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FRAMES WITH A UNIQUE UNIFORMITY

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Abstract. In this paper, we investigate frames that admit a unique uniformity and characterize the completely regular frames which admit a unique uniformity.

1. Introduction

Tychonoff spaces (i.e., completely regular and Hausdorff) with a unique compatible uniform structure were first characterized by R. Doss [6]. R. H. Warren has extended this characterization to the completely regular spaces which are not necessarily Hausdorff in [10] and [11]. Our aim is to establish the analogue of this for frames.

In this paper, we extend characterization of spaces with exactly one compatible uniformity to frames and characterize the completely regular frames for which there is only one uniform structure. We obtain eight characterizations of completely regular frames that admit a unique uniformity. In particular we show that a completely regular frame L admits a unique uniformity if and only if $\{x \in L \mid x \geq a^*\}$ or $\{x \in L \mid x \geq b\}$ is compact whenever $a \prec\prec b$.

We recall some basic notions and facts about frames.

A *frame* is a complete lattice L satisfying the distributive law

$$a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\} \ \ (a \in L,\ S \subseteq L)$$

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and a *frame homomorphism* is a map $h : M \to L$ between frames which preserves finitary meets, including the unit(=top) e, and arbitrary joins, including the zero(=bottom) 0. A frame homomorphism $h : M \to L$ is called *dense* if $h(x) = 0$ implies $x = 0$ for all $x \in M$.

A frame L is called *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, where $x \prec a$ means that $x \wedge y = 0$, $a \vee y = e$ for some $y \in L$, alternatively expressed as $x^* \vee a = e$ with the *pseudocomplement* $x^* = \bigvee\{y \in L \mid x \wedge y = 0\}$ of x.

An element $a \in L$ is called *compact* if $a \leq \bigvee S$ implies $a \leq \bigvee E$ for some finite $E \subseteq S$, for all $S \subseteq L$. L itself is called *compact* if the top $e \in L$ is a compact. A *compactification* of a frame L is a compact regular frame M together with a dense onto homomorphism $h : M \to L$.

Concerning *completely regularity*, we take this to mean that $a = \bigvee\{x \in L \mid x \prec\prec a\}$ for each $a \in L$, where $x \prec\prec a$ (x is *completely below* a, or *really inside* a) means that there exists a sequence $(c_{nk})_{n=0,1,\cdots,k=0,1,\cdots,2^n}$ such that

$$c_{00} = x,\ c_{01} = a,\ c_{nk} = c_{n+1\,2k},\ c_{nk} \prec c_{n\,k+1}$$

for all $n = 0, 1, \cdots$ and $k = 0, 1, \cdots, 2^n$.

A frame L is called *continuous* whenever, for each $a \in L$, $a = \bigvee\{x \in L \mid x << a\}$ where $x << a$ (x is *way below* a) means that, for any $S \subseteq L$ such that $a \leq \bigvee S$ there exists a finite $E \subseteq S$ for which $x \leq \bigvee E$.

For subsets $A, B, \cdots$ and elements $a, b, \cdots, x, y, \cdots$ of a frame L, we use the following notation and terminology:

$A \leq B$(A *refines* B) if each $a \in A$, $a \leq b$ for some $b \in B$.

$Ax = \bigvee\{a \in A \mid a \wedge x \neq 0\}$.

$A \leq^* B$(A *star-refines* B) if $\{Ax \mid x \in A\} \leq B$.

A is *cover* of L if $\bigvee A = e$.

Further, for any set $\mathcal{M}$ of covers of L,

$a \lhd_{\mathcal{M}} b$ if $Ca \leq b$ for some $C \in \mathcal{M}$.

$\mathcal{M}$ is *admissible* if $a = \bigvee\{x \in L \mid x \lhd_{\mathcal{M}} a\}$ for all $a \in L$.

Now, a *uniformity* on L is an admissible set $\mathcal{U}$ of covers of L which is a filter relative to $\leq$ such that, for each $A \in \mathcal{U}$, there exists $B \leq^* A$ in $\mathcal{U}$. In this case, the pair $(L, \mathcal{U})$ is called a *uniform frame* and the relation $\lhd_{\mathcal{M}}$ is the *strong inclusion* for $\mathcal{U}$ (see [1]).

Let $(M, \mathcal{M})$, $(L, \mathcal{U})$ be uniform frames and $h : M \to L$ a frame homomorphism. Then h is said to be *uniform frame homomorphism* if for any $A \in \mathcal{M}$, $h(A) = \{h(a) \mid a \in A\} \in \mathcal{U}$. A uniformity $\mathcal{U}$ is called *totally bounded* if $\mathcal{U}$ is generated by its finite members and we say that $(L, \mathcal{U})$ is a *totally bounded uniform frame.*

For basic results on uniform frames we refer to [7] and [9], and for the general background of frames, we refer to [8].

The following is due to Banaschewski ([2], [3]).

In the following, $\mathbf{Q}$ is the usual ordered set of rational numbers.

Next, the *frame of reals* is the frame $\mathcal{L}(\mathbf{R})$ generated by all ordered pairs (p, q) where $p, q \in \mathbf{Q}$, subject to the relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$
(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$, and
(R4) $e = \bigvee\{(p, q) \mid p, q \in \mathbf{Q}\}$.

Note that the condition $(p, q) = 0$ whenever $p \geq q$ is a consequence of (R3).

We use the following notation in $\mathcal{L}(\mathbf{R})$:

$$(p, -) = \bigvee\{(p, q) \mid q \in \mathbf{Q}\} = \bigvee\{(p, q) \mid p < q \in \mathbf{Q}\}$$

$$(-, q) = \bigvee\{(p, q) \mid p \in \mathbf{Q}\} = \bigvee\{(p, q) \mid q > p \in \mathbf{Q}\}$$

and note that $(p, -) \wedge (-, q) = (p, q)$.

The frame $\mathcal{L}(\mathbf{R})$ carries a natural uniformity, its *metric uniformity,* generated by the covers

$$C_n = \{(p, q) \mid 0 < q - p < \tfrac{1}{n}\}, \quad n = 1, 2, \cdots.$$

Note that C_{3n} is a star-refinement of C_n, for each n.

Now, a *continuous real function* on a frame L is a homomorphism $\mathcal{L}(\mathbf{R}) \to L$. For any frame L, $\varphi : \mathcal{L}(\mathbf{R}) \to L$ is called *bounded* if $\varphi(p, q) = e$ for some $p, q \in \mathbf{Q}$ and L is called *pseudocompact* if all $\varphi : \mathcal{L}(\mathbf{R}) \to L$ are bounded. We observe that a completely regular frame L is pseudocompact if and only if all its uniformities are totally bounded [5, Proposition 3].

2. Frames that admit a unique uniformity

In this section, we characterize the completely regular frames that admit a unique uniformity.

LEMMA 1. *In any frame L, $a \prec\prec b$ if and only if there exists a bounded homomorphism $\varphi : \mathcal{L}(\mathbf{R}) \to L$ such that $a \leq \varphi(-, \frac{1}{2})$ and $\varphi(-, 1) \leq b$.*

PROOF. By Proposition 6 in [3], it is enough to show that φ is bounded. Let $\varphi : \mathcal{L}(\mathbf{R}) \to L$ be a homomorphism which is described in the proof of Proposition 6 in [3]. Then $\varphi(p, q) = \bigvee\{t(p')^* \wedge t(q') \mid p < p' < q' < q\}$, where $(w_{nk})_{n=0,1,\cdots,k=0,1,\cdots,2^n}$ is a sequence witnessing that $a \prec\prec b$, for each $r \in \mathbf{Q}$,

$$t(r) = \begin{cases} 0 & (r < 0) \\ \bigvee\{w_{nk} \mid \frac{k}{2^n} \leq r\} & (0 \leq r \leq 1) \\ e & (1 < r). \end{cases}$$

Now, we show that φ is bounded. Let $m, n \in \mathbf{Q}$ with $m < 0$ and $n > 1$. Since $t(\frac{1}{2}m) = 0$ and $t(\frac{1}{2}(1+n)) = e$, $t(\frac{1}{2}m)^* \wedge t(\frac{1}{2}(1+n)) = 0^* \wedge e = e$. Hence $\varphi(m, n) = e$. □

In the remainder of the section, $\mathcal{L}(\mathbf{R})$ denotes the frame of reals with the metric uniformity and $\mathbf{N}$ denotes the set of all natural numbers.

PROPOSITION 2. *Let $(L, \mathcal{U})$ be a uniform frame. Suppose that $\mathcal{U}$ is totally bounded. Let $\mathcal{F} = \{\varphi | \varphi : \mathcal{L}(\mathbf{R}) \to L$ is a bounded homomorphism$\}$. Then $\mathcal{F}$ determines $\mathcal{U}$ in the following sense. If $B \in \mathcal{U}$ there exist $\varphi_1, \varphi_2, \cdots, \varphi_n \in \mathcal{F}$ and a uniform cover C_r of $\mathcal{L}(\mathbf{R})$ such that $\bigwedge_{k=1}^{n} \varphi_k(C_r) \leq B$.*

PROOF. Take any $B \in \mathcal{U}$. Then B has a finite star-refinement $A \in \mathcal{U}$, say $A = \{a_1, a_2, a_3, \cdots, a_n\}$. For $1 \leq k \leq n$, there is $b_k \in B$ with $a_k \prec\prec b_k$ with Countable Dependent Choice. For each $k = 1, 2, \cdots, n$ there is a bounded homomorphism $\varphi_k : \mathcal{L}(\mathbf{R}) \to L$ such that $a_k \leq \varphi_k(-, \frac{1}{2})$ and $\varphi_k(-, 1) \leq b_k$ by Lemma 1. Since $C_2(-, \frac{1}{2}) \leq (-, 1)$, $\varphi_k(C_2)a_k \leq b_k$ for each $k = 1, 2, \cdots, n$. Put $D = \bigwedge_{k=1}^{n} \varphi_k(C_2)$. We claim that $D \leq B$. Take any $d \in D$. Then we may assume that $d \neq 0$. Since $d = d \wedge (\bigvee_{k=1}^{n} a_k) =$

$\bigvee_{k=1}^{n}(d \wedge a_k)$, there is k_0 with $d \wedge a_{k_0} \neq 0$. Hence $d \leq Da_{k_0} \leq \varphi_{k_0}(C_2)a_{k_0} \leq b_{k_0}$. □

PROPOSITION 3. *Let L be a completely regular frame and let $\mathcal{W} = \{\bigwedge_{k=1}^{n} \varphi_k(C_r) \mid$ for each $k = 1, 2, \cdots, n, \varphi_k : \mathcal{L}(\mathbf{R}) \to L$ is a bounded homomorphism and $r \in \mathrm{N}\}$. Then $\mathcal{W}$ is a base for a totally bounded uniformity on L. Moreover, $\mathcal{W}$ is a base for the finest totally bounded uniformity on L.*

PROOF. For each $k = 1, 2, \cdots, n$, let $\varphi_k : \mathcal{L}(\mathbf{R}) \to L$ be a bounded homomorphism and $r \in \mathbf{N}$. Since $C_{3r} \leq^* C_r$, $\bigwedge_{k=1}^{n} \varphi_k(C_{3r}) \leq^* \bigwedge_{k=1}^{n} \varphi_k(C_r)$. Take any $b \in L$. By Lemma 1, there is a bounded homomorphism $\varphi : \mathcal{L}(\mathbf{R}) \to L$ such that $a \leq \varphi(-, \frac{1}{2})$ and $\varphi(-, 1) \leq b$. Since $C_2(-, \frac{1}{2}) \leq (-, 1)$, $\varphi(C_2)a \leq \varphi(C_2)\varphi(-, \frac{1}{2}) \leq \varphi(-, 1) \leq b$; hence $a \lhd_{\mathcal{W}} b$. Thus $\mathcal{W}$ is admissible. Therefore $\mathcal{W}$ is a base for a uniformity on L. We now show that $\mathcal{W}$ is a base for a totally bounded uniformity on L. For each $k = 1, 2, \cdots, n$, let $\varphi_k : \mathcal{L}(\mathbf{R}) \to L$ be a bounded homomorphism and $r \in \mathbf{N}$. Since $\varphi_k (1 \leq k \leq n)$ is bounded, $\varphi_k(p_k, q_k) = e$ for some $p_k, q_k \in \mathbf{Q}$ $(p_k < q_k)$. For $k = 1, 2, \cdots, n$, there exist $l_0^k < l_1^k < l_2^k < \cdots < l_m^k$ such that $l_0^k = p_k$, $l_m^k = q_k$, $\bigvee_{t=0}^{m-2}(l_t^k, l_{t+2}^k) = (p_k, q_k)$ and $l_t^k - l_{t-1}^k < \frac{1}{2r}$ $(t = 1, 2, \cdots, m)$. For each $k = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, m-2$, $(l_t^k, l_{t+2}^k) \in C_r$ and hence $\varphi_k(l_t^k, l_{t+2}^k) \in \varphi_k(C_r)$. For each $k = 1, 2, \cdots, n$, let $B_k = \{\varphi_k(l_t^k, l_{t+2}^k) \mid t = 0, 1, \cdots, m-2\}$, then B_k is finite and $B_k \leq \varphi_k(C_r)$.
For each $k = 1, 2, \cdots, n$,

$$\begin{aligned} \varphi_k(C_{2r}) &= \varphi_k(C_{2r}) \wedge \varphi_k(p_k, q_k) \\ &= \{\varphi_k(p, q) \wedge \varphi_k(p_k, q_k) \mid (p, q) \in C_{2r}\} \\ &= \{\varphi_k(p \vee p_k, q \wedge q_k) \mid (p, q) \in C_{2r}\}. \end{aligned}$$

It follows that $\varphi_k(C_{2r}) \leq B_k (k = 1, 2, \cdots, n)$. Thus $\bigwedge_{k=1}^{n} \varphi_k(C_{2r}) \leq \bigwedge_{k=1}^{n} B_k \leq \bigwedge_{k=1}^{n} \varphi_k(C_r)$; hence $\bigwedge_{k=1}^{n} B_k \in \mathcal{W}$. In all, $\mathcal{W}$ is a base for a totally bounded uniformity on L.

It follows from Proposition 2 that $\mathcal{W}$ is a base the finest totally bounded uniformity on L. □

Collecting the above, we now have our main theorem.

THEOREM 4. *For any completely regular frame L, the following are equivalent:*

(1) *L admits a unique uniformity.*
(2) *Every frame homomorphism from a uniform frame into L is uniform in every admissible uniform structure on L.*
(3) *If $\varphi : \mathcal{L}(\mathbf{R}) \to L$ is a frame homomorphism, then φ is a uniform frame homomorphism in every admissible uniform structure on L.*
(4) *If $\varphi : \mathcal{L}(\mathbf{R}) \to L$ is a bounded frame homomorphism, then φ is a uniform frame homomorphism in every admissible uniform structure on L.*
(5) *If $\varphi : \mathcal{L}(\mathbf{R}) \to L$ is a bounded frame homomorphism, then φ is a uniform frame homomorphism in every totally bounded uniformity on L.*
(6) *L admits only one totally bounded uniform structure.*
(7) *L admits a unique strong inclusion.*
(8) *L has a unique compactification.*
(9) *For $a \prec\prec b \in L$, $\uparrow a^*$ or $\uparrow b$ is compact.*

PROOF. (1) ⇒ (2). Let $(M, \mathcal{M})$ be a uniform frame and let $f : M \to L$ be a frame homomorphism. Take any uniform cover $A \in \mathcal{M}$. Then $f(A)$ is also a normal cover of L. Thus $f(A)$ belongs to the fine uniformity on L.

(2) ⇒ (3). $\mathcal{L}(\mathbf{R})$ admits the metric uniform structure.

(3) ⇒ (4). It is obvious.

(4) ⇒ (5). Each totally bounded uniformity is a uniformity.

(5) ⇒ (6). Let $\mathcal{U}$ be any totally bounded uniformity on L and let $\mathcal{W}$ be a base for the totally bounded uniformity on L which is described in Proposition 3. Since W is a base for the finest totally bounded uniformity on L, it is enough to show that $\mathcal{W} \subseteq \mathcal{U}$. Let $\varphi_k : \mathcal{L}(\mathbf{R}) \to L$ be a bounded homomorphism for each $k = 1, 2, \cdots, n$ and let $r \in \mathbf{N}$. By hypothesis, for each $k = 1, 2, \cdots, n$, $\varphi_k : \mathcal{L}(\mathbf{R}) \to (L, \mathcal{U})$ is a uniform frame homomorphism. Hence $\bigwedge_{k=1}^{n} \varphi_k(C_r) \in \mathcal{U}$. Thus $\mathcal{W} \subseteq \mathcal{U}$. In all, L admits only one totally bounded uniform structure.

(6) $\Leftrightarrow$ (7) and (7) $\Leftrightarrow$ (8). [2, Proposition 1] and [1, Proposition 2].

(8) $\Rightarrow$ (1). It follows from Corollary 1 of Proposition 5 in [4] that L is pseudocompact. Hence every uniformity on L is totally bounded.

(8) $\Rightarrow$ (9). Suppose that L has a unique compactification. Then it follows from the proof of Proposition 4 in [1] that L is regular continuous and hence $\blacktriangleleft$ is a strong inclusion on L, where $a \blacktriangleleft b$ means that $a \prec b$ and $\uparrow a^*$ or $\uparrow b$ is compact. By (7), $\blacktriangleleft = \prec\!\prec$. If $a \prec\!\prec b$, then $a \blacktriangleleft b$ and hence $\uparrow a^*$ or $\uparrow b$ is compact.

(9) $\Rightarrow$ (8). Since (7) $\Leftrightarrow$ (8), it is enough to show that $\prec\!\prec$ is the unique strong inclusion on L. Take any strong inclusion $\lhd$ on L. Then clearly, with Countable Dependent Choice, $\lhd \subseteq \prec\!\prec$. Let $a \prec\!\prec b$. If $\uparrow a^*$ is compact, then $a \prec b$ and hence $a^* \vee b = e$. Since $\lhd$ is a strong inclusion, $a^* \vee b = a^* \vee (\bigvee\{x \in L \,|\, x \lhd b\}) = \bigvee\{a^* \vee x \,|\, x \lhd b\} = e$. Since $\uparrow a^*$ is compact, there is $z \lhd b$ with $a^* \vee z = e$; hence $a \prec z$. Thus $a \leq z \lhd b$; hence $a \lhd b$. Now, consider the case that $\uparrow b$ is compact. Since $a \prec\!\prec b$, there is $u \in L$ such that $a \wedge u = 0$ and $b \vee u = e$. By the compactness of $\uparrow b$, there is $v \lhd u$ with $b \vee v = e$. Since $a \wedge u = 0$ and $v \lhd u$, $a \leq u^* \lhd v^* \leq b$. Hence $a \lhd b$. We conclude that $\prec\!\prec$ is the only strong inclusion on L. □

REMARK. A completely regular frame that admits a unique uniformity must be regular continuous and hence has a smallest compactification. This frame compactification corresponds to the one-point compactification of locally compact Hausdorff space.

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