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THE CAUCHY PROBLEM FOR A DEGENERATE PARABOLIC EQUATION WITH ABSORPTION

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ABSTRACT. The Cauchy problem for degenerate parabolic equations with absorption is studied. We prove the existence of a fundamental solution. Also a Harnack type inequality is established and the existence and uniqueness of initial trace for nonnegative solutions is shown.

I. Introduction

In this paper we consider the Cauchy problem for the degenerate parabolic equation

(1.1) $$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - bu^q \quad \text{in} \quad \mathbb{R}^n \times (0,\infty).$$

Here $p > 2$, $b \in [0,1]$ and $q \in (0,1)$ are given constants. A measurable function $u(x,t)$ defined in $\mathbb{R}^n \times (0,\infty)$ is said to be a weak solution of (1.1) if for every bounded open set $\Omega \subset \mathbb{R}^n$

$$u \in C(0,T : L^1(\Omega)) \cap L^p(0,T : W^{1,p}(\Omega))$$

and satisfies

(1.2)
$$\int_\Omega u(x,t_2)\eta(x,t_2)dx + \int_{t_1}^{t_2}\int_\Omega -u\eta_t + |\nabla u|^{p-2}\nabla u \nabla\eta dxdt = \int_\Omega u(x,t_1)\eta(x,t_1)dx - b\int_{t_1}^{t_2}\int_\Omega u^q\eta dxdt$$

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for each bounded interval $[t_1, t_2] \subset (0, \infty)$ and all test functions η such that

$$\eta \in W^{1,\infty}(0,T; L^\infty(\Omega)) \cap L^\infty(0,T; W_0^{1,\infty}(\Omega)).$$

A Radon measure μ on $\mathbb{R}^n$ is called the initial trace of u if μ satisfies

$$\lim_{t\to 0} \int_{R^n} u(x,t)\eta(x)dx = \int_{R^n} \eta\, d\mu$$

for all continuous functions η in $\mathbb{R}^n$ with compact support. We then say that $u(x,t)$ is a weak solution to (1.1) with initial trace μ.

When $p = 2$, Brezis and Friedman[1] showed that (1.1) has a fundamental solution if and only if $q \in (0, (n+2)/n)$. Later Brezis, Peletier and Terman[2] proved the existence of a very singular solution such that

$$\lim_{t\to 0} \int_{R^n} u(x,t)dx = \infty,$$

when $q \in (1, (n+2)/n)$. The evolutionary p-Laplace equation

$$\text{(1.3)} \qquad u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T) \quad p > 2$$

has been studied by many authors [4], [5], [8],$\cdots$. DiBenedetto and Herrero[9] showed that for every σ-finite Borel measure μ in $\mathbb{R}^n$ satisfying

$$\|\mu\|_r = \sup_{\rho \ge r} \rho^{-n-\frac{p}{p-2}} \int_{|x|\le\rho} d\mu(x) < \infty \quad \text{for some} \quad r > 0,$$

there exist a weak solution to (1.3) in $\mathbb{R}^n \times (0,T)$, where T is

$$T(\mu) = c_0[\lim_{r\to\infty} \|\mu\|_r]^{-(p-2)} \quad \text{if} \quad \|\mu\|_r > 0$$

and $T(\mu) = \infty$ if $\|\mu\|_r = 0$. Furthermore, they showed that for each nonnegative weak solution u, there is a unique σ-finite Borel measure μ on $\mathbb{R}^n$ such that

$$\lim_{t\to 0} \int_{R^n} u(x,t)\eta(x)dx = \int_{R^n} \eta d\mu$$

for all continuous and compactly supported functions η in $\mathbb{R}^n$. There they used the existence of an explicit Barenblatt solution whose initial trace is the Dirac measure at the origin. For the p-Laplace equation with absorption,

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q \quad \text{in} \quad \mathbb{R}^n \times (0,T] \quad p > 2.$$

Peletier and Wang[13] showed that when $p-1 < q < p-1+p/n$ there exists a very singular solution. They formulate the general type of equation (1.1) as a system of three first order differential equations, from this they show the desired solution corresponds to an orbit connecting two given curves in the three dimensional solution space. They prove the existence of such an orbit by means of a shooting argument, but this does not work in our case $q \in (0,1)$.

At first we will show the existence of fundamental solution for the case $p \in (0,1)$ and we shall establish the Harnack type estimate and prove the existence of initial traces of weak solutions of (1.1). It follows that there is no very singular solution when $q \in (0,1)$.

The paper will be organized as follows. In section 2 we will state some preliminary results of weak maximum properties and the growth rate of weak solution u in terms of t which are useful for our article.

In section 3 we derive a Harnack inequality namely

$$\int_{B_R(x_0)} u(x,\tau_1)dx \le \beta \left[\left(\frac{R^p T_0}{\tau_2 - \tau_1} \right)^{\frac{1}{p-2}} + \left(\frac{\tau_2 - \tau_1}{R^p T_0} \right)^{\frac{n}{p}} u(x_0, \tau_2)^{\frac{n(p-2)+p}{p}} \right]$$

for some constant β. DiBenedetto and Herrero[9] established a similar type inequality for the evolutionary p-Laplace equation without absorption, but their proof depends on the explicit formula of Barenblatt solution and that is not applicable to our problems. We employ here a compactness method and scaling argument, so the explicit formula of Barenblatt solution is not necessary. From this we can prove the existence and uniqueness of the initial trace of any nonnegative weak solution of (1.1).

The following symbols are used in the sequel;

$$\fint_A u dx = \frac{1}{|A|}\int_A u\,dx,$$
$$B_R(x_0) = \{x : |x - x_0| < R\},$$
$$Q_R(x_0,t_0) = B_R(x_0) \times (t_0 - R^p, t_0),$$
$$S_R(x_0,t_0) = B_R(x_0) \times (t_0 - R^p, t_0 + R^p).$$

If there is no confusion, we drop out (x_0, t_0) in various expressions.

2. Preliminaries

In this section we state some a priori estimates which are useful in studying pointwise behavior of u. Some of the results were established in [5], [12] and we state here without proof. First we state a local maximum estimate. Estimate the interior Lipschitz norm in terms of L^p norm of u by the Moser type iteration. After this, the maximum of u can be estimated by L^1 norm of u.

PROPOSITION 2.1. *Suppose u is a nonnegative weak solution of* (1.1) *in Q_R. Then there exist constants c_1 and c_2 depending on p, n, and R such that*

$$\sup_{Q_{\frac{R}{2}}} u \le c_1 \left[\int_{Q_R} u^p dxdt\right]^{\frac{1}{2}} + c_2.$$

PROPOSITION 2.2. *Let u be a nonnegative weak solution of* (1.1) *in S_{2R}. Then there are constants c, γ and σ depending on p, n such that*

$$\sup_{S_{\frac{R_0}{2}}} u \le c\,[I^\sigma + I^\gamma], \tag{2.1}$$

where $I = \sup_t \int_{|x|<2R_0} u(x,t)dx$.

We denote by S the class of all nonnegative weak solutions of (1.1) in $\mathbb{R}^n \times (0,\infty)$ and we define a subclass $P(N)$ by $P(N) = \{u \in S : \sup_t \int_{R^n} u(x,t)dx \le N\}$.

The next proposition gives the growth rate of weak solution u in terms of t.

PROPOSITION 2.3. *Let $u \in P(N)$, then there is a constant $c(N)$ depending on p, n and N such that*

$$u(x,t) \le c(N) t^{\frac{-n}{n(p-2)+p}} \quad \text{for} \quad 0 < t < 1.$$

The proof of proposition 2.3 can be found in [5] and [12]. As a consequence of proposition 2.3 we obtain following two propositions

PROPOSITION 2.4. *Let $u \in P(N)$, then there exists some positive constant c depending on σ, p and n such that*

$$\int_0^\tau \int_{B_R} |\nabla u|^{p-1} dx dt \le c(N) \tau^{\frac{1}{\kappa}} \tag{2.2}$$

for all $\tau \in (0,1)$, where $\kappa = n(p-2)+p$.

PROPOSITION 2.5. *Let $u \in P(N)$, then there exists some positive constant c depending on p, n, N and R such that*

$$\int_0^\tau \int_{B_R} u^q dx dt \le \begin{cases} c\tau^{1-nq/k} & \text{if} \quad q < \frac{k}{n} \\ c\tau^{\frac{2}{3}} & \text{if} \quad q = \frac{k}{n} \\ c\tau^{1-\frac{nq}{2nq-k}} & \text{if} \quad q > \frac{k}{n} \end{cases}$$

for all $\tau \in (0,1)$, where $\kappa = n(p-2)+p$.

3. Harnack inequality and initial traces

In order to show the existence of an initial trace for nonnegative weak solution of (1.1), we need suitable Harnack type inequality.

For the evolutionary p-Laplace equation without absorption

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

DiBenedetto and Herrero[9] established the following inequality

$$\fint_{B_R} u(x,0)dx \le c\left[\left(\frac{R^p}{T}\right)^{\frac{1}{p-2}} + \left(\frac{T}{R^p}\right)^{\frac{n}{p}} [u(0,T)]^{\frac{n(p-2)+p}{p}}\right]$$

by using explicit formula for Barenblatt solution(see [9]).

Here we employ the idea of Dahlberg and Kenig [6] they use the scaling properties of solutions.

Now we prove compactness theorem.

THEOREM 3.1. *Suppose $u_k \in P(N)$ with $b_k \in [0,1]$ and μ_k is the initial trace of u_k at $t=0$. Suppose also that μ_k converges to μ weakly as $t \to 0$ and that b_k converges to b. Then there exists a subsequence u_k which converges to u uniformly on each compact subset of $\mathbb{R}^n \times (0,\infty)$ to a weak solution u of $(1,1)$ and whose initial trace is μ.*

Proof. Suppose $K \subset \mathbb{R}^n \times (0,\infty)$ is compact. From proposition 2.3 we know that $\{u_k\}$ are uniformly bounded in K and hence u_k are uniformly Hölder continuous(see [7],[10]) and hence $\{u_k\}$ converge to a Hölder continuous function u in K. The fact u is a weak solution follows from weak convergence in $L^p(0,t;W^{1,p}(K))$ and equicontinuity of u_k. Hence it is enough to show that whenever u is locally the uniform limit of $u_k, u \in P(N)$ and u has initial trace μ. Let $0 < \tau < t < 1$ and fix $\eta \in C_0^\infty(\mathbb{R}^n)$. From Proposition 2.4 and Proposition 2.5 that

$$\begin{aligned}
&|\int_{R^n} u(x,t)\eta(x)dx - \int_{R^n} \eta(x)d\mu| \\
&= \lim_{k\to\infty}\left|\int_{R^n} u_k(x,t)\eta(x)dx - \int_{R^n}\eta(x)d\mu_k\right| \\
&= \lim_{k\to\infty}\lim_{\tau\to\infty}\left|\int_{R^n} u_k(x,t)\eta(x)dx - \int_{R^n} u_k(x,\tau)\eta(x)dx\right| \\
&\le c\lim_{k\to\infty}\lim_{\tau\to\infty}\left|\int_\tau^t\int_{R^n} |\nabla u_k|^{p-1}|\nabla\eta| + u_k^q\eta\, dx\, ds\right| \qquad (3.1)\\
&\le c(N, \|\nabla\eta\|_\infty, \|\eta\|_\infty, R)t^{\delta_q}
\end{aligned}$$

for some positive constant δ_q depending on n, p and q, in fact,

$$\delta_q = \begin{cases} \min(\frac{1}{k}, 1 - \frac{nq}{k}) & \text{if} \quad q < \frac{k}{n} \\ \min(\frac{1}{k}, \frac{2}{3}) & \text{if} \quad q = \frac{k}{n} \\ \min(\frac{1}{k}, 1 - \frac{nq}{2nq - k}) & \text{if} \quad q > \frac{k}{n} \end{cases}.$$

So we have

$$\left| \int_{R^n} u(x,t)\eta(x)dx - \int_{R^n} \eta(x)d\mu \right| = O(t^{\delta_q}),$$

and this implies that the initial trace of u is μ. □

The existence of the bounded solutions of (1.1) with nonnegative initial data is well established and also solutions are uniformly bounded on every compact set of $\mathbb{R}^n \times (0, \infty)$. Then by constructing a suitable sequence of weak solution, we can see that there exists a fundamental solution of (1.1).

LEMMA 3.2. *There exists a solution $Q(x,t)$ of $u_t = \text{div}\,(|\nabla u|^{p-2}\nabla u)$ such that the initial trace of Q is Dirac measure at origin. Moreover, there is a constant T_0 depending on p, n such that*

$$Q(0, T_0) \geq \frac{1}{2}.$$

Proof. Let initial data $u_0^k(x)$ of u_k as $u_0^k(x) = \frac{1}{\omega_n}k^n\chi_{B(\frac{1}{k})}$ and $\int_{R^n} u_0^k(x)dx = 1$, where ω is the measure of $B_1(0)$.

Then a nonnegative solution u_k of $u_t = \text{div}(|\nabla u|^{p-2}\nabla u)$ with initial data $u(x,0) = u_0^k(x)$ will exist. So the existence of solution with Dirac measure as initial trace follows as Theorem 3.1. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be nonnegative with

$$\eta(0) = \max_{x \in R^n} = 1, \quad \|\nabla\eta\|_{L^\infty} \leq 1 \text{ and } \int_{R^n} \eta(x)dx = 1.$$

From (3.1) we have for each t

$$\left| \int_{R^n} Q(x,t)\eta(x)dx - \eta(0) \right| < ct^{\delta_q}$$

for some c depending on p and n. Furthermore $u(0,t) = \max_{R^n} u(x,t)$ for each t(see [11]). Thus if $T_0 \le (2c)^{-\delta_q}$, then

$$u(0,T_0) = \int_{R^n} u(0,T_0)\eta(x)dx \ge \int_{R^n} u(x,T_0)\eta(x)dx \ge \frac{1}{2}.$$

The proof is completed. □

We next show

LEMMA 3.3. *Suppose that u is a continuous nonnegative weak solution of* (1.1) *in* $\mathbb{R}^n \times (0,\infty)$. *Define*

$$H(s) = \begin{cases} 1 & \text{if } \ 0 < s \le 1 \\ s^{\frac{k}{p}} & \text{if } \ s > 1, \end{cases} \quad \text{where} \quad k = n(p-2) + p$$

then there is a constant c such that

$$\int_{|x|\le 1} u(x,0)dx \le cH\left(u(0,T_0)\right) \tag{3.2}$$

where T_0 is the constant in Lemma 3.2.

proof. Assume first that the following hypotheses are satisfied

$$\operatorname{supp}\{u(x,0)\} \subset \{x : |x| < 1\} \quad \text{and} \quad \sup_{0<t<\infty} \int u(x,t)\,dx < \infty. \tag{3.3}$$

Suppose (3.2) does not hold, then there exist a sequence of continuous nonnegative solutions u_k with $b_k \in [0,1]$ that

$$(u_k)_t = \operatorname{div}(|\nabla u_k|^{p-2}\nabla u_k) - b_k u_k^q$$

satisfying $I_k \equiv \int_{R^n} u_k(x,0)dx \geq kH\left(u(0,T_0)\right)$ for $k = 1, 2, 3, \cdots$. Let

$$v_k(x,t) = \frac{1}{\gamma_k} u_k(\alpha_k x, t)$$

then $v_k(x,t)$ is solution to

$$(v_k)_t = \frac{\gamma_k^{p-2}}{\alpha_k^p} \operatorname{div}\left(|\nabla v_k|^{p-2} \nabla v_k\right) - b_k \gamma_k^{q-1} v_k^q.$$

We define α_k and γ_k such that

$$\alpha_k^{n+\frac{p}{p-2}} = I_k \quad \text{and} \quad \gamma_k = \alpha_k^{\frac{p}{p-2}}.$$

By the definition of v_k we have

$$\int v_k(x,0)dx = \frac{1}{\gamma_k} \int u_k(\alpha_k x, 0)dx = \frac{1}{\gamma_k \alpha_k^n} \int u_k(x,0)dx = 1$$

moreover supp $\{v_k(x,0)\} \subset \{x; \alpha_k|x| < 1\}$, and $v_k(x,0)$ converges weakly to a Dirac delta measure centered at origin. We also note that $b_k \gamma_k^{q-1} = b_k \alpha_k^{\frac{p(q-1)}{p-2}} = b_k \alpha_k^{\frac{-p(1-q)}{p-2}}$ goes to zero as k goes to infinity. By our compactness argument, $v_k(x,t)$ converges uniformly on each compact subset of $\mathbb{R}^n \times (0,\infty)$ to a weak solution $Q(x,t)$ of

$$v(x,t) = \operatorname{div}(|\nabla v|^{p-2} \nabla v),$$

with initial trace Dirac measure. So we have

$$\lim_{k\to\infty} \alpha_k^{-\frac{p}{p-2}} u_k(0,T_0) = \lim_{k\to\infty} v_k(0,T_0) = Q(0,T_0) > 0$$

by Lemma 3.2, where T_0 denotes the constant in Lemma 3.2.

Since $\lim_{k\to\infty} \alpha_k = 0$ it follows that $\lim_{k\to\infty} u_k(0,T_0) = \infty$, and hence for sufficiently large k, we have

$$[u_k(0,T_0)]^{\frac{k}{p}} = H(u_k(0,T_0)) \leq \frac{1}{k} \alpha_k^{\frac{p}{p-2}\frac{k}{p}}.$$

It follows that $Q(0,T_0) = 0$, which is contradiction. To remove the assumption (3.3), we introduce w_R and $h_R(x)$ such that $h_R \geq 0$ is smooth on $\mathbb{R}^n$ with $h_R(x) = 0$ for $|x| \geq R$ and $0 \leq h_R \leq 1$ and w_R solves the equation

$$(w_R)_t = \operatorname{div}(|\nabla w_R|^{p-2}\nabla w_R) - bw_R^q$$

in $S_R \equiv \{(x,t) : |x| < R, 0 < t < R\}$ with $w_R(x,0) = h_R(x)u(x,0)$ for $|x| \leq R$ and hence $w_R(x,0) = 0$ on $\{|x| \geq R\}$. Thus from the maximum principle we have $w_R \leq w_\rho \leq u$ if $R < \rho$ and $w_R \leq w_\rho$ in S_R. Using compactness argument as in Theorem 3.1, it follows that

$$\lim_{R\to\infty} w_R \in P(N) \quad \text{for some} \quad N > 0.$$

Considering the previous result for w we obtain

$$\int w(x,0)dx = \int h(x)u(x,0)dx \leq cH(w(0,T_0)) \leq cH(u(0,T_0)).$$

Since h_R is an arbitrary smooth function, the lemma follows. □

Now we prove Harnack estimate.

THEOREM 3.4. *Suppose u is a nonnegative weak solution to (1.1). Then there is a constant β and T_0 depending only on n,p,q and n such that*

(3.4)

$$\int_{B_R(x_0)} u(x,\tau_1)dx \leq \beta \left[\left(\frac{R^pT_0}{\tau_2-\tau_1}\right)^{\frac{1}{p-2}} + \left(\frac{\tau_2-\tau_1}{R^pT_0}\right)^{\frac{n}{p}} u(x_0,\tau_2)^{\frac{n(p-2)+p}{p}} \right]$$

where $0 < \tau_1 < \tau_2 < \infty$, and $R \geq [\frac{\tau_2-\tau_1}{T_0}]^{\frac{p-q-1}{p(1-q)}}$.

Proof. Let $\gamma = (\frac{R^pT_0}{\tau_2-\tau_1})^{\frac{1}{p-2}}$ and $v(x,t) = \frac{1}{\gamma}u(x_0 + Rx, \tau_1 + \frac{\tau_2-\tau_1}{T_0}t)$. Then v is solution to

$$v_t(x,t) = \operatorname{div}(|\nabla v|^{p-2}\nabla v) - b\left(\frac{\tau_2-\tau_1}{T_0}\right)^{\frac{p-q-1}{p-2}} R^{\frac{p(q-1)}{p-2}} v^q,$$

and v is continuous on $\mathbb{R}^n \times (0, \infty)$. We note that the coefficient of v^q is less than or equal to 1 under our hypothesis $R \geq [\frac{\tau_2 - \tau_1}{T_0}]^{\frac{p-q-1}{p(1-q)}}$.

Hence from Lemma 3.3 we obtain

$$\begin{aligned}\frac{1}{R^n\gamma}\int_{B_R(x_0)} u(x,\tau_1)dx = \int_{B_1} v(x,0)dx &\leq \beta H(v(0,T_0)) \\ &\leq \beta\left[1 + v(0,T_0)^{\frac{\kappa}{p}}\right] \\ &= \beta\left[1 + (\frac{u(x_0,\tau_2)}{\gamma})^{\frac{\kappa}{p}}\right], \\ &\text{where} \quad \kappa = n(p-2)+p.\end{aligned}$$

Therefore we obtain that

$$\fint_{B_R(x_0)} u(x,\tau_1)dx \leq \beta\left[\left(\frac{R^pT_0}{\tau_2-\tau_1}\right)^{\frac{1}{p-2}} + \left(\frac{\tau_2-\tau_1}{R^pT_0}\right)^{\frac{n}{p}} u(x_0,\tau_2)^{\frac{n(p-2)+p}{p}}\right].$$

The proof is now complete. □

Now we can show the existence of an initial trace for any nonnegative weak solution u in $\mathbb{R}^n \times (0, \infty)$. The Harnack inequality and the compactness argument are the main ingredients in the proof.

THEOREM 3.5. *Suppose that u is a nonnegative weak solution of (1.1). Then there is a unique Radon measure μ on $\mathbb{R}^n$ such that*

$$\lim_{t\to 0}\int_{R^n} u(x,t)\eta(x)\,dx = \int \eta d\mu,$$

for all $\eta(x) \in C_0^\infty(\mathbb{R}^n)$.

Furthermore μ satisfies

$$\sup_{R \geq T_0^{\frac{p-q-1}{p(1-q)}}} R^{-\frac{\kappa}{p-2}} \int_{B_R} d\mu < c(u(0,T))$$

for some constant c.

Proof. As a consequence of our Harnack inequality (Theorem 3.4) we get

$$\sup_t \int_{B_R} u(x,t)dx \le c(T,p,R,u(0,T)) < \infty,$$

for each $t \in (0,T)$. Thus there exists a sequence $t_j \to 0$ and a Radon measure μ on $\mathbb{R}^n$ such that $u(x,t_j)$ converges weakly to μ on $\mathbb{R}^n$. If $\eta \in C_0^\infty(\mathbb{R}^n)$, then for $0<\tau<t$

$$\begin{aligned}\int_{R^n} u(x,t)\eta(x)\,dx - \int_{R^n} u(x,\tau)\eta(x)\,dx \\ = \int_\tau^t \int_{R^n} -|\nabla u|^{p-2}\nabla u \nabla \eta - bu^q \eta\, dx\, ds.\end{aligned} \tag{3.5}$$

From Proposition 2.4 and Proposition 2.5 it follows from (3.5)

$$\left|\int_{R^n} u(x,t)\eta(x)dx - \int_{R^n} u(x,\tau)\eta(x)dx\right| \le c\left[(t-\tau)^{\frac{1}{\kappa}} + (t-\tau)^{\delta_q}\right], \tag{3.6}$$

where δ_q is the power of τ in the statement of Proposition 2.4. Hence taking τ along t_j we get

$$\lim_{t\to 0}\int_{R^n} u(x,t)\eta(x)dx = \int \eta d\mu.$$

Now we assume that there exists another Radon measure ν and a sequence $s_j \to 0$ such that

$$\lim_{j\to\infty}\int_{R^n} u(x,s_j)\eta(x)dx = \int \eta d\nu$$

for all $\eta \in c_0^\infty(\mathbb{R}^n)$. Let ζ be a nonnegative cutoff function in $B_{(1+\varepsilon)R}$ such that $\zeta \equiv 1$ in B_R and $|\nabla\zeta| \le \frac{c}{\varepsilon R}$. Taking ζ as a test function to (1.1) we deduce that

$$\begin{aligned}\int_{B_{(1+\varepsilon)R}} u(x,t)\zeta(x)dx - \int_{B_{(1+\varepsilon)R}} u(x,\tau)\zeta(x)dx \\ = \int_\tau^t \int_{B_{(1+\varepsilon)R}} -|\nabla u|^{p-2}\nabla u\nabla\zeta - bu^q\zeta dxds\end{aligned}$$

and hence

$$\int_{B_{(1+\varepsilon)R}} u(x,t)\zeta(x)dx - \int_{B_R} u(x,\tau)\zeta(x)dx$$
$$\geq -\frac{c}{\varepsilon R}\int_\tau^t \int_{B_{(1+\varepsilon)R}} |\nabla u|^{p-2}|\nabla u| dxds - c(t-\tau)^{\delta_q}$$

where

$$\int_{B_{(1+\varepsilon)R}} u(x,t)\zeta(x)dx \geq \int_{B_R} u(x,\tau)\zeta(x)dx - \frac{c}{\varepsilon R}(t-\tau)^{\frac{1}{\kappa}} - c(t-\tau)^{\delta_q}.$$

Taking $\tau = t_j$ and sending j to infinity, we see that

$$\int_{B_{(1+\varepsilon)R}} u(x,t)\zeta(x)dx \geq \int_{B_R} d\mu - \frac{c}{\varepsilon R}t^{\frac{1}{\kappa}} - ct^{\delta_q}.$$

Then taking t along s_j, we find that

$$\int_{B_{(1+\varepsilon)R}} d\nu \geq \int_{B_R} d\mu \quad \text{for all} \quad R > 0.$$

Since this inequality holds for all R and ε, letting $\varepsilon \to 0$ and interchanging the role of μ and ν we conclude that $\nu = \mu$. This completes the proof. □

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