Bull. Korean Math. Soc. **37** (2000), No. 2, pp. 347–352

A PROPAGATION OF QUADRATICALLY HYPONORMAL WEIGHTED SHIFTS

YONG BIN CHOI

ABSTRACT. In this note we answer to a question of Curto: Non-first two equal weights in the weighted shift force subnormality in the presence of quadratic hyponormality. Also it is shown that every hyponormal weighted shift with two equal weights cannot be polynomially hyponormal without being flat.

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([**2**],[**3**, III.1.9]). If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be k-*hyponormal* whenever the $k \times k$ operator matrix

Received February 10, 1999.
2000 Mathematics Subject Classification: 47B20, 47B37.
Key words and phrases: Quadratically hyponormal operators, polynomially hyponormal operators, weighted shifts, propagations.

$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$ is positive, then the Bram-Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([**6**]). Recall ([**1**],[**6**]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if $T + \alpha_1 T^2 + \cdots + \alpha_{k-1} T^k$ is hyponormal for every $\alpha_1, \cdots, \alpha_{k-1} \in \mathbb{C}$. If $k = 2$ then it is said to be *quadratically hyponormal* Also $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, and the converse is not true in general (cf. [**4**],[**5**]).

J. Stampfli [**7**] showed that for subnormal weighted shifts W_α, a *propagation* phenomenon occurs which forces the flatness of W_α whenever two equal weights are present.

PROPAGATION OF SUBNORMALITY ([**7**, THEOREM 6]). *Let W_α be a subnormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.*

In [**4**] it was shown that Stampfli's propagation for subnormality can be extended for 2-hyponormality.

PROPAGATION OF 2-HYPONORMALITY ([**4**, COROLLARY 6]). *Let W_α be a 2-hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.*

On the other hand, it was shown in [**4**, Theorem 2] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat: *If W_α is a quadratically hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$ and if $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.*

Furthermore, in [**4**, Proposition 11], it was shown that, in the presence of quadratic hyponormality, two consecutive pairs of equal weights again force subnormality: *If W_α is a quadratically hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$, and if $\alpha_n = \alpha_{n+1}$ and $\alpha_{n+2} = \alpha_{n+3}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.*

In [**4**] the following question was raised: *Whether two non-consecutive pairs of equal weights force subnormality in the presence of quadratic hyponormality ?* We now answer it affirmatively.

THEOREM 1. *Let W_α be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$, and assume that W_α is quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.*

Proof. Without loss of generality (the restriction of a quadratically hyponormal operator to an invariant subspace is also quadratically hyponormal), we may assume that $n = 1$ and $\alpha_1 = \alpha_2 = 1$. We will show that either $\alpha_0 = 1$ or $\alpha_3 = 1$. Then three equal weights are present, so that by [**4**, Theorem 2], we have that $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$. For this we need to consider the selfcommutator $[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$. Let W_α be a hyponormal weighted shift. For $s \in \mathbb{C}$, we write

$$D(s) := [(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$$

and we let

$$D_n(s) := P_n[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]P_n$$

$$= \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{pmatrix},$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_0, \cdots, e_n\}$,

$$\begin{aligned} q_n &:= u_n + |s|^2 v_n, \quad r_n := s\sqrt{w_n}, \\ u_n &:= \alpha_n^2 - \alpha_{n-1}^2, \\ v_n &:= \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2, \\ w_n &:= \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{aligned}$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all

$n \geq 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then d_n satisfies the following 2–step recursive formula:

$$d_0 = q_0, \quad d_1 = q_0q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2}d_{n+1} - |r_{n+1}|^2 d_n;$$

if we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n+1$. For our purpose we assume $s \in \mathbb{R}$. Then a straightforward calculation shows that if we let $t := s^2$ and $d_n = \sum_{i=0}^{n+1} c(n,i)t^i$, then

$$\begin{aligned}
&d_0(t) = \alpha_0^2 + \alpha_0^2 t;\\
&d_1(t) = (\alpha_0^2 - \alpha_0^4) + c(1,1)t + c(1,2)t^2;\\
&d_2(t) = \alpha_0^2(1-\alpha_0^2)(\alpha_3^2 - 1)t + c(2,2)t^2 + c(2,3)t^3;\\
&d_3(t) = \alpha_0^2(1-\alpha_0^2)(\alpha_3^2 - 1)(\alpha_3^2\alpha_4^2 - 1)t^2 + c(3,3)t^3 + c(3,4)t^4;\\
&d_4(t) = \alpha_0^2\alpha_4^2(\alpha_0^2 - 1)(\alpha_3^2 - 1)^3 t^2 + c(4,3)t^3 + c(4,4)t^4 + c(4,5)t^5,
\end{aligned}$$

so that

$$\lim_{t \to 0+} \frac{d_4(t)}{t^2} = \alpha_0^2\alpha_4^2(\alpha_0^2 - 1)(\alpha_3^2 - 1)^3.$$

Note that $\alpha_0^2 \leq 1 \leq \alpha_3^2$. But since $d_4(t) \geq 0$ for all $t \geq 0$, we must have that either $\alpha_0 = 1$ or $\alpha_3 = 1$. This completes the proof. □

As a corollary of Theorem 1, we can see that two pairs of equal weights force subnormality because the condition "*two* pairs of equal weights" guarantees $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, and in turn, by Theorem 1, subnormality. However, in Theorem 1, note that the condition "$n \geq 1$" cannot be relaxed to the condition "$n \geq 0$". For example, if (cf. [**4**, Proposition 7])

$$\text{(1.1.1)} \qquad \alpha_0 = \alpha_1 = \sqrt{\frac{2}{3}}, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}}\ (n \geq 2),$$

then W_α is quadratically hyponormal but not subnormal.

Also we have:

THEOREM 2. *If W_α is a polynomially hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$ with $\alpha_0 = \alpha_1$, then $\alpha_0 = \alpha_1 = \alpha_2 = \cdots =$.*

Proof. Without loss of generality we may assume $\alpha_0 = \alpha_1 = 1$. We first claim that if W_α is a hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$ and if $\alpha_0 = \alpha_1 = 1$, then

(2.1)

$$W_\alpha \text{ is weakly } k\text{-hyponormal} \Longrightarrow (2-\alpha_{k-1}^2)\,\alpha_k^2 \geq 1 \quad \text{for all } k \geq 3.$$

For (2.1) suppose W_α is weakly k-hyponormal. Then $T_k := W_\alpha + s\,W_\alpha^k$ is hyponormal for every $s \in \mathbb{R}$. For $k \geq 3$ we have

$$D_k := P_k\,[T_k,\,T_k^*]\,P_k = \begin{pmatrix} q_{k,0} & 0 & 0 & \dots & r_{k,0} & 0 \\ 0 & q_{k,1} & 0 & \dots & 0 & r_{k,1} \\ 0 & 0 & q_{k,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{k,0} & 0 & 0 & \dots & q_{k,k-1} & 0 \\ 0 & r_{k,1} & 0 & \dots & 0 & q_{k,k} \end{pmatrix},$$

where

$$q_{k,j} = \begin{cases} (\alpha_j^2 - \alpha_{j-1}^2) + s^2\,(\alpha_{k+j-1}^2\alpha_{k+j-2}^2\cdots\alpha_j^2)\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad (0 \leq j \leq k-1) \\ (\alpha_k^2 - \alpha_{k-1}^2) + s^2\,(\alpha_{2k-1}^2\alpha_{2k-2}^2\cdots\alpha_k^2 - \alpha_{k-1}^2\alpha_{k-2}^2\cdots\alpha_0^2)\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad (j = k) \end{cases};$$

$$r_{k,0} = s\,\alpha_0\alpha_1\cdots\alpha_{k-2}\alpha_{k-1}^2;$$

$$r_{k,1} = s\,\alpha_1\alpha_2\cdots\alpha_{k-1}(\alpha_k^2 - \alpha_0^2).$$

Thus

$$\det D_k = \begin{cases} (q_{k,k}q_{k,1} - r_{k,1}^2)(q_{k,k-1}q_{k,0} - r_{k,0}^2)q_{k,k-2}q_{k,k-3}\cdots q_{k,2} & (k \geq 4) \\ (q_{3,3}q_{3,1} - r_{3,1}^2)(q_{3,2}q_{3,0} - r_{3,0}^2) & (k = 3). \end{cases}$$

If $\alpha_0 = \alpha_1 = 1$ and if we let $t := s^2$ then by a straightforward calculation shows that

$$\lim_{t\to 0+} \frac{\det D_k}{t^k} = (2\alpha_k^2 - \alpha_{k-1}^2\alpha_k^2 - 1)\prod_{j=2}^{k-1}\alpha_j^2(\alpha_j^2 - \alpha_{j-1}^2).$$

Since if W_α is weakly k-hyponormal then $\det D_k \geq 0$, it follows that $(2-\alpha_{k-1}^2)\,\alpha_k^2 - 1 \geq 0$, which proves (2.1). Suppose W_α is polynomially hyponormal. Since $\{\alpha_n\}$ is bounded and increasing it follows that $\lim \alpha_n = \alpha$ for some $\alpha \in \mathbb{C}$. Therefore taking $\lim_n$ on both sides of the inequality in (2.1), we have that $(\alpha-1)^2 \leq 0$, i.e., $\alpha = 1$, which forces $\alpha_0 = \alpha_1 = \alpha_2 = \cdots = 1$. This completes the proof. □

COROLLARY 3. *If W_α is the weighted shift with weight sequence $\{\alpha_n\}$ which is polynomially hyponormal but not subnormal then the weight sequence $\{\alpha_n\}$ is strictly increasing.*

Proof. This immediately follows from Theorem 2. □

Acknowledgement. The author would like to express his thanks to Professor W. Y. Lee and Dr. In Ho Jeon for their encouragements related to the material in this note.

References

[1] A. Athavale, *On joint hyponormality of operators*, Proc. Amer. Math. Soc. **103** (1988), 417–423.

[2] J. Bram, *Subnormal operators*, Duke Math. J. **22** (1955), 75–94.

[3] J. B. Conway, *Subnormal Operators*, Research Notes in Mathematics, vol. 51, Pitman Publ. Co., London, 1981.

[4] R. E. Curto, *Quadratically hyponormal weighted shifts*, Int. Eq. Op. Th. **13** (1990), 49–66.

[5] R. E. Curto and L. A. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem, II*, Int. Eq. Op. Th. **18** (1994), 369–426.

[6] R. E. Curto, P. Muhly and J. Xia, *Hyponormal pairs of commuting operators*, Operator Th. : Adv. Appl. **35** (1988), 1–22.

[7] J. Stampfli, *Which weighted shifts are subnormal*, Pacific J. Math. **17** (1966), 367–379.

DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON 440-746, KOREA