

ON PURE-STRATEGY EQUILIBRIA IN MATRIX GAMES

TAE-HWAN YOON AND O-HUN KWON

ABSTRACT. In this paper we find a sufficient condition to guarantee the existence of pure-strategy equilibria in matrix games. In the process of formulating our condition, the alternative theorem of Farkas is used. The formulated condition is necessary and sufficient to the existence of pure-strategy equilibria in undominated matrix games.

1. Introduction

The minimax theorem of a matrix game was first developed by von Neumann [8] in 1928 using Brouwer's fixed point theorem. Dantzig [3] in 1956 proved this theorem via linear programming in a constructive way leading to a computational algorithm. Panik [6] in 1994 proved the minimax theorem using the specialization of the alternative theorem of Farkas which involves the expression of one vector as a convex combination of a set of vectors.

We analyse an equilibrium point in mixed strategies to get the situation in which an equilibrium point in mixed strategies simultaneously becomes an equilibrium point in pure strategies. We find a sufficient condition to guarantee the existence of an equilibrium point in pure strategies. Our sufficient condition is formulated using the alternative theorem of Farkas [5]. Furthermore, in case of a undominated matrix game, our condition is necessary and sufficient to the existence of an equilibrium point in pure strategies. Therefore this paper presents a condition for the existence of an equilibrium point in pure strategies, and conversely, this

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condition is always satisfied by entries of matrix whenever there exists an equilibrium point in pure strategies.

2. Preliminaries

Let Γ_A be an $(m \times n)$ matrix game, where $A = (a_{ij})$, $a_{ij} > 0$. Denote A^t the transpose of A , \mathbf{a}^j (respectively, \mathbf{a}_i) the column (respectively, row) vector of the matrix A , and $\mathbf{x} \cdot \mathbf{y}$ the inner product of vectors \mathbf{x} and \mathbf{y} . Let $\mathbf{1}_m = (1, 1, \dots, 1) \in \mathbf{R}^m$ and $\mathbf{0}_m$ the zero vector in \mathbf{R}^m . For convenience, we denote $\mathbf{x} \geq \mathbf{b}$ if $x_i \geq b_i$ for all i ($1 \leq i \leq n$).

The following is the fundamental theorem of game theory by von Neumann [8].

THEOREM 1. (Minimax Theorem) *A matrix game Γ_A has an equilibrium point in mixed strategies.*

Given an m -vector \mathbf{b} and an n -vector \mathbf{c} , we have the following two problems called *dual linear programs* in standard form.

$$\begin{array}{ll} \text{Primal:} & \text{Minimize } \mathbf{x} \cdot \mathbf{b} \\ & \text{subject to } \mathbf{x}A \geq \mathbf{c}, \mathbf{x} \geq \mathbf{0}_m. \\ \text{Dual:} & \text{Maximize } \mathbf{c} \cdot \mathbf{y} \\ & \text{subject to } A\mathbf{y} \leq \mathbf{b}, \mathbf{y} \geq \mathbf{0}_n. \end{array}$$

Any \mathbf{x} satisfying constraints of the primal is called a *feasible solution* to the primal. A feasible solution to the dual is similarly defined. Here $\mathbf{x} \cdot \mathbf{b}$ and $\mathbf{c} \cdot \mathbf{y}$ are called *objective functions* for the primal and dual.

The following is a version of the fundamental theorem of linear programming due to von Neumann [2].

THEOREM 2. (Duality Theorem) *If the primal and dual problems have at least one feasible solution, then two problems have optimal solutions. Furthermore, at any optimal solution, the value of two objective functions coincide.*

Indeed, the duality theorem and the minimax theorem are equivalent [2]. Thus the solution algorithm of the minimax theorem can be described as follows [3]. First, we find objective functions $\bar{\mathbf{x}} \cdot \mathbf{1}_m$ and $\mathbf{1}_n \cdot \bar{\mathbf{y}}$ for the following primal and dual problems.

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- (1) Primal: Minimize $\mathbf{x} \cdot \mathbf{1}_m$
 subject to $\mathbf{x}A \geq \mathbf{1}_n, \mathbf{x} \geq \mathbf{0}_m$.
- (2) Dual: Maximize $\mathbf{1}_n \cdot \mathbf{y}$
 subject to $A\mathbf{y} \leq \mathbf{1}_m, \mathbf{y} \geq \mathbf{0}_n$.

Second, we set

$$\mathbf{x}^* = \frac{\bar{\mathbf{x}}}{\bar{\mathbf{x}} \cdot \mathbf{1}_m}$$

and

$$\mathbf{y}^* = \frac{\bar{\mathbf{y}}}{\mathbf{1}_n \cdot \bar{\mathbf{y}}}.$$

Then $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium point in mixed strategies.

Consider the mixed strategy $\mathbf{x}^* = (x_1, x_2, \dots, x_m)$, where $x_k = 1$ for some k and $x_i = 0$ for $i \neq k$. We can identify \mathbf{x}^* with the i -th pure strategy of player 1. Similarly, we can identify the mixed strategy $\mathbf{y}^* = (y_1, y_2, \dots, y_n)$, where $y_k = 1$ for some k and $y_j = 0$ for $j \neq k$, with the j -th pure strategy of player 2.

We have an equilibrium point in pure strategies if the sets of optimal strategies of player 1 and 2 include the pure strategies of player 1 and 2, respectively.

The following is the alternative theorem of Farkas [5].

THEOREM 3. (Alternative Theorem) For each $(m \times n)$ matrix A and each vector \mathbf{b} in \mathbf{R}^n ,

either

(A) $A\mathbf{x} \leq \mathbf{0}_m, \mathbf{b} \cdot \mathbf{x} > 0$ has a solution $\mathbf{x}_0 \in \mathbf{R}^m$

or

(B) $A^t\mathbf{y} = \mathbf{b}, \mathbf{y} \geq \mathbf{0}_n$ has a solution $\mathbf{y}_0 \in \mathbf{R}^n$

but never both.

We may rewrite the above theorem as follows,

either

(A) $\mathbf{a}_i \cdot \mathbf{x}_0 \leq 0, \mathbf{b} \cdot \mathbf{x}_0 > 0$ for $i = 1, 2, \dots, m$

or

$$(B) \sum_{i=1}^m a_i y_i = \mathbf{b}, y_i \geq 0 \text{ for } i = 1, 2, \dots, m$$

but never both.

Additionally, the alternative theorem can be described as follows by changing the role of column vectors and row vectors.

THEOREM 4. For each $(m \times n)$ matrix A and each vector \mathbf{c} in \mathbf{R}^m ,

either

$$(A) \mathbf{x}A \leq \mathbf{0}_n, \mathbf{x} \cdot \mathbf{c} > 0 \text{ has a solution } \mathbf{x}_0 \in \mathbf{R}^m$$

or

$$(B) \mathbf{y}A^t = \mathbf{c}, \mathbf{y} \geq \mathbf{0}_n \text{ has a solution } \mathbf{y}_0 \in \mathbf{R}^n$$

but never both.

We need the following definition. The p -th strategy of player 1 *dominates* the q -th strategy of player 1 if

$$a_{pj} \geq a_{qj} \text{ for } j = 1, 2, \dots, n$$

and

$$a_{pk} > a_{qk} \text{ for some } k \in \{1, 2, \dots, n\}.$$

Similarly, the p -th strategy of player 2 *dominates* the q -th strategy of player 2 if

$$a_{ip} \leq a_{iq} \text{ for } i = 1, 2, \dots, m$$

and

$$a_{kp} < a_{kq} \text{ for some } k \in \{1, 2, \dots, m\}.$$

Γ_A is called a *undominated matrix game* if all strategies of player 1 are not dominated relative to each other and all strategies of player 2 are not dominated relative to each other.

3. Main Theorems

Denote

$$\mathbf{R}_-^n = \{(x_1, x_2, \dots, x_n) : x_i < 0 \text{ for } i = 1, 2, \dots, n\}$$

and

$$\mathbf{R}_+^n = \{(x_1, x_2, \dots, x_n) : x_i > 0 \text{ for } i = 1, 2, \dots, n\}.$$

We use a version of the alternative theorem of Farkas (Theorem 4) in Preliminaries to get the following Lemma.

LEMMA 1. For the existence of a solution \mathbf{x}_0 in \mathbf{R}^m of the system

$$\begin{cases} \mathbf{x}A \in \mathbf{R}_-^n \cup \{\mathbf{0}_n\} \\ \mathbf{x} \cdot \mathbf{1}_m > 0, \end{cases}$$

it is necessary and sufficient that

$$(4) \quad \mathbf{1}_m \notin \text{cone}\{\mathbf{a}^j : 1 \leq j \leq n\}$$

where $\text{cone}\{\mathbf{a}^j : 1 \leq j \leq n\}$ is the cone generated by column vectors \mathbf{a}^j of A .

Proof. We may rewrite Theorem 4 as follows,

either

$$(A) \quad \mathbf{x}_0 \cdot \mathbf{a}^j \leq 0, \quad \mathbf{x}_0 \cdot \mathbf{c} > 0 \quad \text{for } j = 1, 2, \dots, n$$

or

$$(B) \quad \sum_{j=1}^n y_j \mathbf{a}^j = \mathbf{c}, \quad y_j \geq 0 \quad \text{for } j = 1, 2, \dots, n$$

but never both.

Now assume that there exists a solution \mathbf{x}_0 in \mathbf{R}^m of the system

$$\begin{cases} \mathbf{x}A \in \mathbf{R}_-^n \cup \{\mathbf{0}_n\} \\ \mathbf{x} \cdot \mathbf{1}_m > 0. \end{cases}$$

We replace \mathbf{c} by $\mathbf{1}_m \in \mathbf{R}^m$ in conditions (A) and (B). Then there exists no solution \mathbf{y}_0 of the condition (B). This implies that there is no $\mathbf{y}_0 \in \mathbf{R}_+^n \cup \{\mathbf{0}_n\}$ such that

$$\mathbf{1}_m = \sum_{j=1}^n y_j \mathbf{a}^j.$$

Thus

$$\mathbf{1}_m \notin \text{cone}\{\mathbf{a}^j : 1 \leq j \leq n\}.$$

Conversely, assume that (3) is satisfied. Then there is no $\mathbf{y}_0 \in \mathbf{R}_+^n \cup \{\mathbf{0}_n\}$ such that

$$\mathbf{1}_m = \sum_{j=1}^n y_j \mathbf{a}^j.$$

This implies that there exists no solution \mathbf{y}_0 of the condition (B). Therefore we can find a solution \mathbf{x}_0 of the condition (A). Hence there exists a solution \mathbf{x}_0 in \mathbf{R}^m of the system

$$\begin{cases} \mathbf{x}A \in \mathbf{R}_-^n \cup \{\mathbf{0}_n\} \\ \mathbf{x} \cdot \mathbf{1}_m > 0. \end{cases}$$

This completes the proof. □

Now we formulate a sufficient condition for the existence of an equilibrium point in pure strategies for a matrix game.

THEOREM 5. *For the existence of an equilibrium point in pure strategies in Γ_A , it is sufficient that there exist a solution \mathbf{x}_0 in \mathbf{R}^m of the system*

$$(5) \quad \begin{cases} \mathbf{x}A \in \mathbf{R}_-^n \cup \{\mathbf{0}_n\} \\ \mathbf{x} \cdot \mathbf{1}_m > 0 \end{cases}$$

and a solution \mathbf{y}_0 in \mathbf{R}^n of the system

$$(6) \quad \begin{cases} A\mathbf{y} \in \mathbf{R}_+^m \cup \{\mathbf{0}_m\} \\ \mathbf{1}_n \cdot \mathbf{y} < 0. \end{cases}$$

Proof. Assume that there exists a solution \mathbf{x}_0 in \mathbf{R}^m of the system

$$\begin{cases} \mathbf{x}A \in \mathbf{R}_-^n \cup \{\mathbf{0}_n\} \\ \mathbf{x} \cdot \mathbf{1}_m > 0. \end{cases}$$

By Lemma 1, this implies that

$$\mathbf{1}_m \notin \text{cone}\{\mathbf{a}^j : 1 \leq j \leq n\}.$$

For our proof, we exclude dominated columns in the solution algorithm in Preliminaries. For this, we define

$$a_{ij_i} = \min\{a_{ij} : 1 \leq j \leq n\}$$

for each i . That is, a_{ij_i} is the entry of i -th row and j_i -th column having the smallest number for the i -th strategy of player 1. Then we have

$$\mathbf{1}_m \notin \text{cone}\{\mathbf{a}^{j_i} : 1 \leq i \leq m\},$$

since

$$\text{cone}\{\mathbf{a}^{j_i} : 1 \leq i \leq m\} \subseteq \text{cone}\{\mathbf{a}^j : 1 \leq j \leq n\},$$

where \mathbf{a}^j is the j -th column vector. This implies that there exists a pure strategy in the set of solutions for the primal problem (1) in Preliminaries.

To show this, we can rewrite (1) as follows.

$$(7) \quad \begin{array}{l} \text{Minimize} \quad \sum_{i=1}^m x_i \\ \text{subject to} \quad \sum_{i=1}^m x_i a_{ij} \geq 1 \text{ for } j = 1, 2, \dots, n, \\ \quad \quad \quad x_i \geq 0 \text{ for } i = 1, 2, \dots, m. \end{array}$$

Let $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m)$ be an optimal solution for the minimization problem (6). The column vector $\mathbf{a}^j = (a_{1j}, a_{2j}, \dots, a_{mj})$ is the normal vector of the equation

$$\sum_{i=1}^m a_{ij} x_i = 1$$

at the point $\bar{\mathbf{x}}$ for $j = 1, 2, \dots, n$. Consider the point $\bar{\mathbf{x}}$ as the origin in \mathbf{R}^m . Then we have the cone

$$\text{cone}\{\mathbf{a}^i : 1 \leq i \leq m\}$$

generated by column vectors \mathbf{a}^i having the point $\bar{\mathbf{x}}$ as the origin. If the vector $\mathbf{1}_m$ at the point $\bar{\mathbf{x}}$ is not in the cone generated by column vectors \mathbf{a}^i , $\bar{\mathbf{x}}$ is an extreme point of the feasible region of (6) on the k -axis for some k . Then $\bar{x}_k = 1$ and $\bar{x}_i = 0$ for $i \neq k$. Therefore there exists a pure strategy in the set of solutions for the primal problem (1).

Hence the existence of an optimal pure strategy for the first player is guaranteed. A similar argument applies for the second player but for using the dual problem (2) in Preliminaries. This completes the proof. \square

In a undominated matrix game, the converse of Theorem 5 is also valid.

THEOREM 6. *For the existence of an equilibrium point in pure strategies in a undominated matrix game Γ_A , it is necessary and sufficient that there exist a solution \mathbf{x}_0 in \mathbf{R}^m of the system (4) and a solution \mathbf{y}_0 in \mathbf{R}^n of the system (5).*

Proof. We have only to show the necessity (sufficiency: by Theorem 5). Assume that there exists an equilibrium point in pure strategies. Then the set of solutions for the primal problem (1) in Preliminaries includes a pure strategy.

Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m)$ be an optimal pure strategy. Then it follows that $\bar{x}_k = 1$ for some k and $\bar{x}_i = 0$ for $i \neq k$. Thus, since any strategy of player 1 is not dominated by another, the vector $\mathbf{1}_m$ at the point \bar{x} is not in the cone generated by column vectors. That is,

$$\mathbf{1}_m \notin \text{cone}\{\mathbf{a}^j : 1 \leq j \leq n\}.$$

This implies the existence of a solution \mathbf{x}_0 in \mathbf{R}^m of the system (4) by Lemma 1. The same for the second player but for using the dual problem (2), that is, there exists a solution \mathbf{y}_0 in \mathbf{R}^n of the system (5). This completes the proof. \square

We combine Theorem 6 and Lemma 1 to get the following Proposition.

PROPOSITION 1. *For the existence of an equilibrium point in pure strategies in a undominated matrix game Γ_A , it is necessary and sufficient that there does not exist*

$$(s_1, s_2, \dots, s_n) \in \mathbf{R}_+^n \cup \{\mathbf{0}_n\}$$

satisfying

$$\begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = s_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} + s_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix} + \dots + s_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix}$$

and there does not exist

$$(t_1, t_2, \dots, t_m) \in \mathbf{R}_+^m \cup \{\mathbf{0}_m\}$$

satisfying

$$\begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = t_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{1n} \end{bmatrix} + t_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{2n} \end{bmatrix} + \cdots + t_m \begin{bmatrix} a_{m1} \\ a_{m2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} .$$

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References

- [1] A. Cegielski, *Geometrically convergent projection method in matrix games*, J. Optim. Theory Appl. **85** (1995), no. 2, 249-264.
- [2] G. Dantzig, *A proof of the equivalence of the programming problem and the game problem*, In activity analysis of production and allocation, Cowles Commission Monograph **13**(1951), 333-335.
- [3] ———, *Constructive proof of the mini-max theorem*, Pacific Journal of Mathematics, 1956, pp. 25-33.
- [4] I. Kaplansky, *A contribution to von Neumann's theory of games II*, Linear Algebra Appl. **226/228**(1995), 371-373.
- [5] O. Mangasarian, *Nonlinear programming*, McGraw-hill, New York, 1982.
- [6] M. Panik, *The fundamental theorem of game theory revisited*, Appl. Math. Lett. **7**, (1994), no. 2, 77-79.
- [7] L. Petrosjan and N. Zenkevich, *Game theory*, World Scientific Publishing, London, 1996.
- [8] J. von Neumann, *Zur Theorie der Gesellschaftsspiele*, Mathematische Annalen **100**(1928), 295-320.
- [9] V. Zakharov and O. Kwon, *Linear programming approach in cooperative games*, J. Kor. Math. Soc. **34** (1997), no. 2, 423-435.

TAE-HWAN YOON, DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, SEOUL, 136-701, KOREA
E-mail: yoonth@bessel.korea.ac.kr

O-HUN KWON, DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, SEOUL, 136-701, KOREA
E-mail: kwon@semi.korea.ac.kr