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ON LAG INCREMENTS OF A GAUSSIAN PROCESS

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Abstract. In this paper the limit theorems on lag increments of a Wiener process due to Chen, Kong and Lin [1] are developed to the case of a Gaussian process via estimating upper bounds of large deviation probabilities on suprema of the Gaussian process.

1. Introduction

Limit theorems on the increments of Wiener processes and Gaussian processes are deeply related to the properties of their sample paths. So the convergence properties of increments of Wiener processes and Gaussian processes attract the attention of many probabilists in a last few decades. Our interest in this paper is to obtain some limit theorems on lag increments of Gaussian processes. The limit results on lag increments of Wiener processes were initially presented and discussed by Hanson and Russo [5]. Since then, several results on Wiener processes in various directions have been investigated by the following authors: Chen, Kong and Lin [1], Liu [10], Shao [13], He and Chen [7], Hanson and Russo [6], Lin and Lu [9] and Lu [11], etc.

Among the above many results, we are interested in Chen, Kong and Lin [1] whose results are the following fundamental limit theorems on the lag increments of a Wiener process.

Theorem A. ([1]) *Let* $\{W(t), 0 \le t < \infty\}$ *be a Wiener process.*

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Then

$$\limsup_{T\to\infty} \sup_{0<t\le T} |W(T)-W(T-t)|/D(T,t)=1 \qquad \text{a.s.},$$

$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{0\le s\le t} |W(T)-W(T-s)|/D(T,t)=1 \qquad \text{a.s.},$$

$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{t\le s\le T} |W(s)-W(s-t)|/D(T,t)=1 \qquad \text{a.s.},$$

where $D(T,t)=\{2t(\log(T/t)+\log\log t)\}^{1/2}$.

The main aim of this paper is to extend Theorem A to the general form of a Gaussian process.

Throughout this paper we shall always assume the following statements: Let $\{X(t), 0\le t<\infty\}$ be a centered Gaussian process on the probability space $(\Omega, \mathcal{S}, P)$ with $X(0)=0$ and stationary increments $E\{X(t)-X(s)\}^2=\sigma^2(|t-s|)$, where $\sigma(y)$ is a function of $y\ge 0$. For some $C_0>0$, let $\sigma(t)=C_0t^\alpha, 0<\alpha<1$. Denote $d(T,t)=\{2\sigma^2(t)(\log(T/t)+\log\log t)\}^{1/2}$, where $\log t=\ln(t\vee 1)$ and $m\vee n=\max\{m,n\}$. When $\alpha=1/2$ and $C_0=1$, $\{X(t), 0\le t<\infty\}$ is a Wiener process $\{W(t), 0\le t<\infty\}$.

The main results are as follows:

THEOREM 1.1. *We have*

(1.1) $\limsup_{T\to\infty}\sup_{0<t\le T}|X(T)-X(T-t)|/d(T,t)=1$ a.s.,

(1.2) $\lim_{T\to\infty}\sup_{0<t\le T}\sup_{0\le s\le t}|X(T)-X(T-s)|/d(T,t)=1$ a.s.,

(1.3) $\lim_{T\to\infty}\sup_{0<t\le T}\sup_{t\le s\le T}|X(s)-X(s-t)|/d(T,t)=1$ a.s.,

(1.4) $\lim_{T\to\infty}\sup_{0<t\le T}\sup_{t\le s\le T}\sup_{0\le h\le t}|X(s)-X(s-h)|/d(T,t)$ $=1$ a.s.

REMARK. Theorem A is immediate by putting $\sigma(t)=\sqrt{t}$ in Theorem 1.1. It is interesting to compare (1.1) with the law of the iterated logarithm:

$$\limsup_{T\to\infty}|X(T)|/d(T,T)=1 \qquad \text{a.s.} \tag{1.5}$$

Here (1.5) follows by setting $a_T=T$ in the next Lemma 2.1.

Using Theorem 1.1 and (1.5), we can obtain the following:

COROLLARY 1.1. *Let $a_T(0 < T < \infty)$ be a function of T such that $0 < a_T \le T$. Then we have*

$$\begin{aligned}
&\limsup_{T\to\infty} \sup_{a_T\le t\le T} |X(T) - X(T-t)|/d(T,t) \\
&= \limsup_{T\to\infty} \sup_{a_T\le t\le T} \sup_{0\le s\le t} |X(T) - X(T-s)|/d(T,t) \\
&= \limsup_{T\to\infty} \sup_{a_T\le t\le T} \sup_{t\le s\le T} |X(s) - X(s-t)|/d(T,t) \\
&= \limsup_{T\to\infty} \sup_{a_T\le t\le T} \sup_{t\le s\le T} \sup_{0\le h\le t} |X(s) - X(s-h)|/d(T,t) \\
&= 1 \quad \text{a.s.}
\end{aligned}$$

In Corollary 1.1, we can illustrate such kinds of functions as $a_T = 1,\ \log T,\ T/\log\log T,\ cT(0 < c < 1)$ and etc.

2. Proofs

We shall accomplish the proofs of Theorems 1.1 and Corollary 1.1 through the following several lemmas. Lemmas 2.2~2.6 are mainly related to the estimation for upper bounds of large deviation probabilities on suprema of the Gaussian process.

LEMMA 2.1 (Ortega [12]). *Let $\{X(t), 0 \le t < \infty\}$ be a centered Gaussian process with $\sigma^2(h) = E\{X(t+h) - X(t)\}^2 = C_0 h^{2\alpha}$ for $0 < \alpha < 1$ and some constant $C_0 > 0$. Let $0 < a_T \le T$ be a function of T for which*

(i) a_T is non-decreasing,

(ii) T/a_T is non-decreasing.

Then

$$\limsup_{T\to\infty} |X(T) - X(T-a_T)| \cdot \beta_T = 1 \quad \text{a.s.},$$

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |X(t+s) - X(t)| \cdot \beta_T = 1 \quad \text{a.s.},$$

where $\beta_T = \{2\sigma^2(a_T)(\log(T/a_T) + \log\log T)\}^{-1/2}$.

LEMMA 2.2 (Csáki et al. [3], Choi [2]). *Let* $\{X(t), -\infty < t < \infty\}$ *be an almost surely continuous Gaussian process with* $E\{X(t)\} = 0$ *and* $E\{X(s)-X(s-t)\}^2 = \sigma^2(t)$, $\sigma(t) = t^\alpha \sigma_1(t)$ *for some* $\alpha > 0$, *where* $\sigma_1(t)$ *is a nondecreasing function. Then, for any* $\varepsilon > 0$, *there exist positive constants* $C = C_\varepsilon$ *and* $a_\varepsilon > 0$ *such that*

$$P\Big\{\sup_{0\le s-h, s\le T}\ \sup_{0\le h\le a} |X(s) - X(s-h)| > \nu\sigma(a)\Big\} \le \frac{CT}{a}\exp\Big(\frac{-\nu^2}{2+\varepsilon}\Big)$$

for every positive ν *and* $a \ge a_\varepsilon$.

LEMMA 2.3 (Slepian [14]). *Suppose that* $\{V_i, i = 1, 2, \cdots, n\}$ *and* $\{W_i,\ i = 1, 2, \cdots, n\}$ *are jointly standardized normal random variables with*

$$\mathrm{Cov}(V_i, V_j) \le \mathrm{Cov}\ (W_i, W_j), \quad i \ne j.$$

Then, for any real u,

$$P\{\max_{1\le i\le n} V_i \le u\} \le P\{\max_{1\le i\le n} W_i \le u\}.$$

LEMMA 2.4. *Let* $\{X(t), 0 \le t < \infty\}$ *be a centered Gaussian process such that* $X(0) = 0$ *and* $E\{X(t) - X(s)\}^2 = \sigma^2(|t-s|) = C_0^2|t-s|^{2\alpha}$ *for* $0 < \alpha \le \frac{1}{2}$ *and some* $C_0 > 0$. *Then, for any* $u_n \ge 0$,

$$P\Big\{\sup_{1\le s\le n}\frac{X(s) - X(s-1)}{\sigma(1)} \le u_n\Big\} \le \exp\Big(\frac{-n}{\sqrt{2\pi}(u_n+1)}e^{-u_n{}^2/2}\Big).$$

PROOF. From the Fernique inequality ([4], p.71), we have

$$\Big(\frac{1}{\sqrt{2\pi}(u_n+1)}e^{-u_n{}^2/2}\Big) \le \Phi(u_n) \le \frac{4}{3}\Big(\frac{1}{\sqrt{2\pi}(u_n+1)}e^{-u_n{}^2/2}\Big), \quad u_n \ge 0,$$

where $\Phi(t) = \int_t^\infty \frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,dy$. For $i = 1, 2, \cdots, n$, define $Z(i) = X(i) - X(i-1)$. It follows from the relation $ab = (a^2 + b^2 - (a-b)^2)/2$ that, for $l := i - j \geq 1$, without loss of generality,

$$\begin{aligned}
&\mathrm{Cov}(Z(i), Z(j)) \\
&\quad = E\{(X(i) - X(i-1))(X(j) - X(j-1))\} \\
&\quad = -\frac{1}{2}E\{X(i) - X(j)\}^2 + \frac{1}{2}E\{X(i) - X(j-1)\}^2 \\
&\qquad + \frac{1}{2}E\{X(i-1) - X(j)\}^2 + \frac{1}{2}E\{X(i-1) - X(j-1)\}^2 \\
&\quad = \frac{1}{2}\big\{(\sigma^2(l+1) - \sigma^2(l)) - (\sigma^2(l) - \sigma^2(l-1))\big\} \leq 0.
\end{aligned}$$

In order to apply Lemma 2.3, let $V_i = Z(i)/\sigma(1)$, and let W_i be independent standard normal random variables. Since

$$\mathrm{Cov}(V_i, V_j) \leq \mathrm{Cov}\ (W_i, W_j) = 0, \quad i \neq j,$$

it follows from Lemma 2.3 and the Fernique inequality that

$$\begin{aligned}
&P\Big\{\sup_{1\leq s\leq n} \frac{|X(s) - X(s-1)|}{\sigma(1)} \leq u_n\Big\} \leq P\Big\{\max_{1\leq i\leq n} V_i \leq u_n\Big\} \\
&\leq P\Big\{\max_{1\leq i\leq n} W_i \leq u_n\Big\} = \prod_{i=1}^{n} P\{W_i \leq u_n\} \\
&= (1 - \Phi(u_n))^n \leq \exp(-n\Phi(u_n)) \leq \exp\Big(\frac{-n}{\sqrt{2\pi}(u_n+1)}e^{-u_n{}^2/2}\Big).
\end{aligned}$$

□

Lemma 2.5 (Leadbetter et al. [8]). *Let $\{\xi_j, j = 1, 2, \cdots, n\}$ be jointly standardized normal random variables with $\Lambda_{ij} = \mathrm{Corr}(\xi_i, \xi_j)$ such that*

$$\delta := \max_{i\neq j} |\Lambda_{ij}| < 1.$$

Then, for any real u_n and integers $1 \leq l_1 < l_2 < \cdots < l_{k_n} \leq n$,

$$P\Big\{\max_{1\leq i\leq k_n} \xi_{l_i} \leq u_n\Big\} \leq \big(1 - \Phi(u_n)\big)^{k_n} + K \sum_{1\leq i<j\leq k_n} |r_{ij}| \exp\Big(\frac{-u_n{}^2}{1 + |r_{ij}|}\Big), \tag{2.1}$$

where $r_{ij} = \Lambda_{l_i l_j}, K = K_\delta$ *is a constant independent of* n, u_n *and* k_n.

In order to estimate an upper bound for the second term of the right hand side of (2.1), we establish the following lemma, which is easily proved by emulating the proof of Lemma 4.4 in Choi [2].

LEMMA 2.6. *Let* $\xi_j (j = 1, 2, \cdots, n)$, δ, k_n *and* r_{ij} *be as in* Lemma 2.5. *Assume that* $|r_{ij}| \le \rho_{|i-j|} < 1$ $(i \ne j)$ *and, for some* $\nu > 0$

$$\rho_m < m^{-\nu}, \quad m = 1, 2, \cdots, k_n - 1.$$

Let $u_n = \{(2 - 2\epsilon)n\}^{1/2}$ *and* $k_n = [e^n/M]$ *for some* $M > 0$, *where* $[\cdot]$ *denotes the integer part. Then there exist constants* $\delta_0 = \delta_0(\varepsilon, \delta, \nu) > 0$ *and* $C > 0$ *such that*

$$\sum_n := \sum_{1 \le i < j \le k_n} |r_{ij}| \exp\Big(-\frac{{u_n}^2}{1 + |r_{ij}|}\Big) \le C e^{-\delta_0 n}.$$

Hereafter c and C denote positive constants which can be changed in different lines if necessary. We are now ready to prove Theorem 1.1. The main stream of the proof is similar to the proof of Theorem A.

PROOF OF THEOREM 1.1. Step 1. From Lemma 2.1, we have

$$\begin{aligned} &\limsup_{T\to\infty} \sup_{0<t\le T} |X(T) - X(T-t)|/d(T,t) \\ &\quad \ge \limsup_{T\to\infty} |X(T)|/d(T,T) = 1 \qquad \text{a.s.} \end{aligned} \tag{2.2}$$

Step 2. (1.1) follows from (2.2) if we show that

$$\limsup_{T\to\infty} \sup_{0<t\le T} \sup_{t\le s\le T} \sup_{0\le h\le t} |X(s) - X(s-h)|/d(T,t) \le 1 \qquad \text{a.s.} \tag{2.3}$$

Take $\theta > 1$ so that $1 < 2(1+\varepsilon)^2/((2+\varepsilon)\theta^{2\alpha}) =: 1 + 2\varepsilon'$ for any small $\varepsilon > 0$. For $n = 1, 2, \cdots$, let $k = \cdots, -2, -1, 0, 1, 2, \cdots, k_n$, where $k_n = [(n+1)/\log\theta]$. Set $T_n = e^n, t_k = \theta^k$, $k_\theta = [1/\log\theta]$ and $k'_n = \big[(n + 1 -$

$\log n^{1/\varepsilon'})/\log\theta]$. When $T_n \le T \le T_{n+1}$, we have

$$
\begin{aligned}
&\sup_{0<t\le T}\ \sup_{t\le s\le T}\ \sup_{0\le h\le t} |X(s)-X(s-h)|/d(T,t) \\
&\le \sup_{-\infty<k\le k_n-1}\ \sup_{t_k\le t\le t_{k+1}}\ \sup_{t\le s\le T_{n+1}}\ \sup_{0\le h\le t} \frac{|X(s)-X(s-h)|}{d(T_n,t_k)} \\
&\le \sup_{-\infty<k\le k_n-1}\ \sup_{0\le s-h,s\le T_{n+1}}\ \sup_{0\le h\le t_{k+1}} \frac{|X(s)-X(s-h)|}{d(T_n,t_k)} \\
&=: \sup_{-\infty<k\le k_n-1} A_{nk}.
\end{aligned}
\tag{2.4}
$$

From Lemma 2.2, we have

$$
\begin{aligned}
&P\Big\{A_{nk}\ge 1+\varepsilon\Big\} \\
&= P\Big\{\sup_{\substack{0\le s-h,s\le T_{n+1}\\ 0\le h\le t_{k+1}}} \frac{|X(s)-X(s-h)|}{\sigma(t_{k+1})} \ge (1+\varepsilon)\frac{\sigma(t_k)}{\sigma(t_{k+1})} \\
&\qquad\qquad \times \{2(\log(T_n/t_k)+\log\log t_k)\}^{1/2}\Big\} \\
&\le \frac{CT_{n+1}}{t_{k+1}} \exp\Big(-\frac{(1+\varepsilon)^2}{2+\varepsilon}\Big(\frac{\sigma(t_k)}{\sigma(t_{k+1})}\Big)^2 \{2(\log(T_n/t_k)+\log\log t_k)\}\Big) \\
&\le C\frac{T_{n+1}}{t_{k+1}}\Big(\frac{T_n\log t_k}{t_k}\Big)^{-\frac{2(1+\varepsilon)^2}{2+\varepsilon}\theta^{-2\alpha}} \le C\Big(\frac{T_n}{t_k}\Big)^{-2\varepsilon'}(\log t_k)^{-1-2\varepsilon'}.
\end{aligned}
\tag{2.5}
$$

Hence, for $-\infty<k\le k_\theta$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{-\infty<k\le k_\theta} P\Big\{A_{nk}\ge 1+\varepsilon\Big\} \le \sum_{n=1}^{\infty}\sum_{-\infty<k\le k_\theta} C\Big(\frac{T_n}{t_k}\Big)^{-2\varepsilon'} \\
&= C\sum_{n=1}^{\infty}\sum_{-\infty<k<0}\Big(\frac{T_n}{t_k}\Big)^{-2\varepsilon'} + C\sum_{n=1}^{\infty}\sum_{k=0}^{k_\theta}\Big(\frac{T_n}{t_k}\Big)^{-2\varepsilon'} \\
&\le C\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\Big(\frac{e^n}{\theta^{-k}}\Big)^{-2\varepsilon'} + C\sum_{n=1}^{\infty}(e^n)^{-2\varepsilon'}(k_\theta+1)e^{2\varepsilon'} \\
&< \infty.
\end{aligned}
\tag{2.6}
$$

For the case $k_\theta < k \le k_n - 1$, we have, as in (2.5),

$$P\Big\{A_{nk} \ge 1+\varepsilon\Big\} \le C\Big(\frac{T_{n+1}}{t_{k+1}}\Big)^{-2\varepsilon'} \big(\log t_k\big)^{-(1+2\varepsilon')}. \tag{2.7}$$

Note that, when $k_\theta < k \le {k_n}'$,

$$(t_{k+1})^{2\varepsilon'} \le (\theta^{{k_n}'+1})^{2\varepsilon'} \le \Big\{\frac{\theta T_{n+1}}{(\log T_n)^{1/\varepsilon'}}\Big\}^{2\varepsilon'}.$$

From (2.7), it follows that

(2.8)

$$\sum_{n=1}^{\infty}\sum_{k=k_\theta+1}^{{k_n}'} P\{A_{nk} \ge 1+\varepsilon\} \le C\sum_{n=1}^{\infty}\sum_{k=k_\theta+1}^{{k_n}'} \frac{\theta^{2\varepsilon'}}{(\log T_n)^2}(\log t_k)^{-(1+2}$$

$$< C\sum_{n=1}^{\infty} n^{-2} \sum_{k=k_\theta+1}^{{k_n}'} k^{-(1+2\varepsilon')} < \infty.$$

For the case ${k_n}' < k \le k_n - 1$, we have, for n large enough,

$$\begin{aligned} &{T_n}^{1/2} \le t_{k+1} \le \theta T_{n+1}, \\ &k_n - {k_n}' \le (\varepsilon' \log\theta)^{-1} \log n + 2 =: {k_n}''. \end{aligned}$$

Using (2.7) again, one see that

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{k={k_n}'+1}^{k_n-1} P\{A_{nk} \ge 1+\varepsilon\} \\ &\qquad \le C\sum_{n=1}^{\infty}\sum_{k={k_n}'+1}^{k_n-1} \Big(\frac{T_{n+1}}{t_{k+1}}\Big)^{-2\varepsilon'} (\log t_{k+1})^{-(1+2\varepsilon')} \\ &\qquad \le C\sum_{n=1}^{\infty}(k_n - k'_n - 1)\theta^{2\varepsilon'}(\log {T_n}^{1/2})^{-(1+2\varepsilon')} \\ &\qquad \le C\sum_{n=1}^{\infty} {k_n}'' n^{-(1+2\varepsilon')} \le C\sum_{n=1}^{\infty} n^{-(1+\varepsilon')} < \infty. \end{aligned} \tag{2.9}$$

Finally, merging (2.6), (2.8) and (2.9) together, we get

$$\sum_{n=1}^{\infty} P\Big\{\sup_{-\infty<k\le k_n-1} A_{nk} \ge 1+\varepsilon\Big\} \le \sum_{n=1}^{\infty}\sum_{-\infty<k\le k_n-1} P\{A_{nk}\ge 1+\varepsilon\}$$

$$=\sum_{n=1}^{\infty}\sum_{-\infty<k\le k_\theta} P\{A_{nk}\ge 1+\varepsilon\} + \sum_{n=1}^{\infty}\sum_{k=k_\theta+1}^{k_n'} P\{A_{nk}\ge 1+\varepsilon\}$$

$$+\sum_{n=1}^{\infty}\sum_{k=k_n'+1}^{k_n-1} P\{A_{nk}\ge 1+\epsilon\}$$

$$<\infty.$$

Thus, by the Borel-Cantelli lemma, (2.3) follows from (2.4).

Step 3. (1.4) follows from (2.3) if we show that

$$\liminf_{T\to\infty} \sup_{0<t\le T}\sup_{t\le s\le T}\sup_{0\le h\le t} |X(s)-X(s-h)|/d(T,t)\ge 1 \quad \text{a.s.} \tag{2.10}$$

For $n=1,2,\cdots$, set $T_n=e^n$, and let T be in $T_n\le T\le T_{n+1}$. Then

$$\begin{aligned}
&\sup_{0<t\le T}\sup_{t\le s\le T}\sup_{0\le h\le t} |X(s)-X(s-h)|/d(T,t)\\
&\quad\ge \sup_{1\le s\le T_n} |X(s)-X(s-1)|/d(T_{n+1},1)\\
&\quad= \sup_{1\le s\le T_n} \frac{|X(s)-X(s-1)|}{\sigma(1)\sqrt{2n}}\Big(\frac{n}{n+1}\Big)^{1/2}\\
&\quad=: B_n\Big(\frac{n}{n+1}\Big)^{1/2}.
\end{aligned}$$

To prove (2.10), it is sufficient to show that, for any $0<\varepsilon<1$,

$$\sum_{n=1}^{\infty} P\{B_n\le\sqrt{1-\varepsilon}\}<\infty, \tag{2.11}$$

because, if (2.11) holds, then

$$\liminf_{n\to\infty} B_n\ge 1 \quad \text{a.s.}$$

by the Borel-Cantelli lemma.

First suppose that $0 < \alpha \leq 1/2$. To apply Lemma 2.4, let $u_n = \{(2-2\varepsilon)n\}^{1/2}$. Then, we have

$$P\{B_n \leq \sqrt{1-\varepsilon}\} \leq P\Big\{\sup_{1\leq s\leq T_n} \frac{X(s)-X(s-1)}{\sigma(1)} \leq u_n\Big\}$$
$$\leq \exp\Big\{\frac{-T_n}{\sqrt{2\pi}(u_n+1)} e^{-u_n{}^2/2}\Big\} \leq \exp(-ce^{\varepsilon n}).$$

This yields (2.11).

Next assume that $1/2 < \alpha < 1$. For given α, there exist a big number $M > 0$ and integers n such that

$$2^{1/(1-\alpha)} < M \leq e^n. \tag{2.12}$$

Consider a sequence of integers $k_n = [e^n/M]$. For $i = 1, 2, \cdots, k_n$, define $Y(i) = \{X(Mi) - X(Mi-1)\}/\sigma(1)$. It follows that, for any $0 < \varepsilon < 1$,

$$\begin{aligned} P\{B_n \leq \sqrt{1-\varepsilon}\} &\leq P\Big\{\sup_{1\leq s\leq T_n} \frac{X(s)-X(s-1)}{\sigma(1)} \leq u_n\Big\} \\ &\leq P\Big\{\max_{1\leq i\leq k_n} Y(i) \leq u_n\Big\}, \end{aligned} \tag{2.13}$$

where $u_n = \{(2-2\varepsilon)n\}^{1/2}$. Let $r(i,j) = \mathrm{Cov}(Y(i), Y(j)), i \neq j$, and let $l := i - j \geq 1$, without loss of generality. Using the relation $ab = (a^2 + b^2 - (a-b)^2)/2$ and the mean-value theorem, we have

$$\begin{aligned} |r(i,j)| &= |E\{Y(i)Y(j)\}| \\ &= \frac{1}{2\sigma^2(1)}\big| -\sigma^2(|Mi - Mj|) + \sigma^2(|Mi - Mj + 1|) \\ &\qquad + \sigma^2(|Mi - Mj - 1|) - \sigma^2(|Mi - Mj|)\big| \\ &= \frac{1}{2}\big|(Ml+1)^{2\alpha} - (Ml)^{2\alpha} - \big((Ml)^{2\alpha} - (Ml-1)^{2\alpha}\big)\big| \\ &\leq 2\alpha(2\alpha-1)(Ml-1)^{2(\alpha-1)} \leq 2\alpha(2\alpha-1)(Ml)^{\alpha-1}. \end{aligned}$$

It follows from (2.12) that

$$|r(i,j)| < \frac{2}{M^{1-\alpha}} l^{\alpha-1} < l^{-\nu},$$

where $\nu = 1 - \alpha > 0$. To estimate the upper bound of (2.13), let us apply Lemmas 2.5 and 2.6 for $\xi_{l_i} = Y(i), i = 1, 2, \cdots, k_n$, and $|r_{ij}| = |r(i,j)| < l^{-\nu}$, $l = i - j \geq 1$. Then

$$P\{B_n \leq \sqrt{1-\varepsilon}\} \leq \{1 - \Phi(u_n)\}^{k_n} + Ce^{-\delta_0 n} \tag{2.14}$$

for some $\delta_0 > 0$. Since $1 - \Phi(u_n) \leq \exp\big(-\Phi(u_n)\big)$, we have, for all large n,

$$\Phi(u_n) \geq \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{u_n} - \frac{1}{u_n^3}\Big)\exp(-u_n{}^2/2) \geq C\exp(-(1-\varepsilon)n)$$

and hence

$$\{1 - \Phi(u_n)\}^{k_n} \leq \exp(-ce^{\varepsilon n}).$$

In the sequel, (2.14) yields

$$P\{B_n \leq \sqrt{1-\varepsilon}\} \leq c\exp(-\delta_0 n)$$

and (2.11) holds true. (1.2) and (1.3) follow immediately from (1.1) and (1.4). □

Proof of Corollary 1.1. From Theorem 1.1 and (1.4), we have

$$\begin{aligned}
1 &\geq \limsup_{T\to\infty} \sup_{a_T \leq t \leq T} \sup_{t \leq s \leq T} \sup_{0 \leq h \leq t} |X(s) - X(s-h)|/d(T,t) \\
&\geq \limsup_{T\to\infty} \sup_{a_T \leq t \leq T} \sup_{t \leq s \leq T} |X(s) - X(s-t)|/d(T,t) \\
&\geq \limsup_{T\to\infty} \sup_{a_T \leq t \leq T} \sup_{0 \leq s \leq t} |X(T) - X(T-s)|/d(T,t) \\
&\geq \limsup_{T\to\infty} \sup_{a_T \leq t \leq T} |X(T) - X(T-t)|/d(T,t) \\
&\geq \limsup_{T\to\infty} |X(T)|/d(T,T) = 1 \quad \text{a.s.}
\end{aligned}$$

□

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