. M. Stein—see [14] for the details) is needed. One lets U be a tubular neighborhood of $\partial\Omega$ and covers $U \cap \Omega$ with sub-domains Ω_j, where each Ω_j has the following properties:

1. Ω_j has smooth boundary (as smooth as Ω);
2. For each j there is a unit outward "normal" vector ν_j such that, for $\epsilon > 0$ small, the closure of the domain $\Omega_j - \epsilon\nu_j = \{x - \epsilon\nu_j : x \in \Omega_j\}$ lies in Ω.

See Figure 3.

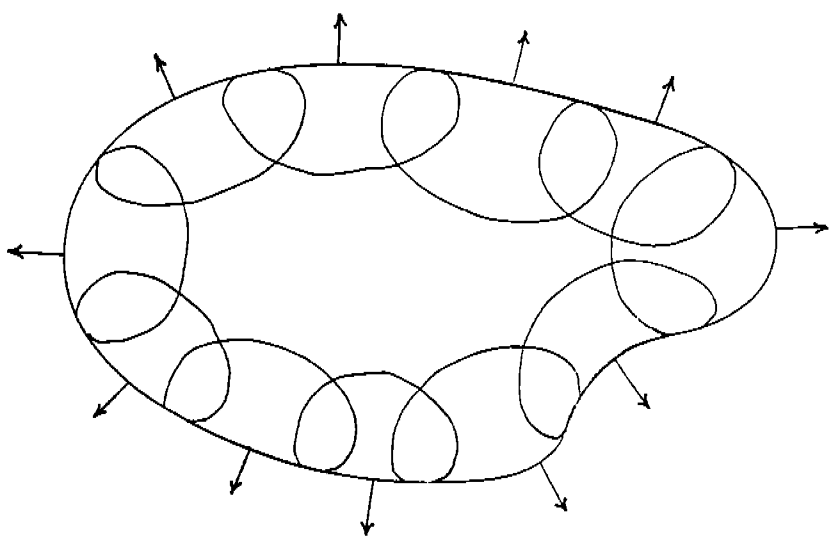

Figure 3

Then, instead of applying Banach-Alaoglu to $\{f_r\}$, one instead applies it to $\{f_{j,\epsilon}\}$ on Ω_j, where $f_{j,\epsilon}(x) = f(x - \epsilon\nu_j)$. The rest of the analysis proceeds essentially as in the two cases (the disc and the ball) that we have already discussed. The result is the following theorem:

THEOREM 4.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with smooth (at least C^2) boundary. Assume that $1 < p \leq \infty$. Let $f \in \mathrm{h}^p(\Omega)$. Fix $\alpha > 1$. Then*

$$\lim_{\Gamma_\alpha(P)\ni x\to P} f(x) = f^*(P) \qquad \text{a.e. } P \in \partial\Omega.$$

There is no hope now of extending p below 1 by passing to holomorphic functions. If $N = 3$, for instance, then there is no way to make sense of "holomorphic". There are additional problems, for there is no Blaschke or other factorization in higher dimensions. The next section will begin to make inroads into this new set of difficulties.

5. Analysis on the Unit Ball in Complex Space

Now let $\mathbb{B} = \{z \in \mathbb{C}^n : \sum_j |z_j|^2 < 1\}$. Let f be a holomorphic function on $\mathbb{B}$ that also lies in h^p for some p. We say that $f \in H^p(\mathbb{B})$. We can use the "forgetful functor" and observe that f is also harmonic. Therefore, according to Theorem 4.2, if $1 < p \leq \infty$ then f has non-tangential boundary limits almost everywhere. What can be done to amplify or extend this basic fact? Can anything about holomorphicity be exploited?

To address the fact (see [14]) that there is no canonical factorization in several complex variables, one can use subharmonicity. In fact if an f as in the last paragraph is holomorphic then $|f|^p$ will be subharmonic, and it can therefore be proved that it will have a harmonic majorant. As a result, arguments (similar, but more complicated than, the proof of Theorem 3.2) due to E. M. Stein can be applied to show that f will still have non-tangential boundary limits almost everywhere. A. Koranyi, in 1969 (see [12, 13]), made the following profound observation.

If one wants to take full advantage of the holomorphicity of f then one should use a reproducing kernel that is special for holomorphic functions (after all, the Poisson kernel was designed only for *harmonic* function theory). The right kernel to use is the so-called Poisson-Szegö kernel. Let us say a few words about that kernel now.

Consider the Hardy space $H^2(\mathbb{B})$. A function $f \in H^2(\mathbb{B})$ is holomorphic and satisfies

$$\sup_{0<r<1} \int_{\partial\mathbb{B}} |f(r\xi)|^2 \, d\sigma(\xi) < \infty.$$

According to the preceding discussion, we may associate to such an f its boundary limit function f^*, and Fatou's lemma guarantees that $f^* \in L^2(\partial\mathbb{B})$. So we may think of $H^2(\mathbb{B})$ as a closed subspace of $L^2(\partial\mathbb{B})$. We use the norm $\|f\|_{H^2(\mathbb{B})} \equiv \|f^*\|_{L^2(\partial B)}$. Let $z \in \mathbb{B}$ be a fixed point; the Poisson integral formula shows that the map

$$\varphi_z : H^2(\mathbb{B}) \ni f \longmapsto f(z)$$

is a bounded linear functional. Elementary Hilbert space theory then guarantees that there is an element $k_z \in H^2(\mathbb{B})$ such that

$$\varphi_z(f) = \langle f, k_z \rangle$$

for all $f \in H^2(\mathbb{B})$.

We set $S(z,\zeta) = \overline{k_z(\zeta)}$. Of course k_z is an element of $H^2(\mathbb{B})$ so it, too, has a boundary limit function that is in L^2. It can be shown that $S(\,\cdot\,,\,\cdot\,)$ is a conjugate symmetric function of its two variables. So we may think of S as a function that is L^2 in each of the variables z and ζ separately. See [14] for a full development of the theory of this important kernel, which is known as the Szegö kernel.

A formal construction, coming from representation theory, suggests that we define an associated positive kernel by

$$\mathcal{P}(z,\zeta) = \frac{|S(z,\zeta)|^2}{S(z,z)}.$$

This is the *Poisson-Szegö kernel.* The kernel $\mathcal{P}$ is still a reproducing kernel for $H^2(\mathbb{B})$. On the unit ball $\mathbb{B} \subseteq \mathbb{C}^n$, this kernel is given explicitly by

$$(*) \qquad \mathcal{P}(z,\zeta) = c_n \frac{(1-|z|^2)^n}{|1 - z\cdot\overline{\zeta}|^{2n}}.$$

Here $z \cdot \overline{\zeta} \equiv z_1\overline{\zeta_1} + \cdots + z_n\overline{\zeta_n}$. The new kernel $\mathcal{P}$ is positive and it still reproduces H^2 in the sense that

$$f(z) = \int_{\partial\Omega} \mathcal{P}(z,\zeta) f(\zeta) d\sigma(\zeta) \qquad \text{for all } f \in H^2(\mathbb{B}).$$

At this point it is natural to attempt to (formally) imitate the arguments given in §§4 using the Poisson kernel with harmonic functions, but now to use the Poisson-Szegö kernel and holomorphic functions. How will things change? The main difference will be in the way that we estimate the kernel. An examination of the proof of Proposition 3.1 shows that that argument was successful because the kernel P meshes nicely with the approach region Γ_α. This point is easiest to see if we use the notation from $\mathbb{R}^N$: the key fact about the ball Poisson kernel

$$P(x,y) = c_n \cdot \frac{1-|x|^2}{|x-y|^N}$$

is that the denominator $|x-y|^N$ can be controlled by a power of the numerator $1-|x|^2$ on the region $\Gamma_\alpha = \{x : |x-y| < \alpha(1-|x|)\}$.

Now refer to formula $(*)$. Proceeding by analogy, we define, for $\alpha > 1$ and $P \in \partial\mathbb{B}$, *new approach regions*

$$\mathcal{A}_\alpha(P) = \{z \in \mathbb{B} : |1 - z \cdot \overline{P}| < \alpha(1-|z|)\}.$$

The approach region $\mathcal{A}_\alpha(P)$ is called an *admissible approach region.* To see that this new approach region is fundamentally different from Γ_α, we now do a calculation.

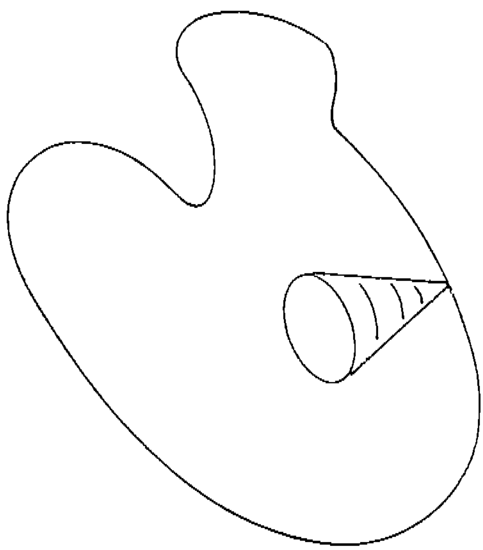

Figure 4

First we "calculate the shape" of Γ_α. Restrict attention to complex dimension 2. Fix the point $\mathbf{1} = (1,0) \in \partial\mathbb{B}$. Then $\Gamma_\alpha(\mathbf{1})$ is defined by the condition

$$|z - \mathbf{1}| < \alpha(1 - |z|).$$

More explicitly, this is

$$|z_1 - 1|^2 + |z_2|^2 < \alpha^2 \cdot (1 - |z|)^2.$$

In particular,

$$|z_1 - 1| < \alpha(1 - |z|)$$

and

$$|z_2| < \alpha(1 - |z|).$$

The thing to see is that the condition on z_1 (i.e., that it deviate from 1 by not more than $\alpha(1-|z|)$) is just the same as the condition on z_2 (that it deviate from 0 by not more than $\alpha(1-|z|)$). Thus Γ_α is isotropic: its extent in each direction is the same. It is a *cone*. See Figure 4.

Now we will "calculate the shape" of $\mathcal{A}_\alpha(P)$. Again restrict attention to complex dimension 2. Fix the point $\mathbf{1} = (1,0) \in \partial\mathbb{B}$. Then $\mathcal{A}_\alpha(\mathbf{1})$ is defined by the condition

$$|1 - z \cdot \overline{\mathbf{1}}| < \alpha(1 - |z|).$$

More explicitly, this is

$$(**) \qquad |1 - z_1| < \alpha(1 - |z|).$$

But we are restricted to study only points in $\mathbb{B}$. This says that

$$|z_1|^2 + |z_2|^2 < 1$$

or

$$\begin{aligned} |z_2|^2 &= 1 - |z_1|^2 \\ &= (1 + |z_1|)(1 - |z_1|) \\ &\leq 2|1 - z_1|. \end{aligned}$$

Substituting $(**)$ into this last inequality gives

$$|z_2| < \sqrt{2\alpha}\sqrt{1 - |z_1|}.$$

Thus the condition on z_2 is now *the square root of* the condition on z_1. A moment's thought reveals that this non-isotropic condition forces $\mathcal{A}_\alpha$ to be conical in the z_1 direction but *parabolic* in the z_2 direction.

Once we understand that the Poisson-Szegö kernel can be estimated on an admissible approach region in just same way that the classical

Poisson kernel can be estimated on a non-tangential approach region, then it is natural to ask whether one can in fact estimate the Poisson-Szegö integral in terms of a maximal function. The answer is "yes", but the correct maximal function cannot be the isotropic Euclidean maximal function that we have been considering thus far. An examination of the proof of Proposition 3.1 reveals that the balls used to define the maximal function there were in fact sublevel sets of the denominator of the Poisson kernel: the geometries must mesh.

The situation is the same in the present context. The Poisson-Szegö kernel of the unit ball has denominator that is a power of the non-isotropic expression $|1 - z \cdot \overline{\zeta}|$. Thus we need to consider balls

$$\beta_2(\zeta, r) \equiv \{\xi \in \partial\mathbb{B} : |1 - \xi \cdot \overline{\zeta}| < r\}.$$

The calculation that we have already done will show that such a ball has extent r in the complex normal direction but has extent $\sqrt{r}$ in the complex tangential directions. Again, it is non-isotropic.

Now the key fact, pursuant to the point of view that we have been developing here, is that the space $X = \partial\mathbb{B}$, equipped with the balls $\beta_2(\zeta, r)$, the standard surface measure $d\sigma$, and the metric[1]

$$\rho(\zeta, \xi) = \sqrt{|1 - \xi \cdot \overline{\zeta}|}$$

is a space of homogeneous type. Thus there is a Hardy-Littlewood maximal function, and it is weakly bounded on L^1 and strongly bounded on L^p for $1 < p \leq \infty$. Now we have all the necessary ingredients for a theorem: **(i)** the right reproducing kernel, **(ii)** the right balls in the boundary, **(iii)** the right maximal function. Putting them together according to the paradigm that we have laid out for the disc and for the ball and for general domains Ω in $\mathbb{R}^N$, we obtain the following theorem:

THEOREM 5.1 ([12, 13]). *Let $0 < p \leq \infty$ and suppose that $f \in H^p(\mathbb{B})$. Fix $\alpha > 1$. Then*

$$\lim_{\mathcal{A}_\alpha(P) \ni z \to P} f(z) = f^*(P)$$

for almost every $P \in \partial\mathbb{B}$.

Koranyi's theorem was considered to be quite a surprise. The classical literature on the disc demonstrates in a variety of senses (see [6]

[1]The proof that this is a genuine metric is non-trivial, and appears in [19], p. 149.

or [38]) that boundary limits along a tangential (that is, a non-non-tangential) trajectory are impossible. Certainly Koranyi's theorem does admit some tangential boundary limits. But it must be understood that these tangential limits are along the *complex tangential direction*, and that direction *does not exist* in one complex dimension!

Next we look at the theory of Stein for general domains in multi-dimensional complex space.

6. Fatou Theorems on Arbitrary Domains in Complex Space

The good thing about Koranyi's approach to the boundary behavior of holomorphic functions is that it shows that there is more to life than non-tangential approach. The bad thing is that it is too heavily dependent on formulas. In order to really understand what is going on, we require some (differential) geometry.

Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly (at least C^2) bounded domain. Given what we have done up to this point, it is natural to endeavor to look at the Poisson-Szegö kernel for Ω. There is the rub. At the time that E. M. Stein studied this matter, virtually nothing was known about canonical reproducing kernels on a domain in $\mathbb{C}^n$ that is not a bounded symmetric domain. So Stein had to develop an approach that side-steps the need for information about the Szegö and the Poisson-Szegö kernels.

This is in fact one of the most interesting points in the development of the subject. Typically in mathematics one gives up something in order to gain information somewhere else. By relinquishing the (unattainable) information about the Szegö kernel for an arbitrary smoothly bounded domain[2] in $\mathbb{C}^n$, Stein was able to prove a result—which we shall discuss momentarily—about *any* smoothly bounded domain in any $\mathbb{C}^n$. But in fact his theorem is not optimal unless the domain is either the ball or is strongly pseudoconvex.

It turns out that, in order to prove the sharp Fatou theorem for a domain in $\mathbb{C}^n$, it is necessary to obtain sharp information about the boundary Levi geometry of the point $P \in \partial\Omega$ that is being approached. As is explained in [16], [17], there are several paths to the needed information. One is by way of the theory of finite type that has been

[2]Indeed, this information is still unavailable except for certain special classes of domains—strongly pseudoconvex domains, finite type domains in $\mathbb{C}^2$, and finite type convex domains in $\mathbb{C}^n$.

developed by Catlin, Kohn, and D'Angelo (see [14] for a thoroughgoing discussion of finite type). Another is by way of detailed information about the singularity of the Szegö kernel (indeed one may use sublevel sets of the Szegö kernel to define the needed balls on the boundary). Yet another is to utilize invariant metrics (see [18]).

Let us now make the necessary definitions so that we may formulate Stein's theorem. Then we shall explain how the ideas may be pushed much further.

If z, w are vectors in $\mathbb{C}^n$, we continue to write $z \cdot \overline{w}$ to denote $\sum_j z\overline{w}_j$. (*Warning:* It is also common in the literature to use the notation $z \cdot w = \sum_j z_j\overline{w}_j$.) Also, for $\Omega \subseteq \mathbb{C}^n$ a domain with C^1 boundary, $P \in \partial\Omega$, we let ν_P be the unit outward normal at P. Let $\mathbb{C}\nu_P$ denote the complex line generated by $\nu_P : \mathbb{C}\nu_P = \{\zeta\nu_P : \zeta \in \mathbb{C}\}$.

By dimension considerations, if $T_P(\partial\Omega)$ is the $(2n-1)$-dimensional real tangent space to $\partial\Omega$ at P, then $\ell = \mathbb{C}\nu_P \cap T_P(\partial\Omega)$ is a (one-dimensional) real line. Let

$$\begin{aligned} \mathcal{T}_P(\partial\Omega) &= \{z \in \mathbb{C}^n : z \cdot \overline{\nu}_P = 0\} \\ &= \{z \in \mathbb{C}^n : z \cdot \overline{w} = 0 \; \forall w \in \mathbb{C}\nu_P\}. \end{aligned}$$

A fortiori, $\mathcal{T}_P(\partial\Omega) \subseteq T_P(\partial\Omega)$. If $z \in \mathcal{T}_P(\partial\Omega)$, then $iz \in \mathcal{T}_P(\partial\Omega)$. Therefore $\mathcal{T}_P(\partial\Omega)$ may be thought of as an $(n-1)$-dimensional complex subspace of $T_P(\partial\Omega)$. Clearly, $\mathcal{T}_P(\partial\Omega)$ is the complex subspace of $T_P(\partial\Omega)$ of maximal dimension. It contains all complex subspaces of $T_P(\partial\Omega)$. We may think of $\mathcal{T}_P(\partial\Omega)$ as the real orthogonal complement in $T_P(\partial\Omega)$ of ℓ.

Now let us examine the matter from another point of view. The complex structure is nothing other than a linear operator J on $\mathbb{R}^{2n}$ that assigns to $(x_1, x_2, \ldots, x_{2n-1}, x_{2n})$ the vector $(-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1})$ (think of multiplication by i). With this in mind, we have that $J : \mathcal{T}_P(\partial\Omega) \to \mathcal{T}_P(\partial\Omega)$ both injectively and surjectively. Notice that $J\nu_P \in T_P(\partial\Omega)$ while $J(J\nu_P) = -\nu_P \notin T_P(\partial\Omega)$. We call $\mathbb{C}\nu_P$ the *complex normal* space to $\partial\Omega$ at P and $\mathcal{T}_P(\partial\Omega)$ the *complex tangent space* to $\partial\Omega$ at P. Let $\mathcal{N}_P = \mathbb{C}\nu_P$. Then we have $\mathcal{N}_P \perp \mathcal{T}_P$ and

$$\begin{aligned} \mathbb{C}^n &= \mathcal{N}_P \oplus_{\mathbb{C}} \mathcal{T}_P \\ T_P &= \mathbb{R}J\nu_P \oplus_{\mathbb{R}} \mathcal{T}_P. \end{aligned}$$

The next definition is best understood in light of the foregoing discussion and the definition of $\beta_2(P, r)$ in the boundary of the unit ball

B. Let $\Omega \subset\subset \mathbb{C}^n$ have C^2 boundary. For $P \in \partial\Omega$, let $\pi_P : \mathbb{C}^n \to \mathcal{N}_P$ be (real or complex) orthogonal projection.

DEFINITION 6.1. If $P \in \partial\Omega$ let

$$\begin{aligned} \beta_1(P,r) &= \{\zeta \in \partial\Omega : |\zeta - P| < r\}; \\ \beta_2(P,r) &= \{\zeta \in \partial\Omega : |\pi_P(\zeta - P)| < r, |\zeta - P| < r^{1/2}\}. \end{aligned}$$

NOTE: The ball $\beta_2(P,r)$ has diameter $\sim \sqrt{r}$ in the $(2n-2)$ complex tangential directions and diameter $\sim r$ in the one (normal) direction. Therefore $\sigma(\beta_2(P,r) \approx (\sqrt{r})^{2n-2} \cdot r \approx Cr^n$.

As usual, we let $\delta_\Omega(z) = \delta(z)$ denote the distance of $z \in \Omega$ to $\partial\Omega$. If $z \in \Omega, P \in \partial\Omega$, we let

$$\delta_P(z) = \min\{\operatorname{dist}(z, \partial\Omega), \operatorname{dist}(z, T_P(\Omega))\}.$$

Notice that if Ω is convex, then $\delta_P(z) = \delta_\Omega(z)$.

DEFINITION 6.2. If $P \in \partial\Omega, \alpha > 1$, let

$$\mathcal{A}_\alpha(P) = \{z \in \Omega : |(z - P) \cdot \bar{\nu}_P| < \alpha\delta_P(z), |z - P|^2 < \alpha\delta_P(z)\}.$$

Notice that δ_P is used because near non-convex boundary points we still want $\mathcal{A}_\alpha(P)$ to have the fundamental geometric shape of (paraboloid $\times$ cone) as shown in Figure 5.

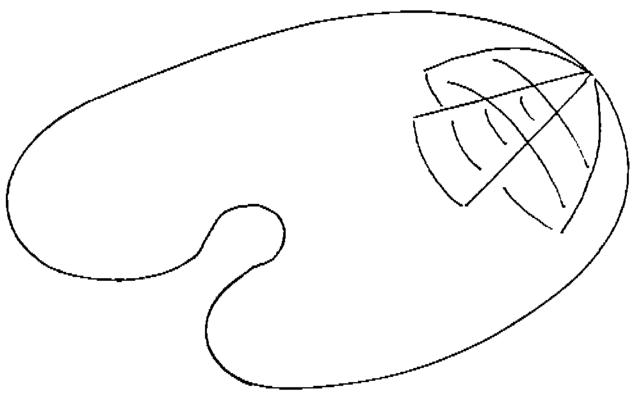

Figure 5

DEFINITION 6.3. If $f \in L^1(\partial\Omega)$ and $P \in \partial\Omega$ then we define

$$M_j f(P) = \sup_{r>0} \sigma(\beta_j(P,r))^{-1} \int_{\beta_j(P,r)} |f(\zeta)| d\sigma(\zeta), \qquad j = 1, 2.$$

DEFINITION 6.4. If $f \in C(\Omega), P \in \partial\Omega$, then we define

$$f_2^{*,\alpha}(P) = \sup_{z \in \mathcal{A}_\alpha(P)} |f(z)|.$$

Now, given the setup that we have pursued thus far, we would like to think that we could majorize $f_2^{*,\alpha}$ by M_2. Unfortunately, we do not know how to do this; because to do so requires a kernel, and we do not have one. So Stein's trick is to intervene with the real variable theory, in which context we do in fact have a kernel (namely, the tried and true Poisson kernel). In fact the correct statement is this:

LEMMA 6.5. *Let $u \in C(\overline{\Omega})$ be non-negative and plurisubharmonic on Ω. Define $f = u|_{\partial\Omega}$. Then*

$$u_2^{*,\alpha}(P) \leq C_\alpha M_2(M_1 f)(P)$$

for all $P \in \partial\Omega$ and any $\alpha > 1$.

We shall not prove this result (but see [14]). Let us say just a few words about why it is true.

Let $z \in \mathcal{A}_\alpha(P)$. Let $\delta(z)$ denote the Euclidean distance of z to $\partial\Omega$. Then $u(z)$ can be written as the average of u over a polydisc $\mathbf{d}$ with extent $\delta(z)/2$ in the complex normal direction and extent $c \cdot \sqrt{\delta(z)}$ in the complex tangential directions. Now, for each $w \in \mathbf{d}$, we may express $u(w)$ as the Poisson integral of $u\big|_{\partial\Omega}$. Thus, using the real variable theory, we may estimate $|u(w)|$ by $C \cdot M_1 u(\widetilde{w})$ (here $\widetilde{w}$ is the projection of w to the boundary). To summarize: **(i)** the average of u over the polydisc $\mathbf{d}$ will be majorized by the maximal function M_2, and **(ii)** the value of each $u(w)$, for $w \in \mathbf{d}$, will be majorized by the maximal function M_1.

Now it turns out that $\partial\Omega$, equipped with the balls β_2 and the standard surface measure $d\sigma$ and with a suitable metric that is constructed with the balls (or, alternatively, with the enveloping property as in Remark (1) following Definition 2.5) is a space of homogeneous type. So the maximal operator M_2 is bounded on the L^p spaces in the usual way. And of course $\partial\Omega$ may be equipped with a different space of homogeneous type structure by instead using the balls β_1, the standard surface measure $d\sigma$, and the Euclidean metric. This observation shows that the maximal operator M_1 is bounded on the L^p spaces in the usual way. Thus Lemma 6.5 may be used, in analogy with what we did in Theorem 3.2, to prove Stein's theorem:

THEOREM 6.6. *Let $\Omega \subseteq \mathbb{C}^n$ be a C^2-smoothly bounded domain and let $f \in H^p(\Omega)$ for some $0 < p \leq \infty$. Fix a number $\alpha > 1$. Then*

$$\lim_{\mathcal{A}_\alpha(P) \ni z \to P} f(z) = f(P) \qquad \textit{for a.e. } P \in \partial\Omega.$$

7. The Shortcomings of Theorem 6.6, and a Broader Perspective

A simple example will suffice to demonstrate why Theorem 6.6 is not the full picture of the boundary behavior of holomorphic functions of several complex variables.

For m a positive integer, let $\Omega_m = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$. Let f be a bounded holomorphic function on Ω_m, and assume that f has a radial limit at the boundary point $\mathbf{1} = (1, 0)$. That is to say, we suppose that

$$\lim_{r \to 1^-} f(r, 0) = \ell.$$

Let $p^j = (p_1^j, p_2^j)$ be a sequence of points approaching $\mathbf{1} = (1, 0)$ with the property that p_1^j is real and positive and

$$(\star) \qquad |p_2^j|^{2m} < \lambda(j)|1 - p_1^j|,$$

where $\lambda(j) > 0$ tend monotonically to zero. For each j, we apply the Cauchy estimates from one complex variable to the function $f(p_1^j, \cdot\,)$ on the disc

$$\mathbf{d}_j \equiv \left\{\zeta \in \mathbb{C} : |\zeta| < \frac{1}{2}|1 - p_1^j|^{1/2m}\right\}.$$

Let K be the supremum of f on Ω_m. Then we have (by a suitable version of the mean value theorem)

$$\left|f(p^j) - f(p_1^j, 0)\right| \leq |p_2^j| \cdot \frac{K \cdot 1!}{(1/2)|1 - p_1^j|^{1/2m}}.$$

Using $(\star)$ we conclude that $|f(p^j) - f(p_1^j, 0)| \to 0$. So the function f has the same limit along the sequence $\{p^j\}$ that it does along the radial ray.

Of course there is nothing special about the particular sequence $\{p^j\}$ that we chose in the preceding discussion. In fact that argument, combined with the classical Lindelöf principle (see [5]), shows that f will have limit ℓ in an approach region that is non-tangential in the complex normal direction and has aperture (nearly) of shape $y = x^{2m}$ in the complex tangential directions. [For instance, when $m = 2$ then the aperture is (nearly) quartic in the complex tangential directions.] Again, the details of the present discussion may be found in [14], §§8.7.

To simplify our discussion, let us now specialize down to Ω_2. It would be incorrect to conclude from the arguments of this section that the correct approach region at *all boundary points* of Ω_2 is (nearly) quartic in complex tangential directions. The fact is that we tacitly used the Levi geometry of the boundary limit point $\mathbf{1} = (1, 0)$ when doing our calculations. This point is *not* strongly pseudoconvex; in fact it is of finite type 4 (that is why the disc $\mathbf{d}$ fits inside the domain Ω_2). And the same is true of all points $(e^{i\theta}, 0)$, $0 \le \theta < 2\pi$. But the other boundary points (which of course form an open, dense subset of the boundary) are strongly pseudoconvex. So the correct analysis for those points is the analysis corresponding to $m = 1$. In other words, for those points the correct shape of an approach region is parabolic in complex tangential direction, just as for the approach regions on the unit ball in $\mathbb{C}^n$.

But the situation is subtle. Let us calculate the Levi form for a boundary point of Ω_2. [Refer to [14, Chapter 3] for background on these matters.] The defining function for Ω_2 is $\rho(z) = |z_1|^2 + |z_2|^4 - 1$. The matrix that defines the Levi form at $z = (z_1, z_2) \in \partial\Omega$ is

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 4|z_2|^2 \end{pmatrix} .$$

A tangent vector at z will have the form

$$A = \left(\frac{-2z_2\overline{z_2}^2\alpha}{\overline{z_1}} \, , \, \alpha \right),$$

for a complex parameter α. Thus

$$\mathcal{L}_z(A, A) = 1 \cdot \frac{4|z_2|^6|\alpha|^2}{|z_1|^2} + 4|z_2|^2 \cdot |\alpha|^2.$$

Now we may see explicitly that points of the form $(e^{i\theta}, 0)$ are weakly pseudoconvex. All other points are strongly pseudoconvex. But we note that the eigenvalue of the Levi form degenerates to zero as the second complex coordinate of the base point z tends to zero. By invariance considerations, it stands to reason that any calculation, using Levi geometry, of approach regions for a Fatou theorem will yield an approach region whose aperture depends on the size of that eigenvalue.

In summary, the parabolic approach regions at points $(z_1, z_2) \in \partial\Omega$ with z_2 non-zero and small will become ever-wider in aperture as $|z_2| \to 0$ and will ultimately pass to the approach region of quartic size that is appropriate for points of the form $(e^{i\theta}, 0)$.

Thus any theory of the boundary behavior of holomorphic functions that takes into account the true complex geometry of the boundary of the domain in question will have to assess how that geometry varies from boundary point to boundary point. In the case of the domain Ω_2, that assessment is fairly simple: it turns out only to depend on $|z_2|$. But for a general domain—especially in higher dimensions—the situation will be quite complicated. Suppose, for example, that $\partial\Omega$ is real analytic. Then the set of strongly pseudoconvex points in the boundary will form a dense open set. The set of weakly pseudoconvex points will form a real analytic variety. By the theory of Lojaciewicz [31], we can be sure that that variety can be stratified into varieties of lower dimension. But the geometry will be technically very complex. And the shapes of the appropriate approach regions for Fatou theory will be correspondingly difficult to calculate.

In fact no program of the sort just described has ever been carried out in generality. Here is a brief description of what is actually known. It can be shown that Theorem 6.6 of Stein is sharp in the case that Ω is strongly pseudoconvex. In the paper [36], Nagel, Stein, and Wainger announced some results along these lines for smoothly bounded, finite type domains in $\mathbb{C}^2$. Many of the technical details for their construction appear in [37]. Some related ideas appear in [18] and [25]. The full story has never been worked out. Some progress in the case of finite type convex domains in $\mathbb{C}^n$ has been made by Fausto di Biase and his collaborators [8]. Other valuable references are [35] and [30].

In the next section we describe a broad approach that unifies the theory for the disc, the ball in $\mathbb{B}^n$, strongly pseudoconvex domains, finite type domains in $\mathbb{C}^2$, finite type convex domains in $\mathbb{C}^n$, and the other types of domains described in the last paragraph. It depends on the use of invariant metrics ([18]).

Invariant metrics are both a useful and attractive device for studying the boundary behavior of holomorphic functions. For one thing, one would certainly like to know that the approach regions being defined are invariant under biholomorphic mappings. This assertion is proved in [41] for the original approach regions $\mathcal{A}_\alpha$, but the arguments are of necessity *ad hoc* just because the definition of the approach regions is *ad hoc*. Checking that the approach regions defined in [36] are invariant would be much more difficult. In fact they *are* known to be invariant, because (by way of work of Aladro [1]) they are equivalent to the metric approach regions described in [18], and those metric approach regions are automatically invariant.

It is also the case that work of I. Graham [10] shows that there is (at least in the strongly pseudoconvex case) a direct connection between the intrinsic geometry of the Kobayashi/Royden and Carathéodory metrics and the Levi geometry of the boundary. [One would expect a similar connection to exist in the finite type case, but this correlation has not been rigorously established.] Thus the development presented in the present paper suggests that invariant metrics might be a useful language for describing approach regions in Fatou theorems. They have also served as a useful predictor for the shapes of approach regions in contexts in which the appropriate theorem was not yet known. Finally, invariant metrics have proved to be the right language in which to formulate the Lindelöf principle in several complex variables (see [5]).

8. Review of Invariant Metrics

We begin with a quick review of the two most important invariant metrics for this discussion. In point of fact we shall only explicitly use the Kobayashi/Royden metric in our applications in §§9 to Fatou theorems. But we also present the Carathéodory metric just to round out the picture, and because the constructions in §§9 using the Kobayashi/Royden metric can be made as well using the Carathéodory metric.

If Ω_1, Ω_2 are domains in $\mathbb{C}^n$ then, following Graham, we define $\Omega_1(\Omega_2)$ to be the collection of all mappings from Ω_2 to Ω_1.

DEFINITION 8.1. If $\Omega \subseteq \mathbb{C}^n$ is open, then the *infinitesimal Carathéodory metric* is given by $F_C : \Omega \times \mathbb{C}^n \to \mathbb{R}$ where

$$F_C(z,\xi) = \sup_{\substack{f\in B(\Omega)\\ f(z)=0}} |f_*(z)\xi| \equiv \sup_{\substack{f\in B(\Omega)\\ f(z)=0}} \left|\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z)\cdot \xi_j\right|.$$

REMARK. In this definition, the $\mathbb{C}^n$ in $\Omega \times \mathbb{C}^n$ should be thought of as the tangent space to Ω at z—in other words, $\Omega \times \mathbb{C}^n$ should be canonically identified with the tangent bundle of Ω at z. We think of $F_C(z,\xi)$ as the length of the tangent vector ξ at the point $z \in \Omega$. In general, $F_C(z,\xi)$ is not given by a quadratic form $(g_{ij}(z))$, hence F_C is not a Riemannian metric.

DEFINITION 8.2. Let $\Omega \subseteq \mathbb{C}^n$ be open and $\gamma : [0,1] \to \Omega$ a C^1 curve. The *Carathéodory length* of γ is defined to be

$$L_C(\gamma) = \int_0^1 F_C(\gamma(t),\gamma'(t))\,dt.$$

This definition parallels the definition of the length of a curve in a Riemannian metric. It would be natural at this point to define the (integrated) Carathéodory distance between two points to be the infimum of lengths of all curves connecting them. One advantage of defining distance in this fashion is that it is then straightforward to verify the triangle inequality.

However we shall not take this approach. One of the most important features of the Carathéodory metric is that it is, in a precise sense, the least metric under which holomorphic mappings are distance decreasing. The notion of distance suggested in the last paragraph is *not* the smallest. It is fortuitous that the following definition *does* result in the least distance-decreasing metric—in particular it *does* satisfy the triangle inequality.

DEFINITION 8.3. Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $z, w \in \Omega$. The *Carathéodory distance* between z and w is defined to be

$$C(z,w) = \sup_{f\in B(\Omega)} \rho(f(z), f(w)),$$

where ρ is the Poincaré-Bergman distance on B.

Notice that, depending on the domain Ω, the Carathéodory distance between two distinct points may be 0.

DEFINITION 8.4. Let $\Omega \subseteq \mathbb{C}^n$ be open. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^n$. The infinitesimal form of the *Kobayashi/Royden metric* is given by $F_K : \Omega \times \mathbb{C}^n \to \mathbb{R}$, where

$$\begin{aligned} &F_K(z, \xi) \\ &\equiv \inf\{\alpha : \alpha > 0 \text{ and } \exists f \in \Omega(B) \text{ with } f(0) = z, (f'(0))(e_1) = \xi/\alpha\} \\ &= \inf\left\{\frac{|\xi|}{|(f'(0))(e_1)|} : f \in \Omega(B), (f'(0))(e_1) \text{is aconstant multiple of } \xi\right\} \\ &= \frac{|\xi|}{\sup\{|(f'(0))(e_1)| : f \in \Omega(B), (f'(0))(e_1) \text{ is a constant multiple of } \xi\}} \end{aligned}$$

We now wish to define an integrated distance based on elements of $\Omega(B)$. The natural analogue for our definition of Carathéodory distance does not satisfy a triangle inequality. Moreover, we want the Kobayashi/Royden distance to be the *greatest* metric under which holomorphic mappings are distance decreasing. Therefore we proceed as follows:

DEFINITION 8.5. Let $\Omega \subseteq \mathbb{C}^n$ be open and $\gamma : [0, 1] \to \Omega$ a piecewise C^1 curve. The *Kobayashi/Royden length* of γ is defined to be

$$L_K(\gamma) = \int_0^1 F_K(\gamma(t), \gamma'(t))dt.$$

DEFINITION 8.6. Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $z, w \in \Omega$. The (integrated) *Kobayashi/Royden distance* between z and w is defined to be

$$K(z, w) = \inf\{L_K(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve connecting } z \text{ and } w\}.$$

Recall that we did not implement a definition like this one for the (integrated) Carathéodory distance because we were able to find a *smaller* distance that satisfied the triangle inequality. Now we are at the other end of the spectrum: we want the Kobayashi/Royden metric to be as large as possible, and to satisfy a triangle inequality as well. This definition serves that dual purpose.

We remark in passing that the use of the ball as a model domain when defining the Carathéodory and Kobayashi/Royden metrics is important, but that choice is not unique. The theory is equally successful if either the disc or the polydisc is used. However, in the current state of the theory, it is essential that the model domain have a transitive group of automorphisms. See also [32], where developments related to other choices of a model domain are presented.

PROPOSITION 8.7 (The Distance Decreasing Properties). *If Ω_1, Ω_2 are domains in $\mathbb{C}^n, z, w \in \Omega_1, \xi \in \mathbb{C}^n$, and if $f : \Omega_1 \to \Omega_2$ is holomorphic, then*

$$\begin{aligned} F_C^{\Omega_1}(z,\xi) &\geq F_C^{\Omega_2}(f(z), f_*(z)\xi) \qquad & F_K^{\Omega_1}(z,\xi) &\geq F_K^{\Omega_2}(f(z), f_*(z)\xi) \\ C_{\Omega_1}(z,w) &\geq C_{\Omega_2}(f(z), f(w)) \qquad & K_{\Omega_1}(z,w) &\geq K_{\Omega_2}(f(z), f(w)). \end{aligned}$$

Proof. This follows by inspection from the definitions. □

COROLLARY 8.8. *If $f : \Omega_1 \to \Omega_2$ is biholomorphic then f is an isometry in both the Carathéodory and the Kobayashi/Royden metrics.*

Proof. Apply the proposition to both f and f^{-1}. □

COROLLARY 8.9. *If $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}^n$ then for any $z, w \in \Omega_1$, any $\xi \in \mathbb{C}^n$, we have*

$$\begin{aligned} F_C^{\Omega_1}(z,\xi) &\geq F_C^{\Omega_2}(z,\xi) \qquad & F_K^{\Omega_1}(z,\xi) &\geq F_K^{\Omega_2}(z,\xi) \\ C_{\Omega_1}(z,w) &\geq C_{\Omega_2}(z,w) \qquad & K_{\Omega_1}(z,w) &\geq K_{\Omega_2}(z,w). \end{aligned}$$

9. Invariant Metrics and the Fatou Theorem

The key idea in [18] is to define the approach regions for a Fatou theorem by way of an invariant metric. For convenience, we specialize now to the Kobayashi/Royden metric (although in many applications it is useful to take another metric). Call it ρ. Fix a domain $\Omega \subseteq \mathbb{C}^n$ with smooth (at least C^2) boundary. Let U be a tubular neighborhood of $\partial\Omega$ and take ϵ_0 to be one fourth of the distance of ${}^cU$ to $\partial\Omega$. As usual, let ν_P be the unit outward normal vector to $\partial\Omega$ at a boundary point $P \in \partial\Omega$. Fix a number $\beta > 0$. If $P \in \partial\Omega$, then we let $n_P = \{P - t\nu_P : 0 < t < \epsilon_0\}$. We set

$$\mathcal{K}_\beta(P) = \{z \in \Omega : \rho(z, n_P) < \beta\}.$$

As in the paper [18], there are a number of technical but plausible geometric hypotheses about the metric ρ that must be put in place. We shall not state these explicitly here. We further define a ball $\beta(P,r)$ in $\partial\Omega$ as follows:

$$\beta(P,r) = \pi\big(\{z \in \mathcal{K}_\beta(P) : \operatorname{dist}(z, \partial\Omega) = r\}.$$

Here, as usual, "dist" denotes Euclidean distance. And π is Euclidean orthogonal projection to the boundary. We use the standard Carathéodory construction (see [29]) to create a $(2n-1)$-dimensional measure σ on $\partial\Omega$ from the balls $\beta(P,r)$. Our plausible geometric hypotheses guarantee that these balls satisfy the enveloping property described in Remark (1) following Definition 2.5. Thus we have equipped $\partial\Omega$ with the structure of a space of homogeneous type. Following (in broad strokes) the general line of attack developed in this paper—with a number of technical changes along the way—we are then able to prove the following theorem:

THEOREM 9.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Let approach regions $\mathcal{K}_\beta$ be defined as above, and assume all the plausible geometric hypotheses as previously indicated* (*see* [18] *for details*). *Let f be an H^p function on Ω, and let f^* be as usual. Then*

$$\lim_{\mathcal{K}_\beta(P)\ni z\to P} f(z) = f^*(P) \qquad \text{for almost every } P \in \partial\Omega\,.$$

Now let us look at several examples to see what this theorem says in cases that we already understand, and also what it says in a few cases that we have not already considered.

EXAMPLE 9.2. Let $\Omega = D$, the unit disc in the complex plane. Fix a number $\beta > 0$. The set of points with Kobayashi/Royden distance less than β from the origin is $\mathbf{d} = D(0,[e^{2\beta}-1]/[e^{2\beta}+1])$. The approach region $\mathcal{K}_\beta$ is then the union of the discs with centers $\alpha+i0$, $0<\alpha<1$, and Kobayashi/Royden radius β. Put in other terms, $\mathcal{K}_\beta$ is the union of the images $\phi_a(\mathbf{d})$, $-1<a<0$. Here the ϕ_a are the Möbius transformations defined in §§3.

To get an idea of what $\mathcal{K}_\beta(1+i0)$ looks like, we calculate the trajectory of the extremal point $0+i[e^{2\beta}-1]/[e^{2\beta}+1]$ on the boundary of $\mathbf{d}$ under the Möbius transformations just indicated. Let $r = [e^{2\beta}-1]/[e^{2\beta}+1]$. Then

$$\phi_a(ir) = \frac{-a(1+r^2)}{1+a^2r^2} + ir\frac{1-a^2}{1+a^2r^2}.$$

We see that

$$\phi_a(ir) \cong -a + ir(1-a).$$

Remembering that a is real (and negative), we find that the real part of $\phi_a(ir)$ has distance $1-|a|$ from the boundary point $1 = 1+i0$ and the imaginary part has distancc (a constant times) $1-|a|$ from the real axis. This is the shape of non-tangential approach.

In summary, the calculation just performed suggests that $\mathcal{K}_\beta$ on the unit disc is the classical non-tangential approach region.

EXAMPLE 9.3. Let $\Omega = B$, the unit ball in $\mathbb{C}^n$. We take $n = 2$ for convenience. Fix a number $\beta > 0$. The set of points with Kobayashi/Royden distance less than β from the origin is $\mathbf{b} = B(0, [e^{2\beta} - 1]/[e^{2\beta} + 1])$. The approach region $\mathcal{K}_\beta$ is then the union of the balls with centers $(\alpha + i0, 0)$, $0 < \alpha < 1$ and Kobayashi/Royden radius β. Put in other terms, $\mathcal{K}_\beta$ is the union of the images $\Phi_a(\mathbf{b})$, $-1 < a < 0$. Here

$$\Phi_a(z_1, z_2) = \left(\frac{z_1 - a}{1 - \bar{a}z_1}, \frac{\sqrt{1 - a^2}z_2}{1 - \bar{a}z_1}\right).$$

We will now do an analysis that suggests the shape of $\mathcal{K}_\beta((1,0))$. As before, we set $r = [e^{2\beta} - 1]/[e^{2\beta} + 1]$. Now a straightforward calculation shows that the image of the extreme point $(0, ir)$ under Φ_a is

$$\left(a\ ,\ \frac{\sqrt{1 - a^2}(r^2 a + ir)}{1 + a^2 r^2}\right).$$

The main thing to notice now is that the first entry has distance $1 - |a|$ from the point $(1, 0)$, while the second entry has size essentially $\sqrt{1 - a^2}$. This is the shape of admissible approach.

In summary, our calculation indicates that $\mathcal{K}_\beta$ is an admissible approach region.

EXAMPLE 9.4. Let $\Omega \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary. Fix a point $P \in \partial\Omega$ and define $\mathcal{K}_\beta(P)$ as usual. Then the results of [10] and the calculations in [1] show that this approach region $\mathcal{K}_\beta$ is comparable to the approach regions defined by Stein in [41]. In particular, it is non-tangential in the complex normal directions and parabolic in the complex tangential directions.

It can be shown that if $\Omega' \subseteq \mathbb{C}^n$ is some smoothly bounded domain and $P \in \partial\Omega$ is *not* strongly pseudoconvex then the approach region $\mathcal{K}_\beta$ will *always* be strictly larger than the parabolic approach regions of Stein. Some of the examples below will explore what can happen in such a case.

EXAMPLE 9.5. Let $\Omega \subseteq \mathbb{C}^2$ be a finite type domain with C^∞ boundary (see [14] for a detailed treatment of the concept of finite type). It is important for this example that $n = 2$. Fix a point $P \in \partial\Omega$—say that P is a point of type m, which says in effect that P is flat to order

m (in a complex-analytic sense) in complex tangential directions. Then the approach region $\mathcal{K}_\beta(P)$ will be comparable to the approach regions defined in [36]. [In fact [36] gives three equivalent definitions of these approach regions.] The details of this equivalence follow from the main results of [3] and the calculations in [1].

In particular, the approach regions $\mathcal{K}_\beta$ in this setting are non-tangential in complex normal directions and have aperture of the shape $y = x^m$ in complex tangential directions (see Figure 6). Of course, as previously indicated, the actual shape of the region will vary in a subtle semi-continuous way as P moves about the boundary.

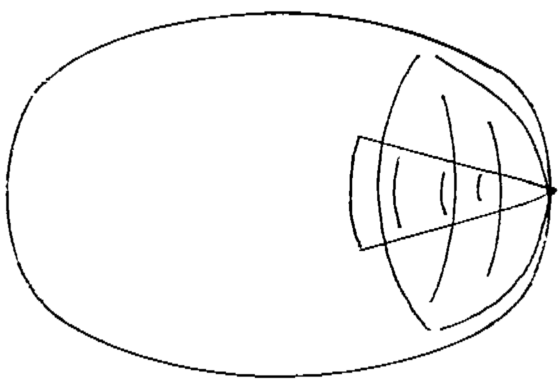

Figure 6

EXAMPLE 9.6. Let $\Omega \subseteq \mathbb{C}^n$ be a domain with C^2 boundary. Let $P \in \partial\Omega$ be a point of strong pseudoconcavity. The Kobayashi/Royden metric near such a point is well understood (see [20]), and the approach region $\mathcal{K}_\beta(P)$ may be seen to contain a full neighborhood $B(P,r) \cap \Omega$. Thus one expects a Hardy space function to have an unrestricted limit at such a point.

Of course the result described in the last paragraph is no surprise, because the Hartogs extension phenomenon (see [14]) tells us that *any* holomorphic function will continue analytically to a full neighborhood of P. In particular, it will be continuous up to P. So of course such a function will have a limit from any direction at P.

EXAMPLE 9.7. Let $\Omega \subseteq \mathbb{C}^n$ be a convex domain of finite type in any complex dimension. Then the results of [33] can be used to estimate the size of the approach regions $\mathcal{K}_\beta$. This program has yet to be carried out. F. di Biase [8] has done some related work.

EXAMPLE 9.8. The Carathéodory construction of measures can be used with different exponents to create boundary measures that measure sets of differing dimensions. In particular, these measures can be used to detect lower dimensional sets on which bounded holomorphic functions have boundary limits. For example, the following remarkable result of Nagel and Rudin [34] can be recovered by using the one-dimensional measure that can be constructed from our metric ideas in this way:

> THEOREM: Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with smooth boundary. Let $\gamma : [0,1] \to \partial\Omega$ be a smooth curve with the property that each of its tangent vectors is transverse to the complex tangential directions. If $f \in H^\infty(\Omega)$ then f will have a (suitable) limit at almost every point of γ. Here "almost every" is measured with respect to arc-length measure.

There is still much to be learned about lower dimensional phenomena and boundary limits of holomorphic functions.

10. Concluding Remarks

The results described in §§9 offer considerable evidence that it is natural to formulate the ideas of harmonic analysis on a domain in $\mathbb{C}^n$ by way of invariant metrics. Especially because each domain has its own analysis—depending on the Levi geometry of the boundary—it is natural to enlist invariant metrics as a tool both in formulating and proving results. Again, the theorem of Graham [10] shows that boundary Levi geometry and intrinsic Kobayashi/Royden geometry are two sides of the same coin.

The paper [25] indicates how results about the Lusin area integral may be related to invariant metric considerations. The papers [21]–[23] give other examples of metric considerations in multi-variable complex function theory. Certainly the very function spaces that we consider should be adapted to the domain under study, and invariant metrics should prove to be the right tools for understanding these spaces. Interpolation theorems, characterizations in terms of derivatives, and boundedness of Hankel, Bergman, Szegö and other canonical operators ought to be studied by way of canonical constructions—particularly invariant metrics (see some of the suggestive evidence in [26], [27]). We hope that this paper will serve as an introduction to the geometric point of view in the harmonic analysis of several complex variables.

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