Bull. Korean Math. Soc. **37** (2000), No. 2, pp. 291–301

HARMONIC MEROMORPHIC STARLIKE FUNCTIONS

JAY M. JAHANGIRI

ABSTRACT. We give sufficient coefficient conditions for a class of meromorphic univalent harmonic functions that are starlike of some order. Furthermore, it is shown that these conditions are also necessary when the coefficients of the analytic part of the function are positive and the coefficients of the co-analytic part of the function are negative. Extreme points, convolution and convex combination conditions for these classes are also determined. Finally, comparable results are given for the convex analogue.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset C$ if both u and v are real harmonic in D. In any simply connected domain we write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D. See [1]. There are numerous papers on univalent harmonic functions defined on the domain $U = \{z : |z| < 1\}$ (see [1], [3], [4], [7], and [8]). In [6], Hengartner and Schober investigated functions harmonic in the exterior of the unit disk $\tilde{U} = \{z : |z| > 1\}$. In particular, they showed that a complex-valued, harmonic, orientation preserving univalent mapping f, defined on $\tilde{U}$ and satisfying $f(\infty) = \infty$, must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A\ log|z|, \tag{1}$$

Received July 7, 1999.
2000 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: harmonic, meromorphic, starlike, and convex functions.
This work was supported by KSU Research Council Grant 1999. Dedicated to KSU Dr. Prof. Richard S. Varga on his seventieth birthday.

where $h(z)=\alpha z+\sum_{n=0}^{\infty}a_nz^{-n}$, $g(z)=\beta z+\sum_{n=1}^{\infty}b_nz^{-n}$, $0\leq|\beta|<|\alpha|$, and $a=\overline{f}_{\bar{z}}/f_z$ is analytic and satisfies $|a(z)|<1$ for $z\in\tilde{U}$. Since the affine transformation

$$\frac{\bar{\alpha}f-\overline{\beta f}-\bar{\alpha}a_o+\overline{\beta a_o}}{|\alpha|^2-|\beta|^2}$$

is again in the class (e.g. see [6]), we may let $\alpha=1$ and $\beta=0$ in the representation (1). We further remove the logarithmic singularity by letting $A=0$ and so focus our attention to the family $\sum_{\mathcal{H}}$ of all harmonic orientation preserving univalent mappings which have the development

$$f(z)=h(z)+\overline{g(z)} \tag{2}$$

where

$$h(z)=z+\sum_{n=1}^{\infty}a_nz^{-n}\ ,\quad g(z)=\sum_{n=1}^{\infty}b_nz^{-n}\ . \tag{3}$$

We call h the analytic part and g the co-analytic part of $f=h+\bar{g}$.

Denote by $\sum_{\mathcal{H}}^*(\gamma)$, $0\leq\gamma<1$, the subclass of $\sum_{\mathcal{H}}$ consisting of functions f of the forms (2) and (3) that are starlike of order γ in $\tilde{U}$.

A necessary and sufficient condition for such f to be in $\sum_{\mathcal{H}}^*(\gamma)$ is (see [2]) that for each z, $|z|=r>1$, we have

$$\frac{\partial}{\partial\theta}arg(f(re^{i\theta}))\geq\gamma,\ 0\leq\gamma<1,\ z=re^{i\theta},\ 0\leq\theta<2\pi,\ r>1. \tag{4}$$

Also denote by $\mathcal{T}_{\mathcal{H}}^*(\gamma)$, $0\leq\gamma<1$ the subfamily of $\sum_{\mathcal{H}}^*(\gamma)$ consisting of functions f of the form (2) for which the functions h and g are restricted by

$$h(z)=z+\sum_{n=1}^{\infty}a_nz^{-n}\ ,\quad g(z)=-\sum_{n=1}^{\infty}b_nz^{-n},\ a_n\geq0,\ b_n\geq0. \tag{5}$$

For $0\leq\gamma_1<\gamma_2<1$, $\sum_{\mathcal{H}}^*(\gamma_2)\subset\sum_{\mathcal{H}}^*(\gamma_1)\subset\sum_{\mathcal{H}}^*(0)$ and $\mathcal{T}_{\mathcal{H}}^*(\gamma_2)\subset\mathcal{T}_{\mathcal{H}}^*(\gamma_1)\subset\mathcal{T}_{\mathcal{H}}^*(0)$. Jahangiri and Silverman [5], among other things, proved the following theorem which we shall use in this paper.

THEOREM A. *Let f be of the forms* (2) *and* (3). *If*

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \le 1 \tag{6}$$

then f is orientation preserving and univalent in $\tilde{U}$.

In [5] it is also proved that the condition (6) is sufficient for f to be in $\sum_{\mathcal{H}}^{*}(0)$. In this paper we extend the above results to the general case $\sum_{\mathcal{H}}^{*}(\gamma)$, $0 \le \gamma < 1$. We also show that our coefficient condition is necessary for f of the form (5) to be in $\mathcal{T}_{\mathcal{H}}^{*}(\gamma)$. Finally, we characterize the extreme points for $\mathcal{T}_{\mathcal{H}}^{*}(\gamma)$ and prove the closure properties under convolution and convex combinations.

2. Coefficient Bounds

THEOREM 1. *Let f be of the forms* (2) *and* (3). *If*

$$\sum_{n=1}^{\infty}[(n+\gamma)|a_n| + (n-\gamma)|b_n|] \le 1-\gamma, \tag{7}$$

then f is harmonic, orientation preserving, univalent in $\tilde{U}$ and $f \in \sum_{\mathcal{H}}^{*}(\gamma)$, $0 \le \gamma < 1$.

Proof. We observe that, by Theorem A, f is harmonic, orientation preserving, and univalent in $\tilde{U}$ since $\sum_{\mathcal{H}}^{*}(\gamma) \subset \sum_{\mathcal{H}}^{*}(0)$ for $0 \le \gamma < 1$. Now it remains to show that the condition (7) is sufficient for f to be in $\sum_{\mathcal{H}}^{*}(\gamma)$. By (4) we must have

$$\begin{aligned}\frac{\partial}{\partial\theta}(arg f(re^{i\theta})) &= Im\frac{\partial}{\partial\theta}(log f(re^{i\theta}))\\ &= Re\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} := Re\frac{A(z)}{B(z)} \ge \gamma.\end{aligned}$$

Or equivalently, we must have

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \ge 0. \tag{8}$$

Substituting for $A(z)$ and $B(z)$ in (8) we obtain

$$
\begin{aligned}
&\left|A(z)+(1-\gamma)B(z)\right|-\left|A(z)-(1+\gamma)B(z)\right|\\
&=\left|(1-\gamma)h(z)+zh'(z)+\overline{(1-\gamma)g(z)-zg'(z)}\right|\\
&\quad-\left|(1+\gamma)h(z)-zh'(z)+\overline{(1+\gamma)g(z)+zg'(z)}\right|\\
&=\Big|(1-\gamma)[z+\sum_{n=1}^{\infty}a_nz^{-n}]+z-\sum_{n=1}^{\infty}na_nz^{-n}\\
&\quad+\overline{(1-\gamma)[\sum_{n=1}^{\infty}b_nz^{-n}]+\sum_{n=1}^{\infty}nb_nz^{-n}}\Big|\\
&\quad-\Big|(1+\gamma)[z+\sum_{n=1}^{\infty}a_nz^{-n}]-z+\sum_{n=1}^{\infty}na_nz^{-n}\\
&\quad+\overline{(1+\gamma)[\sum_{n=1}^{\infty}b_nz^{-n}]-\sum_{n=1}^{\infty}nb_nz^{-n}}\Big|\\
&=\left|(2-\gamma)z+\sum_{n=1}^{\infty}(1-\gamma-n)a_nz^{-n}+\overline{\sum_{n=1}^{\infty}(1-\gamma+n)b_nz^{-n}}\right|\\
&\quad-\left|\gamma z+\sum_{n=1}^{\infty}(1+\gamma+n)a_nz^{-n}+\overline{\sum_{n=1}^{\infty}(1+\gamma-n)b_nz^{-n}}\right|\\
&\geq(2-\gamma)|z|-\sum_{n=1}^{\infty}(n+\gamma-1)|a_n||z|^{-n}-\sum_{n=1}^{\infty}(n-\gamma+1)|b_n||z|^{-n}\\
&\quad-\gamma|z|-\sum_{n=1}^{\infty}(n+\gamma-1)|a_n||z|^{-n}-\sum_{n=1}^{\infty}(n-\gamma+1)|b_n||z|^{-n}\\
&=2(1-\gamma)|z|-\sum_{n=1}^{\infty}2(n+\gamma)|a_n||z|^{-n}-\sum_{n=1}^{\infty}2(n-\gamma)|b_n||z|^{-n}\\
&=2|z|\{1-\gamma-\sum_{n=1}^{\infty}[(n+\gamma)|a_n|+(n-\gamma)|b_n|]|z|^{-n-1}\}
\end{aligned}
$$

$$\geq 2\{1-\gamma-\textstyle\sum_{n=1}^{\infty}[(n+\gamma)|a_n|+(n-\gamma)|b_n|]\}$$
$$\geq 0, \quad by\ (7). \qquad \square$$

We next show that the above sufficient condition for starlikeness is also necessary for functions in $\mathcal{T}^*_{\mathcal{H}}(\gamma)$.

THEOREM 2. *Let $f = h + \bar{g}$ where h and g are of the form (5). A necessary and sufficient condition for f to be in $\mathcal{T}^*_{\mathcal{H}}(\gamma)$ is that*

$$\sum_{n=1}^{\infty}[(n+\gamma)a_n+(n-\gamma)b_n] \leq 1-\gamma. \tag{9}$$

Proof. In view of Theorem 1, we need only show that $f \notin \mathcal{T}^*_{\mathcal{H}}(\gamma)$ if the coefficient inequality (9) does not hold. To this end, we have, if f is starlike of order γ, then

$$\begin{aligned}
&Re\frac{zh'(z)-\overline{zg'(z)}}{h(z)+\overline{g(z)}}-\gamma \\
&= Re\frac{z-\sum_{n=1}^{\infty}na_nz^{-n}-\sum_{n=1}^{\infty}nb_nz^{-n}}{z+\sum_{n=1}^{\infty}a_nz^{-n}-\sum_{n=1}^{\infty}b_nz^{-n}}-\gamma \\
&= Re\frac{z-\sum_{n=1}^{\infty}na_nz^{-n}-\sum_{n=1}^{\infty}nb_nz^{-n}-\gamma z-\sum_{n=1}^{\infty}\gamma a_nz^{-n}}{z+\sum_{n=1}^{\infty}a_nz^{-n}-\sum_{n=1}^{\infty}b_nz^{-n}} \\
&\quad + Re\frac{\sum_{n=1}^{\infty}\gamma b_nz^{-n}}{z+\sum_{n=1}^{\infty}a_nz^{-n}-\sum_{n=1}^{\infty}b_nz^{-n}} \\
&= Re\frac{(1-\gamma)z-\sum_{n=1}^{\infty}(n+\gamma)a_nz^{-n}-\sum_{n=1}^{\infty}(n-\gamma)b_nz^{-n}}{z+\sum_{n=1}^{\infty}a_nz^{-n}-\sum_{n=1}^{\infty}b_nz^{-n}} \\
&\geq 0.
\end{aligned}$$

For $z = r > 1$ the above expression reduces to

$$Re\frac{1-\gamma-\sum_{n=1}^{\infty}[(n+\gamma)a_n+(n-\gamma)b_n]r^{-n-1}}{1+\sum_{n=1}^{\infty}(a_n-b_n)r^{-n-1}} = \frac{A(r)}{B(r)}\geq 0.$$

If the condition (9) does not hold, then $A(r)$ is negative for r sufficiently close to 1. Thus there exists a $z_o = r_o > 1$ for which the quotient $\frac{A(r)}{B(r)}$

is negative. This contradicts the required condition that $\frac{A(r)}{B(r)} \geq 0$, and so the proof is complete. □

We next give a distortion result.

THEOREM 3. *If* $f \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$ *for* $0 \leq \gamma < 1$ *and* $|z| = r > 1$, *then*

$$r - (1-\gamma)r^{-1} \leq |f(z)| \leq r + (1-\gamma)r^{-1}.$$

Proof. Let $f \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$. Taking the absolute value of f we obtain

$$\begin{aligned}
|f(z)| &= |z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \bar{z}^{-n}| \\
&\leq r + \sum_{n=1}^{\infty} (a_n + b_n) r^{-n} \\
&\leq r + \sum_{n=1}^{\infty} (a_n + b_n) r^{-1} \\
&\leq r + \sum_{n=1}^{\infty} [(n+\gamma)a_n + (n-\gamma)b_n] r^{-1} \\
&\leq r + (1-\gamma)r^{-1}\ , \ by\ (9),
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &= |z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \bar{z}^{-n}| \\
&\geq r - \sum_{n=1}^{\infty} (a_n + b_n) r^{-n} \\
&\geq r - \sum_{n=1}^{\infty} (a_n + b_n) r^{-1} \\
&\geq r - \sum_{n=1}^{\infty} [(n+\gamma)a_n + (n-\gamma)b_n] r^{-1} \\
&\geq r - (1-\gamma)r^{-1}\ , \ by\ (9).
\end{aligned}$$

□

3. Extreme Points

For $f = h+\bar{g}$ as in (5) the family $\mathcal{T}^*_{\mathcal{H}}(\gamma)$ is locally uniformly bounded. In this section, we use the coefficient bounds obtained in Section 2 to examine the extreme points for functions in $\mathcal{T}^*_{\mathcal{H}}(\gamma)$ and determine the extreme points of the closed convex hull of $\mathcal{T}^*_{\mathcal{H}}(\gamma)$ denoted by *clco* $\mathcal{T}^*_{\mathcal{H}}(\gamma)$.

THEOREM 4. *$f \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$ if and only if f can be expressed as*

$$f = \sum_{n=0}^{\infty}(x_n h_n + y_n g_n) \tag{10}$$

where $z \in \tilde{U}$, $h_o(z) = z$, $h_n(z) = z + \frac{1-\gamma}{n+\gamma} z^{-n}$, $g_o(z) = z$, $g_n(z) = z - \frac{1-\gamma}{n-\gamma}\bar{z}^{-n}$ $(n = 1, 2, 3, \ ...)$, $\sum_{n=0}^{\infty}(x_n + y_n) = 1$, $x_n \geq 0$, and $y_n \geq 0$.
In particular, the extreme points of clco $\mathcal{T}_{\mathcal{H}^*_o}$ are $\{h_n\}$, $\{g_n\}$, $(n = 0, 1, 2, ...)$.

Proof. Note first that for functions f of the form (10) we may write

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}(x_n h_n + y_n g_n) \\ &= x_o h_o + y_o g_o + \sum_{n=1}^{\infty}[x_n(z + \frac{1-\gamma}{n+\gamma} z^{-n}) + y_n(z - \frac{1-\gamma}{n-\gamma}\bar{z}^{-n})] \\ &= \sum_{n=0}^{\infty}(x_n + y_n)z + \sum_{n=1}^{\infty}\frac{1-\gamma}{n+\gamma} x_n z^{-n} - \sum_{n=1}^{\infty}\frac{1-\gamma}{n-\gamma} y_n \bar{z}^{-n}. \end{aligned}$$

Then, by (9),

$$\begin{aligned} &\sum_{n=1}^{\infty}[(n+\gamma)(\frac{1-\gamma}{n+\gamma}x_n) + (n-\gamma)(\frac{1-\gamma}{n-\gamma}y_n)] \\ &= (1-\gamma)\sum_{n=1}^{\infty}(x_n + y_n) \\ &= (1-\gamma)[1-(x_o + y_o)] \leq 1-\gamma \end{aligned}$$

and so $f \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$.

Conversely, suppose that $f \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$. Then we write $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \bar{z}^{-n}$ where $a_n \geq 0$, $b_n \geq 0$, and $\sum_{n=1}^{\infty} [\frac{n+\gamma}{1-\gamma} a_n + \frac{n-\gamma}{1-\gamma} b_n] \leq 1$. Setting $x_n = \frac{n+\gamma}{1-\gamma} a_n$ and $y_n = \frac{n-\gamma}{1-\gamma} b_n$ $(n = 1, 2, \ldots)$, $0 \leq x_o \leq 1$, and $y_o = 1 - x_o - \sum_{n=1}^{\infty} (x_n + y_n)$, we get $f(z) = \sum_{n=0}^{\infty} (x_n h_n + y_n g_n)$ as required. □

4. Convolution and Convex Combinations

In this section, we show that the class $\mathcal{T}^*_{\mathcal{H}}(\gamma)$ is invariant under convolution and convex combinations of its members.

For harmonic functions $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \bar{z}^{-n}$ and $F(z) = z + \sum_{n=1}^{\infty} A_n z^{-n} - \sum_{n=1}^{\infty} B_n \bar{z}^{-n}$ we define the convolution of f and F as

$$(11) \quad (f * F)(z) = f(z) * F(z) = z + \sum_{n=1}^{\infty} a_n A_n z^{-n} - \sum_{n=1}^{\infty} b_n B_n \bar{z}^{-n}.$$

THEOREM 5. *If f and F belong to $\mathcal{T}^*_{\mathcal{H}}(\gamma)$ so does the convolution function $f * F$.*

Proof. Let $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \bar{z}^{-n}$ and $F(z) = z + \sum_{n=1}^{\infty} A_n z^{-n} - \sum_{n=1}^{\infty} B_n \bar{z}^{-n}$ be in $\mathcal{T}^*_{\mathcal{H}}(\gamma)$. Then the convolution of f and F is given by (11). Note that $A_n \leq 1$ and $B_n \leq 1$ since $F \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$. Therefore we can write

$$\sum_{n=1}^{\infty} [(n+\gamma) a_n A_n + (n-\gamma) b_n B_n] \leq \sum_{n=1}^{\infty} [(n+\gamma) a_n + (n-\gamma) b_n].$$

The right hand side of the above inequality is bounded by $1-\gamma$ because $f \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$. Thus $f * F \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$ by Theorem 2. □

THEOREM 6. *The class $\mathcal{T}^*_{\mathcal{H}}(\gamma)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$ suppose that $f_i(z) \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$ where f_i is given by

$$f_i(z) = z + \sum_{n=1}^{\infty} a_{i_n} z^{-n} - \sum_{n=1}^{\infty} b_{i_n} \bar{z}^{-n}, \ a_{i_n} \geq 0, \ b_{i_n} \geq 0.$$

Then, by (9),

$$\sum_{n=1}^{\infty} [(n+\gamma)a_{i_n} + (n-\gamma)b_{i_n}] \leq 1 - \gamma. \tag{12}$$

For $\sum_{i=1}^{\infty} t_i = 1, \ 0 \leq t_i \leq 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i_n} \right) z^{-n} - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{i_n} \right) \bar{z}^{-n}.$$

Then, by (12),

$$\sum_{n=1}^{\infty} \left[(n+\gamma) \left(\sum_{i=1}^{\infty} t_i a_{i_n} \right) + (n-\gamma) \left(\sum_{i=1}^{\infty} t_i b_{i_n} \right) \right]$$

$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=1}^{\infty} [(n+\gamma)a_{i_n} + (n-\gamma)b_{i_n}] \right\} \leq \sum_{i=1}^{\infty} t_i (1-\gamma) = 1 - \gamma.$$

Thus $\sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{T}^*_{\mathcal{H}}(\gamma)$. □

5. The Convex Case

A function f of the form (2) is said to be convex of order $\gamma, \ 0 \leq \gamma < 1$ in $\tilde{U}$ if for each $z, \ |z| = r > 1$, we have

$$\frac{\partial}{\partial \theta}(arg\{\frac{\partial}{\partial \theta} f(re^{i\theta})\}) \geq \gamma, \ z = re^{i\theta}, \ 0 \leq \theta < 2\pi.$$

One can apply this necessary and sufficient condition for convex functions, much like the characterization for starlike functions led to Theorems 1 through 6. The results for the convex case are found to be similar to those given in Theorems 1 through 6, with an extra "n" added to the coefficient conditions. For completeness, we state an analogous of Theorems 1 and 2.

THEOREM 7. *A sufficient condition for $f = h + \bar{g}$ of the forms* (2) *and* (3) *to be convex of order γ, $0 \le \gamma < 1$, in $\tilde{U}$ is that*

$$\sum_{n=1}^{\infty} n[(n+\gamma)|a_n| + (n-\gamma)|b_n|] \le 1 - \gamma.$$

This condition is also necessary when the coefficients of h and g are restricted by the conditions in (5).

6. A General Case

If we impose no restrictions on α and β in functions f of the form (1) and have only the logarithmic singularity removed, we have

$$f(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n} + \overline{\beta z + \sum_{n=1}^{\infty} b_n z^{-n}}. \tag{13}$$

In this case, we can still use arguments similar to those given to prove Theorems 1 through 7 to obtain analogous results. Here, we only state the results parallel to those of Theorems 1 and 2. The rest follow similarly.

THEOREM 8. *A sufficient condition for functions f of the form* (13) *to be starlike of order γ, $0 \le \gamma \le (|\alpha| - |\beta|)/(|\alpha| + |\beta|)$, is that*

$$\sum_{n=1}^{\infty} [(n+\gamma)|a_n| + (n-\gamma)|b_n|] \le (1-\gamma)|\alpha| - (1+\gamma)|\beta|.$$

This condition is also necessary if $\alpha > \beta \ge 0$, $a_n \ge 0$, and $b_n \le 0$ in (13).

References

[1] J. Clunie and T. Sheil-Small, *Harmonic Univalent Functions*, Ann. Acad. Sci. Fenn. Ser. AI Math. **9** (1984), 3–25.

[2] P. L. Duren, *Univalent Functions*, Springer-Verlag, N.Y., 1983.

[3] J. M. Jahangiri, *Harmonic Functions Starlike in the Unit Disk*, Math. Anal. Appl **235** (1999), 470–477.

[4] ———, *Coefficient Bounds and Univalence Criteria for Harmonic Functions with Negative Coefficients*, Ann. Univ. Marie Curie-Sklodowska Sect. A **52** (1998), no. 2, 57–66.

[5] J. M. Jahangiri and H. Silverman, *Meromorphic Univalent Harmonic Functions with Negative Coefficients*, Bull. Korean Math. Soc. **36** (1999), no. 4, 763–770.

[6] W. Hengartner and G. Schober, *Univalent Harmonic Functions*, Trans. Amer. Math. Soc. **299** (1987), 1–31.

[7] T. Sheil-Small, *Constants for planar harmonic mappings*, J. London Math. Soc. (2) **42** (1990), 237–248.

[8] H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl. **220** (1998), 283–289.

KENT STATE UNIVERSITY, BURTON, OH. 44021, U.S.A.
E-mail: jay@geauga.kent.edu