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RESULTS ON THE RANGE OF DERIVATIONS

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ABSTRACT. Let D be a derivation on a Banach algebra A. Suppose that $[[D(x),x],D(x)]$ lies in the nil radical of A for all $x \in A$. Then $D(A)$ is contained in the Jacobson radical of A.

1. Introduction

Throughout this paper A will represent an algebra over a complex field $\mathbb{C}$. The Jacobson radical of A and the nil radical of A will be denoted by $rad(A)$ and $nil(A)$, respectively. We also write $[x,y]$ for the commutator $xy - yx$. Let I be any closed (2-sided) ideal of a Banach algebra A. Then we will let Q_I denote the canonical quotient map from A onto A/I. Recall that an algebra A is prime if $aAb = \{0\}$ implies that either $a = 0$ or $b = 0$. A linear mapping D from A to A is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in A$.

The Singer-Wermer theorem ([6]) states that every continuous derivation on a commutative Banach algebra has its image in the Jacobson radical. They conjectured that the theorem remains true without assuming the continuity of a derivation in the same paper, which is called the Singer-Wermer conjecture. In 1988 Thomas proved the conjecture ([7]). The so-called noncommutative Singer-Wermer conjecture states that every derivation D on a Banach algebra A such that $[D(x),x] \in rad(A)$ for all $x \in A$ has its image in rad(A). As an evidence for the validity of the conjecture, Mathieu showed that the above conclusion holds if the condition $[D(x),x] \in rad(A)$ for all $x \in A$ is replaced by the condition $[D(x),x] \in nil(A)$ for all $x \in A$ ([4, Theorem 1]). It is the purpose of this paper to show that the condition

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$[[D(x), x], D(x)] \in nil(A)$ for all $x \in A$ also guarantees the range inclusion of a derivation D.

2. Results

For the proof of our main theorem, we need the algebraic ingredient below.

LEMMA 2.1. *Let D be a derivation on a noncommutative prime algebra A. Suppose that $[[D(x), x], D(x)] = 0$ for all $x \in A$. Then $D = 0$ on A.*

In the proof of Lemma 2.1 we will use the next result: A nonzero derivation D of a noncommutative prime algebra A cannot satisfy either $[D(x), x]^2 = 0$ for all $x \in A$ (see the proof in [1, Theorem 3]) or $[D(x), x]D(x) = 0$ for all $x \in A$ ([9, Lemma]).

Proof of Lemma 2.1. Suppose that

$$[[D(x), x], D(x)] = 0 \tag{1}$$

holds for all $x \in A$. The linearization of (1) leads to

$$\begin{aligned} 0 =& 2D(x)yD(x) + 2D(y)xD(x) + 2D(y)yD(x) - xD(y)D(x) \\ &- yD(x)^2 - yD(y)D(x) - D(x)^2y - D(x)D(y)x \\ &- D(x)D(y)y + 2D(x)xD(y) + 2D(x)yD(y) + 2D(y)xD(y) \\ &- xD(x)D(y) - xD(y)^2 - yD(x)D(y) - D(y)D(x)x \\ &- D(y)D(x)y - D(y)^2x, \quad x, y \in A. \end{aligned} \tag{2}$$

Substituting $-y$ for y in (2) we obtain by comparing this new result with (2) that

$$\begin{aligned} &D(x)^2y + D(x)D(y)x + D(y)D(x)x - 2D(x)xD(y) \\ &- 2D(x)yD(x) - 2D(y)xD(x) + xD(x)D(y) + xD(y)D(x) \\ &+ yD(x)^2 = 0, \quad x, y \in A. \end{aligned} \tag{3}$$

Taking xy for y in (3) we then get

$$D(x)^2xy + D(x)xD(y)x + D(x)^2yx + xD(y)D(x)x \\ + D(x)yD(x)x - 2D(x)x^2D(y) - 2D(x)xD(x)y - 2D(x)xyD(x) \\ - 2xD(y)xD(x) - 2D(x)yxD(x) + xD(x)xD(y) + xD(x)^2y \\ + x^2D(y)D(x) + xD(x)yD(x) + xyD(x)^2 = 0, \quad x, y \in A.$$

In view of (1) this relation can be rewritten as

$$\begin{aligned} &D(x)xD(y)x + D(x)^2yx + xD(y)D(x)x + D(x)yD(x)x \\ &- 2D(x)x^2D(y) - 2D(x)xyD(x) - 2xD(y)xD(x) \\ &- 2D(x)yxD(x) + xD(x)xD(y) + x^2D(y)D(x) + xD(x)yD(x) \\ &+ xyD(x)^2 = 0, \quad x, y \in A. \end{aligned} \tag{4}$$

The multiplication of (3) from the left by x gives

$$\begin{aligned} &xD(x)^2y + xD(x)D(y)x + xD(y)D(x)x - 2xD(x)xD(y) \\ &- 2xD(x)yD(x) - 2xD(y)xD(x) + x^2D(x)D(y) + x^2D(y)D(x) \\ &+ xyD(x)^2 = 0, \quad x, y \in A. \end{aligned} \tag{5}$$

Subtracting (5) from (4), we obtain

$$\begin{aligned} &- xD(x)^2y + [D(x), x]D(y)x + D(x)^2yx \\ &+ D(x)y(D(x)x - 2xD(x)) + (x[D(x), x] - 2[D(x), x]x)D(y) \\ &+ (3xD(x) - 2D(x)x)yD(x) = 0, \quad x, y \in A. \end{aligned} \tag{6}$$

Replacing y by yx in (6), we arrive at

$$\begin{aligned} &- xD(x)^2yx + [D(x), x]D(y)x^2 + [D(x), x]yD(x)x + D(x)^2yx^2 \\ &+ D(x)y(xD(x)x - 2x^2D(x)) + (x[D(x), x] - 2[D(x), x]x)D(y)x \\ &+ (x[D(x), x] - 2[D(x), x]x)yD(x) + (3xD(x) - 2D(x)x)yxD(x) \\ &= 0, \quad x, y \in A. \end{aligned} \tag{7}$$

The multiplication of (6) from the right by x gives

$$\begin{aligned}
&-xD(x)^2yx+[D(x),x]D(y)x^2+D(x)^2yx^2\\
&+D(x)y(D(x)x^2-2xD(x)x)+(x[D(x),x]-2[D(x),x]x)D(y)x \qquad (8)\\
&+(3xD(x)-2D(x)x)yD(x)x=0, \quad x,y\in A.
\end{aligned}$$

Subtracting (8) from (7), we obtain

$$\begin{aligned}
&[D(x),x]yD(x)x+D(x)y(3xD(x)x-D(x)x^2\\
&-2x^2D(x))+(x[D(x),x]-2[D(x),x]x)yD(x) \qquad (9)\\
&+(2D(x)x-3xD(x))y[D(x),x]=0, \quad x,y\in A.
\end{aligned}$$

Putting $D(x)y$ instead of y in (9), we have

$$\begin{aligned}
&[D(x),x]D(x)yD(x)x+D(x)^2y(3xD(x)x-D(x)x^2\\
&-2x^2D(x))+(x[D(x),x]-2[D(x),x]x)D(x)yD(x) \qquad (10)\\
&+(2D(x)x-3xD(x))D(x)y[D(x),x]=0, \quad x,y\in A.
\end{aligned}$$

The multiplication of (9) from the left by $D(x)$ gives

$$\begin{aligned}
&D(x)[D(x),x]yD(x)x+D(x)^2y(3xD(x)x-D(x)x^2\\
&-2x^2D(x))+D(x)(x[D(x),x]-2[D(x),x]x)yD(x) \qquad (11)\\
&+D(x)(2D(x)x-3xD(x))y[D(x),x]=0, \quad x,y\in A.
\end{aligned}$$

Subtracting (11) from (10), we obtain

$$\begin{aligned}
&[[D(x),x],D(x)]yD(x)x+[x[D(x),x]-2[D(x),x]x,D(x)] \qquad (12)\\
&\cdot yD(x)+[2D(x)x-3xD(x),D(x)]y[D(x),x]=0, \quad x,y\in A.
\end{aligned}$$

Calculating the relation (12) according to (1), we have

$$[D(x),x]^2yD(x)+[D(x),x]D(x)y[D(x),x]=0, \quad x,y\in A. \qquad (13)$$

Substituting $y[D(x),x]$ for y in (13), we arrive at

(14)

$$[D(x),x]^2y[D(x),x]D(x)+[D(x),x]D(x)y[D(x),x]^2=0, \quad x,y\in A.$$

Replacing y by $y[D(x),x]^2z$, we get

$$(15)\qquad \begin{aligned}&[D(x),x]^2y[D(x),x]^2z[D(x),x]D(x)\\ &+[D(x),x]D(x)y[D(x),x]^2z[D(x),x]^2=0,\quad x,y,z\in A.\end{aligned}$$

By (14), since

$$[D(x),x]^2z[D(x),x]D(x)=-[D(x),x]D(x)z[D(x),x]^2$$

and

$$[D(x),x]D(x)y[D(x),x]^2=-[D(x),x]^2y[D(x),x]D(x),$$

the relation (15) implies that

$$[D(x),x]^2y[D(x),x]D(x)z[D(x),x]^2=0,\quad x,y,z\in A.$$

Since A is prime, it follows that for any $x\in A$, either $[D(x),x]^2=0$ or $[D(x),x]D(x)=0$. Thus A is the union of its subsets $B=\{x\in A:[D(x),x]^2=0\}$ and $C=\{x\in A:[D(x),x]D(x)=0\}$. Suppose that $D\neq 0$. The two results in [1, Theorem 3] and [9, Lemma], respectively, then tell us that $B\neq A$ and $C\neq A$. Hence there exist $x,y\in A$ such that $x\notin B$ and $y\notin C$. Thus $x\in C$ and $y\in B$. Now consider $x+\lambda y$, $\lambda\in\mathbb{C}$. Then we see that either $x+\lambda y\in B$ or $x+\lambda y\in C$. If $x+\lambda y\in B$, then we have

$$(16)\qquad \begin{aligned}&[D(x),x]^2\\ &+\lambda\{[D(x),x][D(x),y]+[D(x),y][D(x),x]\\ &+[D(x),x][D(y),x]+[D(y),x][D(x),x]\}\\ &+\lambda^2\{[D(x),y]^2+[D(x),x][D(y),y]\\ &+[D(x),y][D(y),x]+[D(y),x][D(x),y]\\ &+[D(y),y][D(x),x]+[D(y),x]^2\}+\lambda^3\{[D(x),y][D(y),y]\\ &+[D(y),y][D(x),y]+[D(y),x][D(y),y]\\ &+[D(y),y][D(y),x]\}=0.\end{aligned}$$

Also, if $x + \lambda y \in C$, then we get

$$\begin{aligned} &\lambda\{[D(x),x]D(y) + [D(x),y]D(x) + [D(y),x]D(x)\} \\ (17)\qquad &+ \lambda^2\{[D(x),y]D(y) + [D(y),x]D(y) + [D(y),y]D(x)\} \\ &+ \lambda^3[D(y),y]D(y) = 0. \end{aligned}$$

Therefore, for every $\lambda \in \mathbb{C}$ one of these two possibilities holds. But either (16) has more than three solutions or (17) has more than three solutions. And this contradicts the assumption that $[D(x),x]^2 \neq 0$ and $[D(y),y]D(y) \neq 0$. This proves the theorem. □

The following lemma can be referred in [8, Lemma 1.2].

LEMMA 2.2. *Let D be a derivation on a Banach algebra A and J a primitive ideal of A. If there exists a real constant $K > 0$ such that $\|Q_J D^n\| \leq K^n$ for all $n \in \mathbb{N}$, then $D(J) \subseteq J$.*

Now we prove our main theorem.

THEOREM 2.3. *Let D be a derivation on a Banach algebra A. Suppose that $[[D(x),x],D(x)] \in nil(A)$ for all $x \in A$. Then $D(A) \subseteq rad(A)$.*

Proof. Let J be any primitive ideal of A. Using Zorn's lemma, we find a minimal prime ideal P contained in J, and hence $D(P) \subseteq P$. Suppose first that P is closed. Then a derivation D on A induces a derivation $\bar{D}$ on a Banach algebra A/P defined by $\bar{D}(x+P) = D(x) + P$ $(x \in A)$. In case A/P is commutative, $\bar{D}(A/P)$ is contained in the Jacobson radical of A/P by [7]. In case A/P is noncommutative, Lemma 2.1 implies that $\bar{D} = 0$ on A/P since A/P is prime and $[[\bar{D}(x+P), x+P], \bar{D}(x+P)] = P$ for all $x \in A$. In both cases, $\bar{D}(A/P) \subseteq J/P$. Consequently we see that $D(A) \subseteq J$. If P is not closed, then we see that $S(D) \subset P$ by [2, Lemma 2.3] (where $S(T)$ is the separating space of a linear operator T). Then we have, by [5, Lemma 1.3], $S(Q_{\bar{P}}D) = \overline{Q_{\bar{P}}(S(D))} = \{0\}$ whence $Q_{\bar{P}}D$ is continuous. As a result, $Q_{\bar{P}}D(\bar{P}) = \{0\}$ on $A/\bar{P}$, that is, $D(\bar{P}) \subseteq \bar{P}$. Hence, from a derivation D on A, we can also induce a continuous derivation $\widetilde{D}$ on a Banach algebra $A/\bar{P}$ defined by $\widetilde{D}(x+\bar{P}) = D(x) + \bar{P}$ $(x \in A)$. This shows that we can define a map

$$\Phi\widetilde{D}^n Q_{\bar{P}} : A \to A/\bar{P} \to A/\bar{P} \to A/J$$

by $\Phi\widetilde{D}^n Q_{\bar{P}}(x) = Q_J D^n(x)$ $(x \in A,\ n \in \mathbb{N})$, where Φ is the canonical inclusion map from $A/\bar{P}$ onto A/J (which exists since $\bar{P} \subseteq J$). We therefore conclude that $\|Q_J D^n\| \leq \|\widetilde{D}\|^n$ for all $n \in \mathbb{N}$ since the other maps are norm depressing. By Lemma 2.2, we see that $D(J) \subseteq J$. Then a derivation D on A induces a derivation $\widehat{D}$ on a Banach algebra A/J defined by $\widehat{D}(x + J) = D(x) + J$ $(x \in A)$. The remainder follows the similar argument to the case P is closed since the primitive algebra A/J is prime. So we obtain that $D(A) \subseteq J$. It follows that $D(A) \subseteq J$ for every primitive ideal J, that is, $D(A) \subseteq rad(A)$. We complete the proof. □

REMARK. Combining the techniques used in Theorem 2.3 with a paper by Lanski ([3]), we see that Mathieu's condition $[D(x), x] \in nil(A)$ for all $x \in A$ can be replaced by the condition of some higher commutator

$$[[\cdots[[D(x), x], x]\cdots, x] \in nil(A)$$

for all $x \in A$.

COROLLARY 2.4. *Let D be a derivation on a semisimple Banach algebra A. Suppose that $[[D(x), x], D(x)] = 0$ for all $x \in A$. Then $D = 0$ on A.*

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