Comm. Korean Math. Soc. **15** (2000), No. 2, pp. 309–323

ISHIKAWA-TYPE AND MANN-TYPE ITERATIVE PROCESSES WITH ERRORS FOR m-ACCRETIVE OPERATORS

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ABSTRACT. The purposes of this paper are to revise the definitions of Ishikawa and Mann type iterative processes with errors, to study the unique solution of the m-accretive operator equation $x + Tx = f$ and the convergence problem of Ishikawa and Mann type iterative processes with errors for m-accretive mappings without the Lipschitz condition. The results presented in this paper improve, extend, and unify the corresponding results in [4, 7, 8, 12, 16] in more general setting.

1. Introduction

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, and $\langle \cdot, \cdot \rangle$ denotes the pairing of E and E^*. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

is called the normalized duality mapping.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called accretive [1] if the inequality

$$\|x - y\| \leq \|x - y + t(Tx - Ty)\|$$

holds for each $x, y \in D(T)$ and for all $t > 0$. The operator T is said to be m-accretive if T is accretive and $(I + \lambda T)(D(T)) = E$ for all $\lambda > 0$,

Received November 12, 1999. Revised February 27, 2000.
1991 Mathematics Subject Classification: 47J25.
Key words and phrases: m-accretive operators; Mann and Ishikawa-type iterative processes with errors.

where I is the identity operator on E. T is accretive if and only if for any $x, y \in D(T)$, there is $j \in J(x-y)$ such that

$$\langle Tx - Ty, j \rangle \geq 0.$$

A class of operators closely related to the class of accretive operators is the class of dissipative operators. An operator T is dissipative if and only if $(-T)$ is accretive and T on E is called m-dissipative if $(I - \lambda T)(E) = E$ for each $\lambda > 0$. Browder ([2]) proved that if T is a locally Lipschitzian dissipative operator on $D(T) = E$ then T is m-dissipative.

Accretive operators were introduced independently in 1967 by Browder ([1]) and Kato ([10]). An early fundamental result, due to Browder ([1]), in the theory of accretive operators states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \quad u(0) = u_0$$

is solvable if T is a locally Lipschitzian and accretive operator on E. Utilizing the existence result, Browder ([1]) further proved that if T is locally Lipschitzian and accretive with $D(T) = E$ then T is m-accretive, that is, $(I + T)(E) = E$, so that for any given $f \in X$, the equation $x + Tx = f$ has a solution. The result was generalized by Martin ([14]) to the continuous accretive operators. Recently, Chidume ([4,5]) proved that the Mann type iterative sequence converges strongly to a solution of the equation $x + Tx = f$ where T is a Lipschitz accretive operator defined on the Hilbert space H or the space L_p. In [6], he generalized results in [4,5] to uniformly smooth Banach space and continuous accretive operators. Chidume and Osilike ([7]) extended the above results to the Ishikawa type sequence where T is Lipschitz m-accretive and $D(T)$ is a closed subset of a real Banach space E which is both uniformly convex and q-uniformly smooth. And Ding ([8]) and Zhu ([16]) showed that the Mann and Ishikawa type iterative sequences with errors converge strongly to the unique solution of the equation $x + Tx = f$.

The purposes of this paper are to revise the definitions of Ishikawa and Mann type iterative processeswith errors and to study convergence theorems of Ishikawa and Mann type iterative processes with errors for approximating the unique solution of the equation $x + Tx = f$ where $T : D(T) \subset E \to E$ is an m-accretive operator with closed domain

$D(T)$ which may not be Lipschitz and E is both uniformly convex and uniformly smooth. We also prove convergence theorems of Ishikawa and Mann type iterative processes with errors for approximating the unique solution of the equation $x - \lambda Tx = f$ where $T : D(T) \subset E \to E$ is an m-dissipative operator with closed domain $D(T)$ which may not be Lipschitz. The results presented in this paper improve, extend, and unify the corresponding results in [4, 7, 8, 12, 16] in more general setting.

2. Preliminaries

Let E be a real Banach space with $\|\cdot\|$. The modulus of smoothness $\rho_E(\cdot)$ of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \frac{1}{2}\sup\{\|x+y\| + \|x-y\| - 2 : x, y \in E, \|x\| = 1, \|y\| \leq \tau\}, \tau > 0,$$

and that E is said to be uniformly smooth if $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$. It is well known that E is uniformly convex (smooth) if and only if E^* is uniformly smooth (convex).

We first recall the following iteration process due to Liu ([11]).

(I) The Ishikawa iteration process with errors is defined as follows: For a nonempty subset D of a Banach space E and a mapping $T : D \subset E \to E$, the sequence $\{x_n\}$ in D is defined by

$$x_0 \in D,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n + u_n,$$

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n + v_n, \quad n \geq 0,$$

where $\{u_n\}$ and $\{v_n\}$ are two summable sequences in E, i.e., $\sum_{n=0}^{\infty}\|u_n\| < \infty$, $\sum_{n=0}^{\infty}\|v_n\| < \infty$, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying certain restrictions.

Clearly, the sequence $\{x_n\}$ exists when $D = E$. Note that the Mann ([13]) and Ishikawa ([9]) iteration procedures are all special cases of the Ishikawa iteration process with errors. Inspired by [8,11,12], we introduce the following concept of the Ishikawa type iteration process with errors.

(II) The Ishikawa-type iteration process with errors is defined as follows: For a nonempty subset D of a Banach space E and a mapping $T : D \subset E \to E$, the sequence $\{p_n\}$ in E is defined by

$$\begin{aligned} & x_0 \in D, \\ (2.1) \qquad & p_{n+1} = \alpha_n x_n + \beta_n TQy_n + \gamma_n u_n, \\ & y_n = \hat{\alpha}_n x_n + \hat{\beta}_n Tx_n + \hat{\gamma}_n v_n, \\ & x_{n+1} = Qp_{n+1}, \quad n \geq 0, \end{aligned}$$

where the mapping $Q : E \to D$ is a retraction of E onto D. Hence $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in E; $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha}_n\}$, $\{\hat{\beta}_n\}$, and $\{\hat{\gamma}_n\}$ are six sequences in $[0, 1]$ satisfying the conditions

$$(2.2) \qquad \alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1 \quad \text{for all} \quad n \geq 0.$$

In particular, if $\hat{\beta}_n = \hat{\gamma}_n = 0$ for all $n \geq 0$, the sequence $\{p_n\}$ in E is defined by

$$\begin{aligned} & x_0 \in D, \\ & p_{n+1} = \alpha_n x_n + \beta_n TQx_n + \gamma_n u_n, \\ & x_{n+1} = Qp_{n+1}, \quad n \geq 0, \end{aligned}$$

which is called the Mann-type iteration process with errors.

The following four lemmas play a crucial role in the proofs of our main results.

LEMMA 2.1 ([3]). *Let E be a real Banach space. Then, for all $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $j(x + y) \in J(x + y)$, where $J : E \to 2^{E^*}$ is the normalized duality mapping.

LEMMA 2.2 ([11]). *Let a_n, b_n, and c_n be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

with $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $b_n = O(t_n)$, $\sum_{n=0}^{\infty} c_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

LEMMA 2.3 ([16]). *Let E be a real Banach space and $T : D(T) \subset E \to E$ be an m-accretive operator. Then, for any given $f \in E$, the equation $x + Tx = f$ has a unique solution in $D(T)$.*

LEMMA 2.4 ([15]). *Let E be a real Banach space which is both uniformly convex and uniformly smooth. Let $T : D(T) \subset E \to E$ be m-accretive and let $J_r = (I + rT)^{-1}$. Then for each $x \in E$, the strong limit $\lim_{r\to 0} J_r(x)$ exists. Denote the strong limit by Qx. Then $Q : E \to cl(D(T))$ is a nonexpansive retraction of E onto $cl(D(T))$ where $cl(D(T))$ is the closure of $D(T)$.*

3. Main Results

THEOREM 3.1. *Let E be a real Banach space which is both uniformly convex and uniformly smooth. Let $T : D(T) \subset E \to E$ be an m-accretive operator with a closed domain $D(T)$. Define $S : D(T) \subset E \to E$ by $Sx = f - Tx$, $^{\forall}x \in D(T)$, and suppose that the range of S is bounded. For arbitrary $x_0 \in D(T)$, the iteration sequence $\{p_n\}$ with errors is defined by*

$$p_{n+1} = \alpha_n x_n + \beta_n SQy_n + \gamma_n u_n, \tag{3.1}$$

$$y_n = \hat{\alpha}_n x_n + \hat{\beta}_n Sx_n + \hat{\gamma}_n v_n, \tag{3.2}$$

$$x_{n+1} = Qp_{n+1}, \quad n \geq 0, \tag{3.3}$$

satisfying

$$\text{(i)}\ \lim_{n\to\infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n = +\infty;$$

$$\text{(ii)}\ \lim_{n\to\infty} \hat{\beta}_n = 0;$$

$$\text{(iii)}\ \lim_{n\to\infty} \hat{\gamma}_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n < +\infty,$$

where $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1 (n \geq 0)$, $\{u_n\}$, and $\{v_n\}$ are two bounded subsets of E. Then for any $x_0 \in D(T)$, the iteration sequence $\{p_n\}$ in E converges strongly to the unique solution $x^* \in D(T)$ of the equation $x + Tx = f$.

PROOF. The existence and uniqueness of the solution x^* to the equation $x + Tx = f$ follow from the m-accretiveness of T and Lemma 2.3. By the definition of S, we observe that $Sx^* = x^*$. Moreover, for each $x, y \in D(T)$,

$$\langle Sx - Sy, J(x-y)\rangle \leq 0. \tag{3.4}$$

For any $x_0 \in D(T)$, we first compute $y_0 = \hat{\alpha}_0 x_0 + \hat{\beta}_0 Sx_0 + \hat{\gamma}_0 v_0$ in E and then compute $p_1 = \alpha_0 x_0 + \beta_0 SQy_0 + \gamma_0 u_0$ in E. We can now compute x_1 in $D(T)$ by $x_1 = Qp_1$. With x_1 we compute $y_1 = \hat{\alpha}_1 x_1 + \hat{\beta}_1 Sx_1 + \hat{\gamma}_1 v_1$ in E. Then $p_2 = \alpha_1 x_1 + \beta_1 SQy_1 + \gamma_1 u_1 \in E$ and also $x_2 = Qp_2 \in D(T)$. Continuing this process we generate the sequence $\{p_n\}$ and $\{x_n\}$. Now set

$$\begin{aligned} d = \max\Bigg\{ &\sup_{n\geq 0}\|u_n - x^*\|, \quad \sup_{n\geq 0}\|v_n - x^*\|, \\ &\sup_{x\in D(T)} \|Sx - x^*\|, \|x_0 - x^*\| \Bigg\}. \end{aligned} \tag{3.5}$$

By (3.1), (3.3), (3.5), and the fact Q is a nonexpansive retraction of E onto $D(T)$, we have

$$\begin{aligned} \|x_1 - x^*\| &= \|Qp_1 - Qx^*\| \\ &\leq \|p_1 - x^*\| \\ &\leq \alpha_0\|x_0 - x^*\| + \beta_0\|SQy_0 - x^*\| + \gamma_0\|u_0 - x^*\| \\ &\leq d. \end{aligned}$$

For any $n \geq 0$, using induction, we obtain

$$\|x_n - x^*\| \leq d, \quad n \geq 0. \tag{3.6}$$

Moreover, by using (2.2), (3.1), (3.2), (3.5), (3.6), we have that

$$\begin{aligned} &\|y_n - x^*\| \leq d, \\ &\|p_n - x^*\| \leq d, \quad n \geq 0. \end{aligned} \tag{3.7}$$

By using (3.4) and (3.5), we have

$$
\begin{aligned}
&\langle SQy_n - x^*, J(x_n - x^*)\rangle \\
&\quad = \langle SQy_n - Sx^*, J(Qy_n - x^*)\rangle \\
&\qquad - \langle SQy_n - x^*, J(Qy_n - x^*) - J(x_n - x^*)\rangle \\
&\quad \le |\langle SQy_n - x^*, J(Qy_n - x^*) - J(x_n - x^*)\rangle| \\
&\quad \le \|SQy_n - x^*\|\|J(Qy_n - x^*) - J(x_n - x^*)\| \\
&\quad \le d\|J(Qy_n - x^*) - J(x_n - x^*)\|.
\end{aligned} \tag{3.8}
$$

It then follows from (3.1), (3.5), (3.7), (3.8), and Lemma 2.1 that

$$
\begin{aligned}
&\|p_{n+1} - x^*\|^2 \\
&\le \alpha_n^2\|x_n - x^*\|^2 + 2\beta_n\langle SQy_n - x^*, J(p_{n+1} - x^*)\rangle \\
&\quad + 2\gamma_n\langle u_n - x^*, J(p_{n+1} - x^*)\rangle \\
&= \alpha_n^2\|x_n - x^*\|^2 + 2\beta_n\{\langle SQy_n - x^*, J(x_n - x^*)\rangle \\
&\quad + \langle SQy_n - x^*, J(p_{n+1} - x^*) - J(x_n - x^*)\rangle\} \\
&\quad + 2\gamma_n\langle u_n - x^*, J(p_{n+1} - x^*)\rangle \\
&\le (1-\beta_n)^2\|x_n - x^*\|^2 + 2\beta_n\{d\|J(Qy_n - x^*) - J(x_n - x^*)\| \\
&\quad + \langle SQy_n - x^*, J(p_{n+1} - x^*) - J(x_n - x^*)\rangle\} \\
&\quad + 2\gamma_n\|u_n - x^*\|\|p_{n+1} - x^*\| \\
&\le (1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n\{d\|J(Qy_n - x^*) - J(x_n - x^*)\| \\
&\quad + \langle SQy_n - x^*, J(p_{n+1} - x^*) - J(x_n - x^*)\rangle\} \\
&\quad + 2\gamma_n d^2,
\end{aligned} \tag{3.9}
$$

and from (3.3), and the fact that Q is a nonexpansive retraction of E onto $D(T)$, we have

$$
\|x_n - x^*\| = \|Qp_n - Qx^*\| \le \|p_n - x^*\|. \tag{3.10}
$$

Hence, by (3.9) and (3.10), we have

$$\begin{aligned}\|p_{n+1}-x^*\|^2 &\le (1-\beta_n)\|p_n-x^*\|^2\\ &\quad+2\beta_n\{d\|J(Qy_n-x^*)-J(x_n-x^*)\|\\ &\quad+\langle SQy_n-x^*, J(p_{n+1}-x^*)-J(x_n-x^*)\rangle\}\\ &\quad+2\gamma_n d^2\\ &=(1-\beta_n)\|p_n-x^*\|^2+b_n+c_n,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}b_n &= 2\beta_n\{d\|J(Qy_n-x^*)-J(x_n-x^*)\|\\ &\qquad+\langle SQy_n-x^*, J(p_{n+1}-x^*)-J(x_n-x^*)\rangle\},\\ c_n &= 2\gamma_n d^2.\end{aligned}$$

First, we prove that

$$\|J(Qy_n-x^*)-J(x_n-x^*)\|\to 0 \quad \text{as} \quad n\to\infty,$$
$$\langle SQy_n-x^*, J(p_{n+1}-x^*)-J(x_n-x^*)\rangle\to 0 \quad \text{as} \quad n\to\infty.$$

In fact, by (3.1), (3.2), (3.5), (3.6), the assumptions (ii) and (iii), we have

$$\begin{aligned}&\|(Qy_n-x^*)-(x_n-x^*)\|\\ &\quad=\|Qy_n-Qx_n\|\\ &\quad\le\|y_n-x_n\|\\ &\quad\le\hat{\beta}_n\|Sx_n-x_n\|+\hat{\gamma}_n\|v_n-x_n\|\\ &\quad\le\hat{\beta}_n(\|Sx_n-x^*\|+\|x^*-x_n\|)+\hat{\gamma}_n(\|v_n-x^*\|+\|x^*-x_n\|)\\ &\quad\le 2d(\hat{\beta}_n+\hat{\gamma}_n)\\ &\quad\to 0 \quad \text{as} \quad n\to\infty,\end{aligned}$$

$$\begin{aligned}&\|(p_{n+1}-x^*)-(x_n-x^*)\|\\ &\quad=\|p_{n+1}-x_n\|\\ &\quad=\|\beta_n(SQy_n-x_n)+\gamma_n(u_n-x_n)\|\\ &\quad\le\beta_n(\|SQy_n-x^*\|+\|x^*-x_n\|)+\gamma_n(\|u_n-x^*\|+\|x^*-x_n\|)\\ &\quad=2d(\beta_n+\gamma_n)\\ &\quad\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Since $\{Qy_n - x^*, x_n - x^*, p_{n+1} - x^*\}$ and $\{SQy_n - x^*\}$ are bounded sets and E is uniformly smooth so that J is uniformly continuous on any bounded subset of E, we have that

$$\|J(Qy_n - x^*) - J(x_n - x^*)\| \to 0 \quad \text{as} \quad n \to \infty,$$

$$\|J(p_{n+1} - x^*) - J(x_n - x^*)\| \to 0 \quad \text{as} \quad n \to \infty,$$

and

$$\langle SQy_n - x^*, J(p_{n+1} - x^*) - J(x_n - x^*)\rangle \to 0 \quad \text{as} \quad n \to \infty.$$

So, $b_n = O(\beta_n)$. And we know that c_n is summable.

Now, let $a_n = \|p_n - x^*\|^2$ and $t_n = \beta_n$ for all $n \geq 0$. The inequality (3.11) reduces to

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n.$$

It follows from Lemma 2.2 that $\lim_{n\to\infty} a_n = 0$, so that $\{p_n\}$ converges strongly to the unique solution x^* of the equation $x + Tx = f$. □

COROLLARY 3.1. *Let E, T, and $D(T)$ be as in* Theorem 3.1. *Define $S : D(T) \subset E \to E$ by $Sx = f - Tx$, $^\forall x \in D(T)$. For arbitrary $x_0 \in D(T)$, the iteration sequence $\{p_n\}$ with errors is defined by*

$$p_{n+1} = \alpha_n x_n + \beta_n S x_n + \gamma_n u_n,$$

$$x_{n+1} = Q p_{n+1}, \quad n \geq 0,$$

satisfying

$$\text{(i)} \lim_{n\to\infty} \beta_n = 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} \beta_n = +\infty,$$

$$\text{(ii)} \sum_{n=0}^{\infty} \gamma_n < +\infty,$$

where $\alpha_n + \beta_n + \gamma_n = 1$ $(n \geq 0)$ and $\{u_n\}$ is a bounded subset of E. If the sequence $\{Sx_n\}$ is bounded in E, then the iteration sequence $\{p_n\}$ converges strongly to the unique solution $x^* \in D(T)$ of the equation $x + Tx = f$.

PROOF. This follows from Theorem 3.1 with $\hat{\beta}_n = 0$ and $\hat{\gamma}_n = 0$ for all $n \geq 0$. □

REMARK 3.1. Note that the assumption on the range of S in Theorem 3.1 can be replaced by boundedness of $\{Sx_n\}$.

REMARK 3.2. Theorem 3.1 and Corollary 3.1 improve, generalize, and unify Theorem 3 and 5 of Chidume and Osilike ([7]), Theorem 3 of Zhu ([16]), Theorem 2.1 and Theorem 3.1 of Liu ([12]), and Theorem 3.1 and Corollary 3.1 of Ding ([8]) in the following ways:

(1) that $\{u_n\}$ and $\{v_n\}$ be two summable sequences is replaced by that $\{u_n\}$ and $\{v_n\}$ are two bounded sequences;

(2) T may not be Lipschitz continuous;

(3) the iterative scheme may have errors terms.

If, in Theorem 3.1, $D(T) = E$, the use of the projection operator Q will not be necessary. Moreover X need not be uniformly convex.

THEOREM 3.2. *Let E be a real uniformly smooth Banach space and $T : E \to E$ be a continuous accretive operator. Define $S : E \to E$ by $Sx = f - Tx$, $^\forall x \in E$, and suppose that the range of S is bounded. For arbitrary $x_0 \in E$, the iteration sequence $\{x_n\}$ with errors is defined by*

$$x_{n+1} = \alpha_n x_n + \beta_n S y_n + \gamma_n u_n,$$

$$y_n = \hat{\alpha}_n x_n + \hat{\beta}_n S x_n + \hat{\gamma}_n v_n, \quad n \geq 0,$$

satisfying

$$\text{(i)} \lim_{n\to\infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n = +\infty;$$

$$\text{(ii)} \lim_{n\to\infty} \hat{\beta}_n = 0;$$

$$\text{(iii)} \lim_{n\to\infty} \hat{\gamma}_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n < +\infty,$$

where $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$ $(n \geq 0)$, $\{u_n\}$ and $\{v_n\}$ are two bounded subsets of E. Then for any $x_0 \in E$, the iteration sequence $\{x_n\}$ in E converges strongly to the unique solution $x^* \in E$ of the equation $x + Tx = f$.

PROOF. A result of Martin ([14]) shows that T is m-accretive, and so the equation $x+Tx=f$ has a unique solution $x^* \in E$. Following the technique of the proof of Theorem 3.1 we obtain

$$\|x_{n+1}-x^*\|^2 \le (1-\beta_n)\|x_n-x^*\|^2 + b_n + c_n,$$

where

$$\begin{aligned} b_n &= 2\beta_n\{d\|J(y_n-x^*)-J(x_n-x^*)\| \\ &\quad + \langle Sy_n - x^*, J(x_{n+1}-x^*) - J(x_n-x^*)\rangle\}, \\ c_n &= 2\gamma_n d^2. \end{aligned}$$

The result follows as in Theorem 3.1. □

COROLLARY 3.2. *Let E and T be as in* Theorem 3.2. *Define $S: E \to E$ by $Sx = f - Tx$, $^\forall x \in E$. For arbitrary $x_0 \in E$, the iteration sequence $\{x_n\}$ with errors is defined by*

$$x_{n+1} = \alpha_n x_n + \beta_n S x_n + \gamma_n u_n, \quad n \ge 0,$$

satisfying

$$\text{(i)} \lim_{n\to\infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n = +\infty,$$

$$\text{(ii)} \sum_{n=0}^{\infty} \gamma_n < +\infty,$$

where $\alpha_n + \beta_n + \gamma_n = 1$ $(n \ge 0)$ and $\{u_n\}$ is a bounded subset of E. If the sequence $\{Sx_n\}$ is bounded in E, then the iteration sequence $\{x_n\}$ converges strongly to the unique solution $x^* \in E$ of the equation $x + Tx = f$.

PROOF. This is obvious from Theorem 3.2 with $\hat{\beta}_n = 0$ and $\hat{\gamma}_n = 0$ for all $n \ge 0$. □

REMARK 3.3. Theorem 3.2 and Corollary 3.2 improve, generalize, and unify Theorem 4, Theorem 6, and Corollary 2 of Chidume and Osilike ([7]), Theorem 3.2 and Corollary 3.2 of Liu ([12]), Theorem 3.2 and Corollary 3.2 of Ding ([8]).

We turn our attension to convergence theorems for dissipative operators. We shall be interested in the approximation of a solution of the equation $x - \lambda Tx = f$, where $T : D(T) \subset E \to E$ is m-dissipative and λ is a real positive constant.

THEOREM 3.3. *Let E be a real Banach space which is both uniformly convex and uniformly smooth. Let $T : D(T) \subset E \to E$ be an m-dissipative operator with a closed domain $D(T)$. Define $S : D(T) \subset E \to E$ by $Sx = f + \lambda Tx$, $\forall x \in D(T)$, and suppose that the range of S is bounded. For arbitrary $x_0 \in D(T)$, the iteration sequence $\{p_n\}$ with errors is defined by*

$$p_{n+1} = \alpha_n x_n + \beta_n SQy_n + \gamma_n u_n,$$

$$y_n = \hat{\alpha}_n x_n + \hat{\beta}_n Sx_n + \hat{\gamma}_n v_n,$$

$$x_{n+1} = Qp_{n+1}, \quad n \geq 0,$$

satisfying

$$\text{(i)} \lim_{n\to\infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n = +\infty;$$

$$\text{(ii)} \lim_{n\to\infty} \hat{\beta}_n = 0;$$

$$\text{(iii)} \lim_{n\to\infty} \hat{\gamma}_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n < +\infty,$$

where $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$ $(n \geq 0)$, $\{u_n\}$ and $\{v_n\}$ are two bounded subsets of E. Then for any $x_0 \in D(T)$, the iteration sequence $\{p_n\}$ in E converges strongly to the unique solution $x^* \in D(T)$ of the equation $x - \lambda Tx = f$.

PROOF. Since T is m-dissipative, $(-\lambda T)$ is m-accretive. The result now follows from Theorem 3.1. □

COROLLARY 3.3. *Let E, T, and $D(T)$ be as in Theorem 3.3. Define $S : D(T) \subset E \to E$ by $Sx = f + \lambda Tx$, $\forall x \in D(T)$. For arbitrary $x_0 \in D(T)$, the iteration sequence $\{p_n\}$ with errors is defined by*

$$p_{n+1} = \alpha_n x_n + \beta_n Sx_n + \gamma_n u_n,$$

$$x_{n+1} = Qp_{n+1}, \quad n \geq 0,$$

satisfying

$$\text{(i)} \lim_{n\to\infty} \beta_n = 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} \beta_n = +\infty,$$

$$\text{(ii)} \sum_{n=0}^{\infty} \gamma_n < +\infty,$$

where $\alpha_n + \beta_n + \gamma_n = 1$ $(n \geq 0)$ *and* $\{u_n\}$ *is a bounded subset of* E. *If the sequence* $\{Sx_n\}$ *is bounded in* E, *then the iteration sequence* $\{p_n\}$ *converges strongly to the unique solution* $x^* \in D(T)$ *of the equation* $x - \lambda Tx = f$.

PROOF. The conclusion follows from Theorem 3.3 with $\hat{\beta}_n = 0$ and $\hat{\gamma}_n = 0$ for all $n \geq 0$. □

REMARK 3.4. Theorem 3.3 and Corollary 3.3 improve, generalize, and unify Theorem 7 and Corollary 3 of Chidume and Osilike ([7]), Theorem 3.3 and Corollary 3.3 of Ding ([8]).

THEOREM 3.4. *Let* E *be a real uniformly smooth Banach space and* $T : E \to E$ *be a continuous dissipative operator. Define* $S : E \to E$ *by* $Sx = f + \lambda Tx$, $^\forall x \in E$, *and suppose that the range of* S *is bounded. For arbitrary* $x_0 \in E$, *the iteration sequence* $\{x_n\}$ *with errors is defined by*

$$x_{n+1} = \alpha_n x_n + \beta_n S y_n + \gamma_n u_n,$$

$$y_n = \hat{\alpha}_n x_n + \hat{\beta}_n S x_n + \hat{\gamma}_n v_n, \quad n \geq 0,$$

satisfying

$$\text{(i)} \lim_{n\to\infty} \beta_n = 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} \beta_n = +\infty;$$

$$\text{(ii)} \lim_{n\to\infty} \hat{\beta}_n = 0;$$

$$\text{(iii)} \lim_{n\to\infty} \hat{\gamma}_n = 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} \gamma_n < +\infty,$$

where $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$ $(n \geq 0)$, $\{u_n\}$ *and* $\{v_n\}$ *are two bounded subsets of* E. *Then for any* $x_0 \in E$, *the iteration sequence* $\{x_n\}$ *in* E *converges strongly to the unique solution* $x^* \in E$ *of the equation* $x - \lambda Tx = f$.

PROOF. Since T is continuous and dissipative, $(-\lambda T)$ is continuous and accretive. The conclusion follows from Theorem 3.2. □

REMARK 3.5. Theorem 3.4 improves, generalizes, and unifies Theorem 8 and Corollary 4 of Chidume and Osilike ([7]), Theorem 3.4 of Ding ([8]).

References

[1] F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 875-882.

[2] ———, *Nonlinear monotone and accretive operators in Banach spaces*, Proc. Nat. Acad. Sci. U. S. A. **61** (1968), 388-393.

[3] S. S. Chang, *Some problems and results in the study of nonlinear analysis*, Nonlinear Anal. TMA **30** (1997), no. 7, 4197-4208.

[4] C. E. Chidume, *An approximation method for monotone Lipschitzian operators in Hilbert space*, J. Austral. Math. Soc. **41** (1986), 59-63.

[5] ———, *Iterative solution of nonlinear equations of the monotone and dissipative types*, Appl. Anal. **33** (1989), 79-86.

[6] ———, *Iterative solution of nonlinear equations of the monotone type in Banach spaces*, Bull. Austral. Math. Soc. **42** (1990), 21-31.

[7] C. E. Chidume and M. O. Osilike, *Approximation methods for nonlinear operator equations of the m-accretive type*, J. Math. Anal. Appl. **189** (1995), 225-239.

[8] X. P. Ding, *Iterative process with errors of nonlinear equations involving m-accretive operators*, J. Math. Anal. Appl. **209** (1997), 191-201.

[9] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147-150.

[10] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508-520.

[11] L. S. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114-125.

[12] ———, *Ishikawa type and Mann type iterative processes with errors for constructing solutions of nonlinear equations involving m-accretive operators in Banach spaces*, Nonlinear Anal. **34** (1998), 307-317.

[13] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.

[14] R. H. Martin, Jr, *A global existence theorem for autonomous differential equations in Banach spaces*, Proc. Amer. Math. Soc. **26** (1970), 307-314.
[15] S. Reich, *Strongly convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287-292.
[16] L. Zhu, *Iterative solution of nonlinear equations involving m-accretive operators in Banach spaces*, J. Math. Anal. Appl. **188** (1994), 410-416.

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