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ON RIGHT QUASI-DUO RINGS WHICH ARE Π-REGULAR

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ABSTRACT. This paper is motivated by the results in [2], [10], [13] and [19]. We study some properties of generalizations of commutative rings and relations between them. We also show that for a right quasi-duo right weakly π-regular ring R, R is an $(S,2)$-ring if and only if every idempotent in R is a sum of two units in R, which gives a generalization of [2, Theorem 4] on right quasi-duo rings. Moreover we find a condition which is equivalent to the strongly π-regularity of an abelian right quasi-duo ring.

1. Introduction

Throughout this paper, all rings are associative ones with identity. The prime radical of a ring R and the set of all nilpotent elements in R and the Jacobson radical of R are denoted by $P(R)$, $N(R)$ and $J(R)$, respectively. A ring R is called *right (left) duo* if every right (left) ideal of R is a two-sided ideal. A ring is called *duo* if it is both right and left duo. A ring R is called *weakly right (left) duo* if for each a in R there exists a positive integer $n = n(a)$, depending on a, such that a^nR (Ra^n) is two-sided. A ring R is called *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided. Commutative rings are clearly right and left duo. The study of quasi-duo rings was initiated by Yu in [19], related to the Bass' conjecture in [3]. Right duo rings are obviously

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weakly right duo, and weakly right duo rings are right quasi-duo by [19, Proposition 2.2]. The n by n upper triangular matrix rings over right quasi-duo rings are also right quasi-duo by [19, Proposition 2.1]; but the n by n full matrix rings over right quasi-duo rings are not right quasi-duo when $n \geq 2$. An element a of a ring is called *nilpotent* if $a^n = 0$ for some positive integer n. A subset I of a ring is called *nil* if each element of I is nilpotent. A ring R is called *2-primal* if $P(R) = N(R)$, or equivalently, if $R/P(R)$ is a reduced ring (i.e., a ring without nonzero nilpotent elements). Hirano showed in [9] that given a 2-primal ring R, R is strongly π-regular if and only if $Mat_n(R)$ is strongly π-regular for $n = 1, 2, \ldots$, where $Mat_n(R)$ is the n by n full matrix ring over R. Shin ([16]) proved that for a ring R, R is 2-primal if and only if every minimal prime ideal of R is completely prime. Also Sun ([17]) introduced a condition called *weakly symmetric*, which is equivalent to the 2-primal condition for rings. He showed in [17] that if a ring R is weakly symmetric then each minimal prime ideal of R is completely prime, and that the n by n upper triangular matrix ring over R is also weakly symmetric for each n with $n = 1, 2, \ldots$. Commutative rings are obviously 2-primal and the n by n upper triangular matrix rings over 2-primal rings are also 2-primal by Sun ([17]); but the n by n full matrix rings over 2-primal rings are not 2-primal when $n \geq 2$. Recall that the prime radical is the set of all strongly nilpotent elements (hence it is nil), so reduced rings are clearly 2-primal. A ring R is called an *(S,1)-ring* if for $a, b \in R$, $ab = 0$ implies $aRb = 0$, or equivalently, the right annihilator of a is a two-sided ideal of R for each $a \in R$.

2. Preliminaries

This section concerns some properties of generalizations of commutative rings for our main results in this paper. An element e of a ring is called an *idempotent* if $e^2 = e$. A ring is called *abelian* if every idempotent of it is central.

PROPOSITION 1. *For a ring R we have the following assertions:*

(1) *R is duo if and only if $aR = Ra$ for each element $a \in R$ if and only if for $a, b \in R$ there exist $x, y \in R$ such that $ab = xa$ and $ba = ay$.*

(2) *If R is right (or left) duo then R is an $(S, 1)$-ring.*

(3) *If R is reduced then R is an $(S,1)$-ring.*
(4) *If R is an $(S,1)$-ring then R is 2-primal.*
(5) *If R is an $(S,1)$-ring then R is abelian.*

Proof. (1) By the definition. (2) Since R is right duo, $Rb \subseteq bR$ for $b \in R$; hence $ab = 0$ implies that $aRb = 0$. The proof of the case of left duo is similar. (3) Let $a, b \in R$ such that $ab = 0$. Then since R is reduced, $ba = 0$ and so $aRb = 0$. (4) Let $x \in N(R)$. Then $x^n = 0$ for some positive integer n. Since R is an $(S,1)$-ring, $xRx^{n-1} = 0$ and $xRxRx^{n-2} = 0$; hence inductively we can get $xR_1xR_2x \cdots xR_{n-1}x = 0$ with $R_i = R$ for all i. Consequently, x is strongly nilpotent and therefore $P(R) = N(R)$ because $P(R)$ is the set of all strongly nilpotent elements in R and $P(R) \subseteq N(R)$. Thus R is 2-primal. (5) Let $0 \neq e = e^2 \in R$. Then $eR(1-e) = 0 = (1-e)Re$ because R is an $(S,1)$-ring. So for each $r \in R$, $er(1-e) = 0 = (1-e)re$ implies that e is central. So R is abelian. □

By Proposition 1, right duo rings are $(S,1)$-rings and so we may conjecture that right quasi-duo rings are $(S,1)$-rings. However by the following example there is a right quasi-duo ring which is not 2-primal (hence not an $(S,1)$-ring by Proposition 1).

EXAMPLE 1. We use the ring R in [15, Example 1.1]. Let F be a field and let V be an infinite dimensional left vector space over F with $\{v_1, v_2, \dots\}$ a basis. For the endomorphism ring $A = End_F(V)$, define $A_1 = \{f \in A \mid \text{rank}(f) < \infty \text{ and } f(v_i) = a_1v_1 + \cdots + a_iv_i \text{ for } i = 1, 2, \dots \text{ with } a_j \in F\}$ and let R be the F-subalgebra of A generated by A_1 and 1_A. Let M be a maximal right ideal of R. Then M is of the form $M = \{r \in R \mid (i,i) - \text{entry of } r \text{ is } zero\}$ for some $i \in \{1, 2, \dots\}$. But M is also a two-sided ideal of R and so R is right quasi-duo. By [14, Proposition 6] the formal power series ring $R[[x]]$ over R is also right quasi-duo. However $R[[x]]$ is not 2-primal by the argument in [15, Example 1.1] and hence $R[[x]]$ is not an $(S,1)$-ring.

Notice that Example 3.3 in [5] is another counterexample to the preceding conjecture. The ring $R[[x]]$ in Example 1 is not semiprimitive. We have an affirmative answer to the preceding conjecture when given a ring is semiprimitive.

PROPOSITION 2. *Semiprimitive right (or left) quasi-duo rings are reduced (hence an $(S,1)$-ring).*

Proof. A semiprimitive right (or left) quasi-duo ring is isomorphic to a subdirect product of division rings by [14, Corollary 3]; hence it is reduced. Then it is an $(S,1)$-ring by Proposition 1. $\square$

Also the converses of (2), (3) and (4) in Proposition 1 fail in general as in the following examples.

EXAMPLE 2. The first Weyl algebra $W_1[F]$ over a field F of characteristic zero is the polynomial ring with indeterminates x, y and relation $yx = xy + 1$. Then it is reduced and so an $(S,1)$-ring by Proposition 1; but it is neither right nor left duo.

EXAMPLE 3. For a domain D, the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in Mat_2(D) \mid a, b \in D \right\}$ is an $(S,1)$-ring but not reduced.

EXAMPLE 4. For a 2-primal ring R, the ring $A = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ cannot be an $(S,1)$-ring because $(S,1)$-rings are abelian by Proposition 1; but A is 2-primal by [4, Proposition 2.5].

A ring R is called *von Neumann regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$. A ring R is called *unit-regular* if for each $a \in R$ there exists a unit $u \in R$ such that $a = aua$. A ring R is called *abelian regular* (or, *strongly regular*) if R is von Neumann regular and abelian. A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer $n = n(a)$, depending on a, and $x \in R$ such that $a^n = a^n x a^n$. A ring R is called *strongly π-regular* if for each $a \in R$ there exist a positive integer $m = m(a)$, depending on a, such that $a^m R = a^{m+1} R$. By Azumaya ([1]), strongly π-regular rings are π-regular. Also by Dischinger ([6]) the strongly π-regularity is left-right symmetric. Left or right perfect rings and algebraic algebras over fields are strongly π-regular rings. For a division ring D and a right D-module V, notice that the endomorphism ring of V over D is strongly π-regular if and only if V is finite dimensional over D. A ring R is called *right* (*left*) *weakly π-regular* if for each a in R there exists a positive integer $n = n(a)$, depending on a, such that $a^n \in a^nRa^nR$ ($a^n \in Ra^nRa^n$.) A ring R is called *weakly π-regular* if it is both left and right weakly π-regular. A π-regular ring is clearly weakly π-regular.

PROPOSITION 3. *Let R be a von Neumann regular ring. Then the following statements are equivalent:*
(1) *R is a right duo ring.*
(2) *R is a weakly right duo ring.*
(3) *R is a right quasi-duo ring.*
(4) *R is a reduced ring.*
(5) *R is an $(S,1)$-ring.*
(6) *R is a 2-primal ring.*

Proof. (1)⇒(2) is straightforward, (2)⇒(3) is by [19, Proposition 2.2] and (4)⇒(5)⇒(6) is obtained by Proposition 1. Semiprimitive right quasi-duo rings are reduced by Proposition 2, and so we get (3)⇒(4) because von Neumann regular rings are seimprimitive. Semiprimitive (hence semiprime) 2-primal rings are reduced and so 2-primal von Neumann regular rings are strongly regular; hence we obtain (6)⇒(1) by [8, Theorem 3.2]. □

REMARK. We may obtain the same result for the left version by replacing "right" by "left" in the preceding proposition and its proof.

3. Related Results

The first main result in this paper is the generalizing [2, Theorem 4] on right quasi-duo rings. For doing it, we need more details.

LEMMA 4 ([13, Lemma 4]). *Suppose that a ring R is reduced. Then following statements are equivalent:*
(1) *R is strongly π-regular.*
(2) *R is π-regular.*
(3) *Every prime factor ring of R is a division ring.*
(4) *R is a von Neumann regular ring.*

LEMMA 5. *The Jacobson radicals of right weakly π-regular rings are nil.*

Proof. Let R be a right weakly π-regular ring and $r \in J(R)$. Then since R is right weakly π-regular, $r^n R = r^n R r^n R$ for some positive integer n and so $r^n(1-s) = 0$ for some $s \in Rr^n R$. But $s \in J(R)$ because $r \in J(R)$; hence $1-s$ is invertible and we get $r^n = 0$. □

LEMMA 6. *Let R be a right quasi-duo ring. If R is right weakly π-regular then $R/J(R)$ is von Neumann regular.*

Proof. Since R is right quasi-duo, $R/J(R)$ is reduced by Proposition 2 and so $R/J(R)$ is reduced right weakly π-regular. But $R/J(R)$ is also right quasi-duo by [11, Lemma 7] and hence $R/J(R)$ is reduced strongly π-regular by [10, Proposition 4]. Therefore $R/J(R)$ is von Neumann regular by Lemma 4. □

With these preparations, we may now get the following theorem which is one of our goals, and as a byproduct this gives us a condition under which right quasi-duo rings are $(S, 2)$-rings.

NOTE. For a commutative π-regular ring R, if $2 = 1_R + 1_R$ is a unit (i.e., invertible element) in R then R is an $(S, 2)$-ring. As well-known facts, there are several generalizations of commutativity, for examples, PI-rings, 2-primal rings and quasi-duo rings, and so on. By Fisher-Snider ([7]), the preceding argument is also true for PI-rings. As another generalizations, Badawi ([2]) proved that the result holds for duo rings. We will get the result as a corollary of our main result on right quasi-duo rings, using the method of proof in [2] in part. One may compare this result with [13, Corollary 8]. Moreover we obtain some results on right quasi-duo rings that are similar ones on 2-primal rings in [13].

A ring R is called an *(S,2)-ring* if every element in R is a sum of two units in R. Notice that $\mathbb{Z}_2$, the integers modulo 2, is clearly a right quasi-duo ring; but it is not an $(S, 2)$-ring because 1 cannot be a sum of two units.

THEOREM 7. *For a right quasi-duo right weakly π-regular ring R, R is an $(S, 2)$-ring if and only if every idempotent in R is a sum of two units in R.*

Proof. ($\Rightarrow$) Straightforward. ($\Leftarrow$) Let $x \in R$. Since R is right quasi-duo right weakly π-regular, $R/J(R)$ is von Neumann regular by Lemma 6, and thus $R/J(R)$ is abelian regular because $R/J(R)$ is reduced by Proposition 2; hence $R/J(R)$ is unit-regular by [8, Corollary 4.2]. Then there exists a unit $u + J(R) \in R/J(R)$ such that $x + J(R) = (x + J(R))(u + J(R))(x + J(R))$. In this situation, we can take u so that u is a unit in R. Note that $xu + J(R)$ is an idempotent in $R/J(R)$. But $J(R)$ is a nil ideal of R by Lemma 5, and thus there exists an idempotent $f \in R$ such that $xu + J(R) = f + J(R)$ by [12, Proposition 1, p. 72]. Then $x+J(R) = (xu+J(R))(u^{-1}+J(R)) = (f+J(R))(u^{-1}+J(R)) = fu^{-1}+J(R)$, and hence $x = fu^{-1}+j$ for some $j \in J(R)$, where $u^{-1} \in R$ is the multiplicative inverse of u. Now by hypothesis $f = v+w$ for some units v, w in R and then $x = vu^{-1} + wu^{-1} + j$. Note that $wu^{-1} + j$ is a unit in R. So x is a sum of two units in R, and therefore R is an $(S, 2)$-ring. □

A π-regular ring is right weakly π-regular and so we obtain the following which is a generalization of [2, Theorem 4] on right quasi-duo rings.

COROLLARY 8. *For a right quasi-duo π-regular ring R, R is an $(S, 2)$-ring if and only if every idempotent in R is a sum of two units in R.*

For a ring R, suppose that $2 = 1_R + 1_R$ is a unit in R. Take $e^2 = e \in R$, then $(1-2e)(1-2e) = 1$ and so $1-2e$ is a unit. Put $1-2e = u \in R$, then $e = 2^{-1} - 2^{-1}u$ where 2^{-1} is the multiplicative inverse of 2. Thus e is a sum of two units of R. So we obtain the following corollaries by Theorem 7.

COROLLARY 9. *For a right quasi-duo ring R, if R is right weakly π-regular with $2 = 1_R + 1_R$ a unit in R then R is an $(S, 2)$-ring.*

By the definition, right duo rings are right quasi-duo and so we get the following result.

COROLLARY 10. *For a right duo ring R, if R is right weakly π-regular with $2 = 1_R + 1_R$ a unit in R then R is an $(S, 2)$-ring.*

COROLLARY 11 ([2, Corollary 2]). *For a duo ring R, if R is π-regular with $2 = 1_R + 1_R$ a unit in R then R is an $(S, 2)$-ring.*

The following example shows that there exist right quasi-duo $(S, 2)$-rings which are neither right nor left duo.

EXAMPLE 5. Let R be an n by n upper triangular matrix ring over a field whose charateristic is not 2. Then R is right quasi-duo by [19, Proposition 2.1], 2-primal by [4, Proposition 2.5] and π-regular by [13, Lemma 5]; hence R is an $(S, 2)$-ring by Corollary 9. However R is neither left nor right duo.

Using the methods in the proof of Theorem 7 again, we also obtain the following fact.

LEMMA 12. *For a right quasi-duo ring R, if R is right weakly π-regular then for each $x \in R$ there exist an idempotent f, a unit u in R such that $x = fu + j$ for some $j \in J(R)$.*

Proof. Let $x \in R$. In the proof of Theorem 7, $x = fu + j$ for some $j \in J(R)$, where $f^2 = f \in R$, $u \in R$ is a unit. $\square$

The following result is a similar one to [2, Theorem 3].

THEOREM 13. *For an abelian right quasi-duo ring R, the following statements are equivalent:*

(1) *R is strongly π-regular.*
(2) *R is π-regular.*
(3) *R is weakly π-regular.*
(4) *R is right weakly π-regular.*
(5) *For each $x \in R$ there exist an idempotent f, a unit u in R such that $x = fu + j$ for some $j \in J(R)$ and $J(R)$ is nil.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) are straightforward. (4)$\Rightarrow$(5) comes from Lemma 12. (5)$\Rightarrow$(1) Assume that (5) holds. Then $x + J(R) = fu + J(R)$ and $x + J(R) = (f^2u) + J(R) = (fuu^{-1}fu) + J(R) = (fu + J(R))(u^{-1} + J(R))(fu + J(R)) = (x + J(R))(u^{-1} + J(R))(x + J(R))$, where $u^{-1} \in R$ is the multiplicative inverse of u. So $R/J(R)$ is unit-regular (hence von Neumann regular) and thus R is strongly π-regular by [9, Theorem 2] because $J(R)$ is nil. $\square$

Yao ([18]) proved that weakly right duo rings are abelian, and weakly right duo rings are right quasi-duo rings. So we have the following result.

Corollary 14. *For a weakly right duo ring R, the following statements are equivalent:*

(1) *R is strongly π-regular.*

(2) *R is π-regular.*

(3) *R is weakly π-regular.*

(4) *R is right weakly π-regular.*

(5) *For each $x \in R$ there exist an idempotent f, a unit u in R such that $x = fu + j$ for some $j \in J(R)$ and $J(R)$ is nil.*

Next we study the relations between 2-primal rings and right quasi-duo rings, considering the useful result ([11, Proposition 8]), that is, for a 2-primal right quasi-duo ring R, R is strongly π-regular if and only if R is π-regular if and only if R is weakly π-regular if and only if R is right weakly π-regular if and only if $R/J(R)$ is right weakly π-regular and $J(R)$ is nil if and only if Every prime ideal of R is maximal.

Proposition 15. *For a π-regular ring R, if R is 2-primal then R is right quasi-duo.*

Proof. By Lemma 5, $J(R)$ is nil because R is π-regular (hence right weakly π-regular). But R is 2-primal and so $P(R) = J(R)$; hence $R/P(R) = R/J(R)$ is reduced. Notice that $R/J(R)$ is also π-regular, and so every prime factor ring of $R/J(R)$ is a division ring by Lemma 4. Consequently every right primitive factor ring of $R/J(R)$ is also a division ring. But right primitive factor rings of R and $R/J(R)$ coincide; hence we obtain that every right primitive factor ring of R is a division ring. Now [14, Proposition 1] implies that R is right quasi-duo. $\square$

The converse of Proposition 15 is not true in general by [5, Example 3.3], and the condition "π-regular" is not superfluous by the following example.

Example 6. Let R be the polynomial ring over a noncommutative division ring. Then R is 2-primal because R is reduced. But R is not quasi-duo by [14, Lemma 8], and R is not π-regular.

The *index of nilpotency* of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The *index of nilpotency* of a subset I of R is the supremum of the indices of nilpotency of all nilpotent elements in I. If such a supremum is finite, then I is said to be *of bounded index of nilpotency*.

NOTE. If R is right quasi-duo then R is 2-primal when R is a ring of bounded index of nilpotency with $J(R)$ nil ([10, Theorem 6]). The condition "of bounded index of nilpotency" in the preceding argument is not superfluous by Example 1. The ring $R[[x]]$ in Example 1 is right quasi-duo but not 2-primal. Notice that $R[[x]]$ is not of bounded index of nilpotency. There is another condition in [11, Theorem 6] under which 2-primal rings and right quasi-duo rings coincide. That is, letting R be a right self-injective von Neumann regular ring then the following conditions are equivalent: (1) R is right (left) duo. (2) R is weakly right (left) duo. (3) R is right (left) quasi-duo. (4) R is a reduced ring. (5) R is a 2-primal ring. (6) $R[[x]]$ is right (left) duo. (7) $R[[x]]$ is weakly right (left) duo. (8) $R[[x]]$ is right (left) quasi-duo. (9) $R[[x]]$ is a reduced ring. (10) $R[[x]]$ is a 2-primal ring.

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