Comm. Korean Math. Soc. **15** (2000), No. 2, pp. 285–308

FRAME MULTIRESOLUTION ANALYSIS

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ABSTRACT. We generalize bi-orthogonal (non-orthogonal) MRA to frame MRA in which the family of integer translates of a scaling function forms a frame for the initial ladder space V_0. We investigate the internal structure of frame MRA and establish the existence of a dual scaling function, and show that, unlike bi-orthogonal MRA, there exists a frame MRA that has no (frame) 'wavelet.' Then we prove the existence of a dual wavelet under the assumption of the existence of a wavelet and present easy sufficient conditions for the existence of a wavelet. Finally we give a new proof of an equivalent condition for the translates of a function in $L^2(\mathbb{R})$ to be a frame of its closed linear span.

1. Introduction

We generalize the model of bi-orthogonal (non-orthogonal) multiresolution analyses (MRA) of Feauveau [9] by allowing the collection of the integer translates of the scaling function of the MRA to be a frame of the initial ladder space V_0. The possibility of such generalization was first observed by Benedetto and Li in [1] and by Li in [13]. Given an orthogonal frame MRA $(\{V_j\}, \varphi)$ (see Definition 3.1), their idea is to consider an idempotent $A_0 : V_1 \to V_0$ of the form $A_0 f = \sum_n \langle f, T^n\varphi^* \rangle T^n\varphi$. Everything works more smoothly if we follow Feauveau's model more thoroughly. For example, existence of a dual scaling function and a dual wavelet follows very naturally from Definition 3.1 by considering commutation relations of certain operators. See Proposition 3.11 and Proposition 3.14. These and

Received October 18, 1999. Revised April 2, 2000.

2000 Mathematics Subject Classification: 42C15.

Key words and phrases: Frame expansion, multiresolution analysis, wavelet expansion.

The first author was supported by KOSEF, and the second by KOSEF and Brain Korea 21.

the related results also shed some light on bi-orthogonal MRA situations. Almost everything, except the existence of wavelets, that can be said in a bi-orthogonal MRA can also be said in a frame MRA with even more transparent proof.

The paper is organized in the following manner. Preliminary discussions are gathered in Section 2. In Section 3 we develop the theory of frame MRA by extending Feauveau's non-orthogonal MRA, and investigate the internal structure of a frame MRA. Unlike orthonormal MRA or bi-orthogonal MRA, the existence of a wavelet in the initial difference space W_0 is not readily guaranteed. Actually, there exists a frame MRA that has no frames of translates of a single function for W_0 (Example 3.21). We give sufficient conditions for the existence of a wavelet for W_0 (Theorem 3.19 and Theorem 3.20). In Section A we present a new proof of an equivalent condition for the translates of an element of $L^2(\mathbb{R})$ to be a frame of their closed linear span.

2. Preliminaries

In this section we fix some notations, introduce some terminologies that will be used throughout the discussion, and collect some preliminary facts.

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$, and let I be a countable index set.

DEFINITION 2.1. $\{f_i\}_{i\in I} \subset \mathcal{H}$ is a Riesz basis (with a pair of Riesz bounds A and B) if it is a complete (total) sequence and if there exist A and B with $0 < A \le B < \infty$ such that for any $\{c_i\}_{i\in I} \in \ell^2(I)$

$$A\sum_{i\in I}|c_i|^2 \le \|\sum_{i\in I} c_i f_i\|^2 \le B\sum_{i\in I}|c_i|^2.$$

$\{f_i\}$ is a Bessel sequence (with a Bessel bound B) if there exists $B < \infty$ such that for any $f \in \mathcal{H}$,

$$\sum_{i\in I}|\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

A Bessel sequence $\{f_i\}$ is a frame (with a pair of frame bounds A and B) if there exists $A > 0$ such that for any $f \in \mathcal{H}$,

$$A\|f\|^2 \le \sum_{i\in I}|\langle f, f_i\rangle|^2.$$

The bounds in the above inequalities are not unique. The infimum of all upper Riesz bounds is called the optimal upper Riesz bound. The optimal upper frame bound and optimal Bessel bound are defined in a similar way. Likewise, the supremum of all lower Riesz bounds is called the optimal lower Riesz bound, and the optimal lower frame bound is similarly defined.

For a frame $\{f_i\}$, the frame operator $S : \mathcal{H} \to \mathcal{H}$, $Sf := \sum_i \langle f, f_i \rangle f_i$ defines a bounded positive invertible operator. Suppose now that $\{f_i\}$ is a Bessel sequence and $\{e_i\}$ the standard orthonormal basis of ℓ^2. Then we can define the so-called pre-frame operator [11]:

$$Q : \ell^2 \to \mathcal{H}, \quad Qe_i := f_i, i \in I.$$

Then $\|Q\| \leq \sqrt{B}$, where B is a Bessel bound. The proof of the following characterization of frames can be found in [11] or [4].

THEOREM 2.2. *A sequence $\{f_i\}$ is a frame if and only if the pre-frame operator is bounded and surjective.*

If $\{f_i\}$ is a frame, then $S = QQ^*$, where $*$ denotes the adjoint. Since S is positive and invertible, for any $f \in L^2(\mathbb{R})$, we have a frame expansion, $f = SS^{-1}f = \sum_i \langle f, S^{-1}f_i \rangle f_i$. We call $\{S^{-1}f_i\}$ the dual frame of $\{f_i\}$. It is also a frame, and its dual frame is the original frame. A frame may be over-redundant. Therefore the coefficients in an expansion with respect to a frame are not unique.

DEFINITION 2.3. A sequence $\{f_i\}$ ℓ^2-generates $\mathcal{H}$ if for any $f \in \mathcal{H}$ there exists $(a_i) \in \ell^2$ such that $f = \sum_i a_i f_i$, where the sum converges in the norm topology of $\mathcal{H}$.

LEMMA 2.4. *A Bessel sequence of a subspace of a Hilbert space is a Bessel sequence of the entire space, a subsequence of a Bessel sequence is a Bessel sequence, and the image of a Bessel sequence by a bounded operator is a Bessel sequence. A Bessel sequence satisfies the upper Riesz basis condition of* Definition 2.1.

PROOF. All the statements except the last one is trivial. The proof of the last statement can be found in [15]. □

LEMMA 2.5. *If a Bessel sequence ℓ^2-generates $\mathcal{H}$, then it is a frame for $\mathcal{H}$.*

PROOF. The pre-frame operator is bounded and surjective. The lemma follows by Theorem 2.2. See also Lemma 4.4 in [4]. □

Throughout the rest of the discussion we let $\{e_n\}_{n\in\mathbb{Z}}$ denote the standard orthonormal basis of $\ell^2(\mathbb{Z})$. Let τ be the bilateral right-shift operator such that $\tau : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $\tau e_n = e_{n+1}$. For $\alpha > 0, \beta \in \mathbb{R}$ we define the dilation operator D_α via $D_\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(D_\alpha f)(t) := \alpha^{1/2} f(\alpha t)$, and the translation operator T_β via $T_\beta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(T_\beta f)(t) := f(t-\beta)$. We let $T := T_1$ be the integer translation and $D := D_2$ the dyadic dilation. Then for each $n \in \mathbb{Z}, \beta \in \mathbb{R}$, the following commutation relations hold (Lemma 3.2 [8]).

$$D^n T_\beta = T_{2^{-n}\beta} D^n, \quad T_\beta D^n = D^n T_{2^n \beta}. \tag{2.1}$$

All of these operators are unitary. Notice that $T^* = T^{-1} = T_{-1}$ and $D^* = D^{-1} = D_{1/2}$.

For any $f \in L^2(\mathbb{R})$ let $f_{j,k}(x) := (D^j T^k f)(x) = 2^{j/2} f(2^j x - k)$.

We use the following form of the Fourier transform: for $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $(Ff)(t) = \hat{f}(t) := \int_{\mathbb{R}} f(x) e^{-2\pi i t x}\, dx$, and extend F to a unitary operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ denote the compact group which can conveniently be identified with the interval $[0,1)$. For $f \in L^1(\mathbb{T})$, its k^{th} Fourier coefficient is defined to be $\int_{\mathbb{T}} f(t) e^{2\pi i k t}\, dt$. We let $A'(\mathbb{Z})$ denote the vector space of the Fourier coefficients of the elements of $L^\infty(\mathbb{T})$. Notice that $(a_n) \in A'(\mathbb{Z})$ if and only if $\sum_n a_n e^{-2\pi i n \gamma} \in L^\infty(\mathbb{T})$, and that $\ell^1(\mathbb{Z}) \subset A'(\mathbb{Z})$. All summations without explicit description of the index set is understood to be over $\mathbb{Z}$.

DEFINITION 2.6. Let A and B be two closed subspaces of a Hilbert space $\mathcal{H}$. We say that they are complementary subspaces if $A \cap B = \{0\}$ and $A + B = \mathcal{H}$. In this case we write $\mathcal{H} = A \dot{+} B$. Moreover, if $A \perp B$, then we write $\mathcal{H} = A \oplus B$. We write $\mathcal{H} = \dot{+}_{j\in I} S_j$, where I is a countable index set, if each S_j is a closed subspace of $\mathcal{H}$, $S_i \cap S_j = \{0\}$ $(i \neq j)$, and for each $f \in \mathcal{H}$ there exist $f_j \in S_j$ such that $f = \sum_j f_j$, where the sum converges in $\mathcal{H}$.

DEFINITION 2.7. An idempotent E is a bounded operator satisfying $E^2 = E$.

If E is an idempotent, then it has closed range and its range and kernel are complementary subspace of $\mathcal{H}$. Conversely if A and B are complementary subspaces of a Hilbert space $\mathcal{H}$, then there exists a unique idempotent whose kernel is A and whose range is B. If E is an idempotent, then so are E^* and $I - E$. See [7].

3. Frame multiresolution analysis

In this section we extend Feauveau's model of non-orthogonal multiresolution analysis [9].

DEFINITION 3.1. $(\{A_j\}_{j\in\mathbb{Z}}, \varphi)$ is said to be a frame multiresolution analysis (FMRA) if:

(1) $A_j : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an idempotent;
(2) $V_j \subset V_{j+1}$, $Z_{j+1} \subset Z_j$, where $V_j := \operatorname{ran} A_j$, $Z_j = \ker A_j$;
(3) $D(V_j) = V_{j+1}$, $D(Z_j) = Z_{j+1}$;
(4) $TA_0 = A_0T$;
(5) There exists an $\varphi \in V_0$ such that $\{T^n\varphi\}$ is a frame for V_0;
(6) $\overline{\cup V_j} = L^2(\mathbb{R})$, $\cap V_j = \{0\}$.

If each A_j is an orthogonal projection we say that it is an orthogonal frame multiresolution analysis (OFMRA).

REMARK 3.2. (3) is equivalent to $DA_j = A_{j+1}D$, and (4) implies that $T(V_0) = V_0$ and $T(Z_0) = Z_0$ (p. 157 of [9]). Notice that, for each $j \in \mathbb{Z}$, $A_j = D^jA_0D^{-j}$. Suppose $j \geq 0$. Then $A_jT = D^jA_0D^{-j}T = D^jA_0T^{2^j}D^{-j} = D^jT^{2^j}A_0D^{-j} = TD^jA_0D^{-j} = TA_j$ by (2.1). Therefore $A_jT = TA_j$ for each $j \geq 0$.

REMARK 3.3. We now show that (5) is equivalent to:

(5′) There exists a bounded operator $\xi_0 : V_0 \to l^2(\mathbb{Z})$ which is coercive, i.e., there is $\alpha > 0$ such that $\|\xi_0 f\| \geq \alpha\|f\|$ for each $f \in V_0$, and it satisfies the commutation relation $\xi_0T = \tau\xi_0$ on V_0.

Recall that an operator between Hilbert spaces is coercive if and only if its adjoint is surjective [14]. Suppose that (5′) holds. And let $\varphi = \xi_0^*e_0$. Notice that $T\xi_0^* = \xi_0^*\tau$ and that $\xi_0^* : \ell^2(\mathbb{Z}) \to V_0$ is surjective. $\xi_0^*e_n = \xi_0^*\tau^ne_0 = T^n\xi_0^*e_0 = T^n\varphi$. Hence by Theorem 2.2 ξ_0^* is the pre-frame operator of the frame $\{T^n\varphi : n \in \mathbb{Z}\}$ for V_0. Suppose, on the other hand, that $\{T^n\varphi : n \in \mathbb{Z}\}$ is a frame for V_0 for some $\varphi \in V_0$. Then by Theorem

2.2 there exists a surjective $\zeta_0 : \ell^2(\mathbb{Z}) \to V_0$ such that $\zeta_0 e_n = T^n\varphi$ for each integer n. Then, for each n, $T\zeta_0 e_n = TT^n\varphi = T^{n+1}\varphi = \zeta_0 e_{n+1} = \zeta_0 \tau e_n$. Hence $T\zeta_0 = \zeta_0\tau$. Since ζ_0^* is coercive, ζ_0^* is the desired operator.

In Feauveau's model of non-orthogonal MRA, ξ_0 in $(5')$ is assumed to be a topological isomorphism satisfying the commutation relation in $(5')$, which guarantees the existence of a Riesz basis of translates for V_0. Hence Definition 3.1 is a generalization of his model.

We call any $\varphi \in V_0$ whose integer translates generate a frame for V_0 a scaling function of the FMRA. It is easy to see that $\{D^jT^k\varphi\}_k$ is a frame for V_j for each j with the same frame bounds.

Notice that the definition of OFMRA can be given as follows [1]. $(\{V_j\}, \varphi)$ is an OFMRA if $\{V_j\}$ is a nested sequence of closed subspaces of $L^2(\mathbb{R})$ and $\varphi \in V_0$ such that: $\overline{\cup V_j} = L^2(\mathbb{R}), \cap V_j = \{0\}$; $D(V_j) = V_{j+1}$ for each j; $T(V_0) = V_0$; $\{T^n\varphi\}$ is a frame for V_0. A construction of such an OFMRA is relatively easy. We cite the following construction of an OFMRA which is Theorem 4.6 of [1] for the sake of later reference.

PROPOSITION 3.4. *Suppose $\varphi \in L^2(\mathbb{R})$. Define $V_j := \overline{\text{span}}\{D^jT^n\varphi\}_n$. If it satisfies the following properties, then $(\{V_j\}, \varphi)$ is an OFMRA:*

- *$|\hat{\varphi}|$ is bounded away from zero a.e. on a neighbourhood of zero;*
- *$\{T^n\varphi\}$ is a frame of V_0;*
- *There exists $H_0 \in L^\infty(\mathbb{T})$ such that $\hat{\varphi}(\gamma) = 1/\sqrt{2}H_0(\gamma/2)\hat{\varphi}(\gamma/2)$.*

Definition 3.1 has an added freedom of choosing the kernels Z_j's of A_j's other than $V_j^\perp$'s. We now consider the problem of how to exploit this freedom.

PROPOSITION 3.5. *Suppose that an OFMRA $(\{V_j\}, \varphi)$ is given. Let Z_0 be any complementary subspace of V_0 satisfying $T(Z_0) = Z_0$ and $D(Z_0) \subset Z_0$. Let A_0 be the unique idempotent satisfying* $\operatorname{ran} A_0 = V_0$ *and* $\ker A_0 = Z_0$. *Define $A_j := D^jA_0D^{-j}$ and $Z_j := D^j(Z_0)$. Then $(\{A_j\}, \varphi)$ forms an FMRA.*

PROOF. First notice that such Z_0 always exists. $V_0^\perp$ will do. (5) and (6) of Definition 3.1. are readily satisfied. By its form each A_j is an idempotent.

Obviously, $\operatorname{ran} A_j \subset D^j(V_0) = V_j$. If $f \in V_j = D^j(V_0)$, then there exists $g \in V_0$ such that $f = D^jg$. Hence $A_jf = A_jD^jg = D^jA_0D^{-j}D^jg = D^jg = f$. Therefore, $\operatorname{ran} A_j = V_j$. If $A_jf = 0$, then $D^jA_0D^{-j}f = 0$. So $D^{-j}f \in$

$\ker A_0 = Z_0$, since D^j is unitary. Hence $f \in D^j(Z_0) = Z_j$. If $f \in Z_j$, then there exists $g \in Z_0$ such that $f = D^j g$. So, $A_j f = D^j A_0 D^{-j} D^j g = D^j A_0 g = 0$. Hence $\ker A_j = Z_j$. The condition $D(Z_0) \subset Z_0$ implies that $Z_{j+1} = D^j D(Z_0) \subset D^j(Z_0) = Z_j$. So, the condition (2) follows.

By Remark 3.2 (3) follows from $DA_j = DD^j A_0 D^{-j} = D^{j+1} A_0 D^{-j-1} D = A_{j+1} D$.

It remains to check the condition (4). For any $f \in L^2(\mathbb{R})$, $f = f_1 + f_2$, where $f_1 \in V_0$ and $f_2 \in Z_0$. Notice that $Tf_1 \in V_0$ and $Tf_2 \in Z_0$. Hence $TA_0 f = Tf_1$ and $A_0 Tf = Tf_1$. So $TA_0 = A_0 T$. □

Throughout the rest of this section we assume that an FMRA $(\{A_j\}, \varphi)$ is given.

THEOREM 3.6. *$(\{A_j^*\}, A_0^* \varphi)$ is also an FMRA, called the dual FMRA. In this case, we let $V_j^* := \operatorname{ran} A_j^* = Z_j^\perp$ and $Z_j^* := \ker A_j^* = V_j^\perp$.*

PROOF. The verifications of (1) to (6) except (5) of Definition 3.1 are given in the proof of Theorem 2 in [9]. We show that (5) holds. Let $\zeta_0 = A_0^* \xi_0^*$, where ξ_0 is as in Remark 3.3. Then $\zeta_0 : \ell^2(\mathbb{Z}) \to V_0^*$. We show that ζ_0 is surjective. If $f \in V_0^*$, then there is $g \in L^2(\mathbb{R})$ such that $f = A_0^* g$. Decompose g such that $g = g_1 + g_2 \in V_0 \oplus V_0^\perp$. Notice that $V_0^\perp = Z_0^*$. Hence $f = A_0^* g = A_0^* g_1$. Since $g_1 \in V_0$ and ξ_0^* is surjective, there is $e \in \ell^2(\mathbb{Z})$ such that $g_1 = \xi_0^* e$. Hence $\zeta_0 e = f$. Hence by Theorem 2.2 $\{\zeta_0 e_n\}$ is a frame for V_0^*. Notice that $\zeta_0 \tau = A_0^* \xi_0^* \tau = A_0^* T \xi_0^* = T A_0^* \xi_0^* = T\zeta_0$. Therefore $\zeta e_n = \zeta \tau^n e_0 = T^n A_0^* \xi_0^* e_0 = T^n A_0^* \varphi$. Hence $\{T^n A_0^* \varphi\}$ is a frame for V_0^*. □

We collect some immediate consequences of Definition 3.1.

PROPOSITION 3.7. (1) *$\|A_j\| = \|A_k\|, \|I - A_j\| = \|I - A_k\|$, for any $j, k \in \mathbb{Z}$.*

(2) *$\|f - A_j f\| \to 0$ as $j \to \infty$ for any $f \in L^2(\mathbb{R})$.*

(3) *$A_j A_k = A_k A_j = A_{\min(j,k)}$,*
$(I - A_j)(I - A_k) = (I - A_k)(I - A_j) = I - A_{\max(j,k)}$.

(4) *$A_j f - A_k f \in Z_{\min(j,k)} \cap V_{\max(j,k)}$.*

(5) *$\cap Z_j = \{0\}$.*

(6) *$\overline{\cup Z_j} = L^2(\mathbb{R})$.*

PROOF. (1): For any j, $A_j = D^* A_{j+1} D$. Since D is unitary, $\|A_j\| = \|A_{j+1}\|$. Similarly, $I - A_j = D^*(I - A_{j+1})D$; so $\|I - A_j\| = \|I - A_{j+1}\|$.

(2): Let P_j be the orthogonal projection onto V_j. Then $\|f - P_j f\| \to 0$ as $j \to \infty$ and $\|P_j f\| \to 0$ as $j \to -\infty$ by (6) of Definition 3.1. Hence

$$\begin{aligned}\|f - A_j f\| &\leq \|f - P_j f\| + \|P_j f - A_j f\| \\ &= \|f - P_j f\| + \|A_j P_j f - A_j f\| \\ &\leq \|f - P_j f\| + \|A_j\| \|P_j f - f\| \\ &= (1 + \|A_j\|)\|P_j f - f\| \\ &= (1 + \|A_0\|)\|P_j f - f\| \\ &\to 0, \text{ as } j \to \infty \quad \text{by (1)}.\end{aligned}$$

(3): If $j > k$, then $A_j A_k f = A_k f$, since $V_k \subset V_j$. If $j < k$, then $f = A_k f + f - A_k f$. So, $A_j f = A_j(A_k f + f - A_k f) = A_j A_k f$ since $f - A_k f \in Z_k \subset Z_j$. Hence $A_j A_k = A_k A_j = A_{\min(j,k)}$. On the other hand, $(I - A_j)(I - A_k) = I - A_k - A_j + A_j A_k = I - A_{\max(j,k)}$. Since A_j and A_k commute, so do $I - A_j$ and $I - A_k$.

(4): If $j \geq k$, then $A_k(A_j f - A_k f) = 0$ by (3). So $A_j f - A_k f \in Z_k$. Clearly, it is in V_j since $V_k \subset V_j$.

(5): If $f \in \cap Z_j$, then $A_j f = 0$ for any j. Thus $\|f\| = \|f - A_j f\| \to 0$ as $j \to \infty$ by (2). Hence $f = 0$.

(6): If $f \in (\overline{\cup Z_j})^\perp = (\cup Z_j)^\perp = \cap Z_j^\perp = \cap V_j^*$, then $f = 0$ by Theorem 3.6.

□

We now consider bi-orthogonal wavelet decomposition of $L^2(\mathbb{R})$.

DEFINITION 3.8. For each $j \in \mathbb{Z}$, let $W_j := V_{j+1} \cap Z_j$ and $W_j^* := V_{j+1}^* \cap Z_j^*$.

These are closed subspaces of $L^2(\mathbb{R})$. If A_j is an orthogonal projection, then $Z_j = V_j^\perp$. Hence $W_j = V_{j+1} \ominus V_j :=$ the orthogonal complement of V_j in V_{j+1}.

PROPOSITION 3.9. (1) $V_{j+1} = V_j \dot{+} W_j$ *and* $V_{j+1}^* = V_j^* \dot{+} W_j^*$.

(2) $Z_j = W_j \dot{+} Z_{j+1}$ *and* $Z_j^* = W_j^* \dot{+} Z_{j+1}^*$.

(3) $W_j \cap W_k = \{0\}$ *and* $W_j^* \cap W_k^* = \{0\}$ *if* $j \neq k$.

(4) $W_j = (I - A_j)(V_{j+1})$ *and* $W_j^* = (I - A_j^*)(V_{j+1}^*)$.

(5) $W_j = D^j(W_0)$ *and* $W_j^* = D^j(W_0^*)$.

PROOF. By duality we only need to prove the first part of each statement.

(1): For any $f \in V_{j+1}$, $f = A_j f + (I - A_j)f$. So $(I - A_j)f = f - A_j f \in Z_j \cap V_{j+1} = W_j$. Since $W_j \cap V_j = \{0\}$, $V_{j+1} = V_j \dot{+} W_j$.

(2): For $f \in Z_j$ $f = A_{j+1}f + (I - A_{j+1})f$. So $A_{j+1}f = f - (I - A_{j+1})f \in V_{j+1} \cap Z_j = W_j$. Since $W_j \cap Z_{j+1} = \{0\}$, $Z_j = W_j \dot{+} Z_{j+1}$.

(3): If $j < k$, then $W_j \subset V_{j+1}$ and $W_k \subset Z_k \subset Z_{j+1}$. Hence $W_j \cap W_k = \{0\}$.

(4): Trivially, $(I - A_j)(V_{j+1}) \subset V_{j+1} \cap Z_j = W_j$. Suppose that $f \in W_j = V_{j+1} \cap Z_j = V_{j+1} \cap \mathrm{ran}(I - A_j)$. Then there exists $g \in L^2(\mathbb{R})$ such that $f = (I - A_j)g$. Then $f = A_{j+1}f = A_{j+1}(I - A_j)g = A_{j+1}g - A_{j+1}A_j g = A_{j+1}g - A_j A_{j+1} g = (I - A_j)(A_{j+1}g) \in (I - A_j)V_{j+1}$.

(5): $f \in W_j$ if and only if there exists $g \in V_{j+1}$ such that $f = (I - A_j)g$ by (4). Hence $D^{-j}f = D^{-j}(I - A_j)g = (I - A_0)D^{-j}g \in W_0$. Therefore, $W_j = D^j(W_0)$. □

LEMMA 3.10. *$L^2(\mathbb{R}) = \dot{+}W_{j\in\mathbb{Z}}$ if and only if for any $f \in L^2(\mathbb{R})$ $\|A_j f\| \to 0$ as $j \to -\infty$.*

PROOF. We prove the 'if' part of the statement. Fix $f \in L^2(\mathbb{R})$. Then

$$\begin{aligned} f &= A_0 f + (I - A_0)f \\ &= A_{-1}A_0 f + (I - A_{-1})A_0 f + A_1(I - A_0)f + (I - A_1)(I - A_0)f \\ &= A_{-1}f + (A_0 - A_{-1})f + (A_1 - A_0)f + (I - A_1)f \\ &= \cdots \\ &= A_{-k}f + \sum_{j=-k+1}^{n} (A_j - A_{j-1})f + (I - A_n)f. \end{aligned}$$

By Proposition 3.7, $(A_j - A_{j-1})f \in W_j$ and $\|(I - A_n)f\| \to 0$ as $n \to \infty$. Hence the series $\sum_{j=-k+1}^{n}(A_j - A_{j-1})f$ converges to f as $k, n \to \infty$.

Now suppose $f = \sum_j f_j = \sum_j g_j$, where the sum converges in the norm topology and f_j and g_j are in W_j. Then $0 = \sum_j (f_j - g_j)$. So for any k, $0 = A_k(\sum_j (f_j - g_j)) = \sum_j A_k(f_j - g_j) = \sum_{j<k} A_k(f_j - g_j) = \sum_{j<k}(f_j - g_j)$, since $W_j = Z_j \cap V_{j+1}$. Hence $0 = (A_{k+1} - A_k)(\sum_j (f_j - g_j)) = f_k - g_k$. So the series expansion is unique.

We now prove the 'only if' part of the statement. Let $f \in L^2(\mathbb{R})$ be arbitrary. Then we have a unique expansion $f = \sum_j f_j$ with $f_j \in W_j$. Then $A_k f_j = 0$ for each $j \geq k$. Hence $A_k f = A_k(\sum_{|j|>|k|} f_j)$ if $k < 0$. So

$\|A_k f\| \le \|A_k\| \, \| \sum_{|j|>|k|} f_j \| = \|A_0\| \, \| \sum_{|j|>|k|} f_j \| \to 0$ as $k \to -\infty$, since the series converges. □

Later, we will see that the condition that $A_j f \to 0$ as $j \to -\infty$ for each f holds in any FMRA.

The following proposition shows the existence of a dual scaling function, and thereby generalizes Proposition 9 of [9].

PROPOSITION 3.11. *There exists $\varphi^* \in V_0^*$, called a dual scaling function, such that, for each j,*

$$A_j f = \sum_k \langle f, D^j T^k \varphi^* \rangle D^j T^k \varphi \quad \text{and}$$
$$A_j^* f = \sum_k \langle f, D^j T^k \varphi \rangle D^j T^k \varphi^* .$$

Moreover, $\{D^j T^k \varphi^* : k \in \mathbb{Z}\}$ is a frame for V_j^*, for each j.

PROOF. Let $S_0 : V_0 \to V_0$ be the frame operator with respect to the frame $\{T^n \varphi\}$. Then $S_0 T = T S_0$ on V_0 by Proposition 4.7 in [1]. Hence $T S_0^{-1} = S_0^{-1} T$ on V_0. Then, since $A_0 f \in V_0$,

$$\begin{aligned} A_0 f &= \sum_n \langle A_0 f, S_0^{-1} T^n \varphi \rangle T^n \varphi \\ &= \sum_n \langle A_0 f, T^n S_0^{-1} \varphi \rangle T^n \varphi \\ &= \sum_n \langle f, A_0^* T^n S_0^{-1} \varphi \rangle T^n \varphi \\ &= \sum_n \langle f, T^n A_0^* S_0^{-1} \varphi \rangle T^n \varphi \\ &= \sum_n \langle f, T^n \varphi^* \rangle T^n \varphi , \end{aligned}$$

where $\varphi^* := A_0^* S_0^{-1} \varphi$. By Remark 3.2 $A_j = D^j A_0 D^{-j}$ for any $j \in \mathbb{Z}$. Hence

$$\begin{aligned} A_j f &= D^j A_0 D^{-j} f \\ &= D^j \Big(\sum_n \langle D^{-j} f, T^n \varphi^* \rangle T^n \varphi\Big) \\ &= \sum_n \langle f, D^j T^n \varphi^* \rangle D^j T^n \varphi. \end{aligned}$$

Notice that $\{T^n \varphi^* = T^n A_0^* S_0^{-1} \varphi = A_0^* S_0^{-1} T^n \varphi\} \subset V_0^*$ is a Bessel sequence for $L^2(\mathbb{R})$ (hence for V_0^*) by Lemma 2.4.

Hence the operator $B_0 g := \sum_n \langle g, T^n \varphi \rangle T^n \varphi^*$ is bounded by Lemma 2.4 and Definition 2.1. For each $f, g \in L^2(\mathbb{R})$,

$$\begin{aligned} \langle A_0 f, g \rangle &= \Big\langle \sum_n \langle f, T^n \varphi^* \rangle T^n \varphi, g \Big\rangle \\ &= \sum_n \langle f, T^n \varphi^* \rangle \langle T^n \varphi, g \rangle \\ &= \Big\langle f, \sum_n \langle g, T^n \varphi \rangle T^n \varphi^* \Big\rangle \\ &= \langle f, B_0 g \rangle. \end{aligned}$$

Therefore $A_0^* g = \sum_n \langle g, T^n \varphi \rangle T^n \varphi^*$. Since $A_j^* = D^j A_0^* D^{-j}$, we have, as above, that $A_j^* g = \sum_n \langle g, D^j T^n \varphi \rangle D^j T^n \varphi^*$. For any $g \in V_0^*$, $g = A_0^* g = \sum_n \langle g, T^n \varphi \rangle T^n \varphi^*$. Hence $\{T^n \varphi^*\}$ is a Bessel sequence that ℓ^2-generates V_0^*. Therefore, it is a frame for V_0^* by Lemma 2.5. By a simple dilation argument $\{D^j T^n \varphi^*\}$ is a frame for V_j^*. □

Using the previous proposition we have a decomposition of $L^2(\mathbb{R})$ via W_j's.

PROPOSITION 3.12. *$L^2(\mathbb{R}) = \dot{+}_{j \in \mathbb{Z}} W_j = \dot{+}_{j \in \mathbb{Z}} W_j^*$. In particular, for any $f \in L^2(\mathbb{R})$, $f = \sum_j (A_{j+1} - A_j) f = \sum_j (A_{j+1}^* - A_j^*) f$, where the sum converges in $L^2(\mathbb{R})$.*

PROOF. According to Lemma 3.10 it is enough to show that $A_j f \to 0$ as $j \to -\infty$ for each f. Since simple functions are dense in $L^2(\mathbb{R})$ and $\|A_j\| = \|A_0\|$ for any j, it is enough to show that the convergence condition holds for any simple functions f. By linearity we may assume that f is a characteristic function $\chi_{[a,b]}$. Since $\{T^k \varphi\}$ is a frame for V_0, there exist

$0 < A \leq B$ such that $A\|f\|^2 \leq \sum_k |\langle f, T^k\varphi\rangle|^2 \leq B\|f\|^2$ for any $f \in V_0$. Let $j > 0$.

$$\begin{aligned}
\|A_{-j}f\|^2 &= \|D^jA_{-j}f\|^2 = \|A_0D^jf\|^2 \\
&\leq \frac{1}{A}\sum_k |\langle A_0D^jf, T^k\varphi\rangle|^2 \\
&= \frac{1}{A}\sum_k |\langle D^jf, A_0^*T^k\varphi\rangle|^2 \\
&= \frac{1}{A}\sum_k |\langle D^jf, T^kA_0^*\varphi\rangle|^2 \\
&= \frac{1}{A}\sum_k |\langle f, D^{-j}T^kA_0^*\varphi\rangle|^2 \\
&= \frac{1}{A}\sum_k \left|\int_a^b 2^{-j/2}\overline{(A_0^*\varphi)(2^{-j}t-k)}\,dt\right|^2 \\
&\leq \frac{1}{A}\sum_k (b-a)\int_a^b 2^{-j}|(A_0^*\varphi)(2^{-j}t-k)|^2\,dt \\
&= \frac{b-a}{A}\sum_k \int_{2^{-j}a-k}^{2^{-j}b-k} |(A_0^*\varphi)(t)|^2\,dt \\
&\to 0 \text{ as } j \to \infty.
\end{aligned}$$

The last convergence follows by the dominated convergence theorem.

The remaining statements hold by the argument in the proof of Lemma 3.10 and by duality. □

PROPOSITION 3.13. *For any j, $V_j = \dot{+}_{k<j}W_k$ and $Z_j = \dot{+}_{k\geq j}W_k$.*

PROOF. Suppose $f \in V_j$. By Proposition 3.12 $f = \sum_k(A_{k+1} - A_k)f$. Then $f = A_jf = A_j\sum_k(A_{k+1}-A_k)f = \sum_{k<j}(A_{k+1}-A_k)f$, where we used Proposition 3.7 (3). Since $(A_{k+1} - A_k)f \in W_k$ by Proposition 3.7 (4) and Definition 3.8, $V_j \subset \dot{+}_{k<j}W_k$. On the other hand, if $f = \sum_k(A_{k+1}-A_k)f \in Z_j$, then $\sum_{k<j}(A_{k+1} - A_k)f = 0$ according to the previous calculation. Hence $f = \sum_{k\geq j}(A_{k+1} - A_k)f$. This shows that $Z_j \subset \dot{+}_{k\geq j}W_k$. By definition $\dot{+}_{k<j}W_k \subset V_j$ and $\dot{+}_{k\geq j}W_k \subset Z_j$. □

Under the assumption that a frame of translates for the initial difference space W_0 exists we have the following 'weak' wavelet expansion theorem.

PROPOSITION 3.14. *If there exists $\psi \in W_0$ such that $\{T^n\psi\}$ is a frame for W_0, then there exists $\psi^* \in W_0^*$ such that, for each j,*

$$(3.1) \qquad (A_{j+1} - A_j)f = \sum_n \langle f, D^jT^n\psi^*\rangle D^jT^n\psi, \qquad \text{and}$$

$$(A_{j+1}^* - A_j^*)f = \sum_n \langle f, D^jT^n\psi\rangle D^jT^n\psi^*,$$

where the sum converges in $L^2(\mathbb{R})$. Moreover, for each $j \in \mathbb{Z}$, $\{D^jT^k\psi : k \in \mathbb{Z}\}$ is a frame for W_j and $\{D^jT^k\psi^* : k \in \mathbb{Z}\}$ is a frame for W_j^*. Consequently, for any $f \in L^2(\mathbb{R})$,

$$f = \sum_j \sum_n \langle f, D^jT^n\psi^*\rangle D^jT^n\psi = \sum_j \sum_n \langle f, D^jT^n\psi\rangle D^jT^n\psi^*,$$

where the iterated sum, not the double sum, converges in $L^2(\mathbb{R})$.

PROOF. Notice that $T(W_0) = W_0$ by assumption. Let $S_{W_0} : W_0 \to W_0$ be the frame operator with respect to $\{T^n\psi\}$. Then exactly by the same argument as in Proposition 4.7 of [1] $S_{W_0}T = TS_{W_0}$ on W_0. Hence $S_{W_0}T^{-1} = T^{-1}S_{W_0}$ on W_0. By Remark 3.2 $A_1 - A_0$ commutes with T and so does $(A_1 - A_0)^*$. Hence

$$\begin{aligned}(A_1 - A_0)f &= \sum_n \langle (A_1 - A_0)f, S_{W_0}^{-1}T^n\psi\rangle T^n\psi \\ &= \sum_n \langle (A_1 - A_0)f, T^nS_{W_0}^{-1}\psi\rangle T^n\psi \\ &= \sum_n \langle f, (A_1 - A_0)^*T^nS_{W_0}^{-1}\psi\rangle T^n\psi \\ &= \sum_n \langle f, T^n(A_1 - A_0)^*S_{W_0}^{-1}\psi\rangle T^n\psi \\ &= \sum_n \langle f, T^n\psi^*\rangle T^n\psi,\end{aligned}$$

where $\psi^* := (A_1 - A_0)^*S_{W_0}^{-1}\psi \in W_0^*$. Therefore,

$$\begin{aligned}(A_{j+1} - A_j)f &= D^j(A_1 - A_0)D^{-j}f \\ &= D^j \sum_n \langle D^{-j}f, T^n\psi^*\rangle T^n\psi \\ &= \sum_n \langle f, D^jT^n\psi^*\rangle D^jT^n\psi.\end{aligned}$$

By repeating the argument in the proof of Proposition 3.11 we see that $(A_{j+1}^* - A_j^*)f = \sum_n \langle f, T^n\psi\rangle T^n\psi^*$, and that $\{T^n\psi^*\}$ is a frame for W_0^*. A dilation argument shows that $\{D^jT^n\psi\}$ forms a frame for W_j, since $W_j = D^j(W_0)$ by Proposition 3.9. By duality $\{D^jT^n\psi^*\}$ forms a frame for W_j^*. The convergence of the iterated sum follows by Lemma 3.10 and Proposition 3.12. □

The following proposition gives a condition under which the above 'weak' wavelet expansion can be a genuine wavelet expansion.

PROPOSITION 3.15. *If $\{T^k\psi\}$ is a frame for W_0 and if $\{D^jT^k\psi : j, k \in \mathbb{Z}\}$ and $\{D^jT^k\psi^* : j, k \in \mathbb{Z}\}$ are both Bessel sequences of $L^2(\mathbb{R})$, where ψ^* is any element in $L^2(\mathbb{R})$ that satisfies* (3.1), *then they are frames of $L^2(\mathbb{R})$.*

PROOF. See the proof of Theorem 5 of [6]. □

See [5] for characterizations of Bessel sequences of the type $\{D^jT^k\psi\}$ via decay and oscillation of ψ.

REMARK 3.16. Suppose that we are given an OFMRA. So $A_j = A_j^* = P_j$, an orthogonal projection. If there exists $\psi \in W_0$ such that $\{T^n\psi\}$ forms a frame for W_0, then $W_j = W_j^*$ and $L^2(\mathbb{R}) = \oplus_j W_j$. Let $\psi^* := (P_1 - P_0)S_{W_0}^{-1}\psi$ as in the proof of Proposition 3.14. Then by Theorem 5.11 of [1] and Proposition 3.14 $\{D^jT^k\psi : j, k \in \mathbb{Z}\}$ and $\{D^jT^k\psi^* : j, k \in \mathbb{Z}\}$ are frames of $L^2(\mathbb{R})$. Hence, for each $f \in L^2(\mathbb{R})$, $f = \sum_{j,k}\langle f, D^jT^k\psi^*\rangle D^jT^k\psi = \sum_{j,k}\langle f, D^jT^k\psi\rangle D^jT^k\psi^*$, where the double sum converges in $L^2(\mathbb{R})$.

Suppose, on the other hand, that we are given an FMRA and that the integer translates of a scaling function φ is a Riesz basis for V_0. (So we are given Feauveau's non-orthogonal MRA). Then there always exists $\psi \in W_0$ such that $\{T^n\psi\}$ is a Riesz basis of W_0 (Theorem 13 [9]).

Given an FMRA a natural question arises whether there exists $\psi \in W_0$ such that $\{T^n\psi : n \in \mathbb{Z}\}$ is a frame for W_0. The answer is negative as will be seen by Example 3.21, thereby exhibiting a great constrast to the ordinary bi-orthogonal MRA case. See Remark 3.16. Benedetto and Li gave a sufficient condition for the case of an OFMRA (Theorem 5.6 of [1]). Kim and Lim [12] and Bendetto and Treiber [2] found the equivalent conditions for the existence of wavelets in an OFMRA, independently, with different methods. Kim and Lim ([12]) provide an explicit construction of the wavelet of an OFMRA, if it exists, and prove that an OFMRA always has $\psi_1, \psi_2 \in V_1$ such that $\{T^n\psi_1, T^l\psi_2 : n, l \in \mathbb{Z}\}$ is a frame for W_0 and constructs such ψ_1 and ψ_2 explicitly. We now give a simple sufficient condition for an FMRA to have a frame of translates for W_0. Since $V_0 \subset V_1$ and since $\{DT^n\varphi\}$ is a frame for V_0, there exists $H \in L^2(\mathbb{T})$ such that $\hat{\varphi}(\gamma) = 1/\sqrt{2}H(\gamma)\hat{\varphi}(\gamma/2)$. This H is not unique, reflecting the non-uniqueness of the coefficients in a frame expansion. We can find, however, at least one canonical H in the following manner.

Lemma 3.17. *Let $\Phi(\gamma) := \sum_n |\hat{\varphi}(\gamma+n)|^2$, $N :=$ the zero set of Φ in $\mathbb{T}$ modulo measure zero sets and $M := \mathbb{T} \setminus N$. Let $H(\gamma) := \sqrt{2}(\sum_l \hat{\varphi}(2\gamma + 2l)\overline{\hat{\varphi}(\gamma+l)})\cdot(1/\Phi(\gamma))\cdot\chi_M(\gamma)$. Then $H \in L^\infty(\mathbb{T})$ and $\hat{\varphi}(\gamma) = 1/\sqrt{2}H(\gamma/2)$ $\hat{\varphi}(\gamma/2)$.*

Proof. First notice that by Theorem A.3 there exist positive constants A and B such that $A \le \Phi \le B$ on M. Hence by Cauchy-Schwarz inequality

$$\begin{aligned}
H(\gamma) &\le \sqrt{2}(\sum_l |\hat{\varphi}(2\gamma+2l)\overline{\hat{\varphi}(\gamma+l)})|)\frac{1}{\Phi(\gamma)}\chi_M(\gamma) \\
&\le \sqrt{2}\Phi(2\gamma)^{1/2}\Phi(\gamma)^{1/2}\frac{1}{\Phi(\gamma)}\chi_M(\gamma) \\
&\le (2\frac{B}{A})^{1/2}
\end{aligned}$$

Hence $H \in L^\infty(\mathbb{T})$.

Let S_1 be the frame operator with respect to a frame $\{DT^n\varphi\}$ for V_1. Then, for $f \in V_1$, $S_1 f = \sum_n \langle f, DT^n\varphi\rangle DT^n\varphi = D\sum_n\langle D^{-1}f, T^n\varphi\rangle T^n\varphi = DS_0D^{-1}f$, where S_0 is the frame operator with respect to a frame $\{T^n\varphi\}$ for V_0. Since $D^{-1}(V_1) = V_0$, $S_1 = DS_0D^{-1}$ on V_1. Hence $S_1^{-1} = DS_0^{-1}D^{-1}$

on V_1. So $S_1^{-1}D = DS_1^{-1}$ on V_0. Since $\varphi \in V_0 \subset V_1$,

$$\begin{aligned}\varphi &= \sum_n \langle \varphi, S_1^{-1} DT^n \varphi\rangle DT^n \varphi \\ &= D\sum_n \langle \varphi, DS_0^{-1} T^n \varphi\rangle T^n \varphi \\ &= D\sum_n \langle \varphi, DT^n S_0^{-1} \varphi\rangle T^n \varphi.\end{aligned}$$

Hence $\hat{\varphi}(\gamma) = 1/\sqrt{2}(\sum_n \langle \varphi, DT^n S_0^{-1}\varphi\rangle e^{-2\pi i n\gamma/2})\hat{\varphi}(\gamma/2)$. If we let $H(\gamma) := \sum_n \langle \varphi, DT^n S_0^{-1}\varphi\rangle e^{-2\pi i n\gamma}$, then $\hat{\varphi}(\gamma) = 1/\sqrt{2}H(\gamma/2)\hat{\varphi}(\gamma/2)$.

By Theorem 4.8 of [1] $\widehat{S_0^{-1}\varphi}(\gamma) = \hat{\varphi}(\gamma)\cdot 1/\Phi(\gamma)\cdot \chi_{\tilde{M}}(\gamma)$, where $\tilde{M} := \cup_{n\in\mathbb{Z}} M + n$. Therefore

$$\begin{aligned}H(\gamma) &= \sum_n \langle \varphi, DT^n S_0^{-1}\varphi\rangle e^{-2\pi i n\gamma} \\ &= \sum_n \langle \widehat{D^{-1}\varphi}, \widehat{T^n S_0^{-1}\varphi}\rangle e^{-2\pi i n\gamma} \\ &= \sum_n (\sqrt{2}\int_{\mathbb{R}} \hat{\varphi}(2\xi)\frac{\overline{\hat{\varphi}(\xi)}}{\Phi(\xi)}\chi_{\tilde{M}}(\xi) e^{2\pi i n\xi}\, d\xi) e^{-2\pi i n\gamma} \\ &= \sum_n (\sqrt{2}\sum_l \int_l^{l+1} \hat{\varphi}(2\xi)\frac{\overline{\hat{\varphi}(\xi)}}{\Phi(\xi)}\chi_{\tilde{M}}(\xi) e^{2\pi i n\xi}\, d\xi) e^{-2\pi i n\gamma} \\ &= \sum_n (\sqrt{2}\sum_l \int_0^{1} \hat{\varphi}(2\xi+2l)\frac{\overline{\hat{\varphi}(\xi+l)}}{\Phi(\xi)}\chi_{M}(\xi) e^{2\pi i n\xi}\, d\xi) e^{-2\pi i n\gamma} \\ &= \sum_n (\sqrt{2}\int_0^{1} (\sum_l \hat{\varphi}(2\xi+2l)\overline{\hat{\varphi}(\xi+l)})\frac{\chi_M(\xi)}{\Phi(\xi)} e^{2\pi i n\xi}\, d\xi) e^{-2\pi i n\gamma},\end{aligned}$$

where we used the dominated convergence theorem, whose use is guaranteed by the first part of this proof, in the last equality. Therefore by Plancherel theorem we have $H(\gamma) = \sqrt{2}(\sum_l \hat{\varphi}(2\gamma+2l)\overline{\hat{\varphi}(\gamma+l)})\cdot(1/\Phi(\gamma))\cdot \chi_M(\gamma)$. □

We need the following lemma which is proved in the proof of Theorem 5.4 of [1].

LEMMA 3.18. *Suppose $f \in L^2(\mathbb{R})$. If $\sum_n |f(t+n)|^2 \in L^\infty(\mathbb{T})$ and $(a_n) \in \ell^2(\mathbb{Z})$, then $g(t) := (\sum a_n e^{2\pi int}) f(t) \in L^2(\mathbb{R})$ and $g(t) = \sum a_n e^{2\pi int} f(t)$, where the convergence is in $L^2(\mathbb{R})$.*

THEOREM 3.19. *Let $\psi := (I - A_0)D\varphi$. If there exists $(a_n) \in A'(\mathbb{Z})$ such that*

$$(I - A_0)DT\varphi = \sum_n a_n T^n \psi, \tag{3.2}$$

where the convergence is in $L^2(\mathbb{R})$, then $\{T^n\psi : n \in \mathbb{Z}\}$ is a frame for W_0.

PROOF. By (4) of Proposition 3.9 $\psi \in W_0$. For any $k \in \mathbb{Z}$, $T^k\psi = T^k(I - A_0)D\varphi = (I - A_0)T^k D\varphi = (I - A_0)DT^{2k}\varphi$. Therefore $\{T^n\psi\}$ is a Bessel sequence for $L^2(\mathbb{R})$ by Lemma 2.4, since it is a subsequence of a Bessel sequence. Suppose that $f \in W_0$. Then by (4) of Proposition 3.9 there exists $(b_n) \in \ell^2(\mathbb{Z})$ such that

$$\begin{aligned}
f &= (I - A_0)(\sum_n b_n DT^n\varphi) \\
&= \sum_n b_n (I - A_0)DT^n\varphi \\
&= \sum_n b_{2n}(I - A_0)DT^{2n}\varphi + \sum_n b_{2n+1}(I - A_0)DT^{2n+1}\varphi \\
&= \sum_n b_{2n}T^n(I - A_0)D\varphi + \sum_n b_{2n+1}T^n(I - A_0)DT\varphi \\
&= \sum_n b_{2n}T^n\psi + \sum_n b_{2n+1}T^n(I - A_0)DT\varphi.
\end{aligned}$$

Let $g := \sum_n b_{2n+1}T^n(I - A_0)DT\varphi$. Then, by (3.2),

$$\begin{aligned}
\hat{g}(\gamma) &= (\sum_n b_{2n+1}e^{-2\pi in\gamma})(\sum_k a_k e^{-2\pi ik\gamma})\hat{\psi}(\gamma) \\
&= B_{odd}(\gamma)A(\gamma)\hat{\psi}(\gamma) \\
&= C(\gamma)\hat{\psi}(\gamma),
\end{aligned}$$

where $B_{odd}(\gamma) := \sum_n b_{2n+1}e^{-2\pi in\gamma} \in L^2(\mathbb{T})$, $A(\gamma) := \sum_k a_k e^{-2\pi ik\gamma} \in L^\infty(\mathbb{T})$, and, consequently, $C(\gamma) := B_{odd}(\gamma)A(\gamma) \in L^2(\mathbb{T})$. Let $(c_n) \in \ell^2(\mathbb{Z})$ be the Fourier coefficients of $C(\gamma)$. Then $C(\gamma) = \sum_n c_n e^{-2\pi in\gamma}$ in $L^2(\mathbb{T})$. Since

$\{T^n\psi\}$ is a Bessel sequence, $\sum_n |\hat{\psi}(\gamma+n)|^2 \in L^\infty(\mathbb{T})$ by Corollary A.4. Then by Lemma 3.18 we see that $\sum_n c_n e^{-2\pi i n\gamma}\hat{\psi}(\gamma)$ converges in $L^2(\mathbb{R})$. Hence by taking the inverse Fourier transform of $\hat{g}$, $g = \sum_n c_n T^n\psi$. Hence there exists $(d_n) \in \ell^2(\mathbb{Z})$ such that $f = \sum_n d_n T^n\psi$. Therefore $\{T^n\psi\}$ ℓ^2-generates W_0. Thus $\{T^n\psi\}$ is a frame for W_0 by Lemma 2.5. □

We now give a sufficient condition for (3.2), and hence a sufficient condition for the existence of a (frame) wavelet.

THEOREM 3.20. *Suppose* $H \in L^\infty(\mathbb{T})$ *satisfies*

$$|H(\gamma) - H(\gamma+1/2)| \geq \epsilon \quad \text{a.e. } \mathbb{T} \tag{3.3}$$

for some positive ϵ, *where* H *satisfies* $\hat{\varphi}(\gamma) = 1/\sqrt{2}H(\gamma/2)\hat{\varphi}(\gamma/2)$. *If we define*

$$A(\gamma) := \frac{e^{-\pi i\gamma}(H(\gamma/2) + H(\gamma/2+1/2))}{H(\gamma/2) - H(\gamma/2+1/2)}, \tag{3.4}$$

then the Fourier coefficients of A *satisfy (3.2).*

PROOF. By construction $A \in L^\infty(\mathbb{T})$. Hence its Fourier coefficients are in $A'(\mathbb{Z})$. Notice that (3.2) holds if and only if

$$\begin{aligned}(I - A_0)DT\varphi &= \sum_n a_n T^n (I - A_0)D\varphi \\ &= \sum_n a_n (I - A_0)DT^{2n}\varphi \\ &= (I - A_0)\sum_n a_n DT^{2n}\varphi. \qquad (3.5)\end{aligned}$$

This holds if and only if $DT\varphi - \sum_n a_n DT^{2n}\varphi \in \ker(I - A_0) = \operatorname{ran} A_0 = V_0$. Since $\{T^n\varphi\}$ is a frame for V_0, (3.2) holds if and only if there exists $(b_n) \in \ell^2(\mathbb{Z})$ such that

$$DT\varphi - \sum_n a_n DT^{2n}\varphi = \sum_n b_n T^n\varphi. \tag{3.6}$$

By taking the Fourier transform of (3.6), (3.2) holds if and only if

$$(1/\sqrt{2})e^{-\pi i\gamma}\hat{\varphi}(\gamma/2) - (1/\sqrt{2})\sum_n a_n e^{-2\pi i n\gamma}\hat{\varphi}(\gamma/2) = \sum_n b_n e^{-2\pi i n\gamma}\hat{\varphi}(\gamma)$$

$$= (1/\sqrt{2}) \sum_n b_n e^{-2\pi i n\gamma} H(\gamma/2)\hat{\varphi}(\gamma/2).$$

Hence (3.2) holds if and only if there exist $A \in L^\infty(\mathbb{T})$ and $B \in L^2(\mathbb{T})$ such that

$$(e^{-\pi i\gamma} + A(\gamma))\hat{\varphi}(\gamma/2) = B(\gamma)H(\gamma/2)\hat{\varphi}(\gamma/2).$$

This holds if (if and only if when the support of $\hat{\varphi} = \mathbb{R}$)

$$e^{-\pi i\gamma} + A(\gamma) = B(\gamma)H(\gamma/2) \quad \text{a.e. } \mathbb{R}. \tag{3.7}$$

Since A should be 1-periodic,

$$\begin{aligned} A(\gamma) &= -e^{-\pi i\gamma} + B(\gamma)H(\gamma/2) \\ &= A(\gamma+1) \\ &= e^{-\pi i\gamma} + B(\gamma)H(\gamma/2+1/2). \end{aligned}$$

Thus B should satisfy the relation:

$$B(\gamma)(H(\gamma/2) - H(\gamma/2+1/2)) = 2e^{-\pi i\gamma}.$$

That is,

$$B(\gamma) := \frac{2e^{-\pi i\gamma}}{H(\gamma/2) - H(\gamma/2+1/2)}.$$

Then $B \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$. It is easy to check that B is 1-periodic. Hence

$$\begin{aligned} A(\gamma) &= -e^{-\pi i\gamma} + B(\gamma)H(\gamma/2) \\ &= -e^{-\pi i\gamma} + \frac{2e^{-\pi i\gamma}}{H(\gamma/2) - H(\gamma/2+1/2)}H(\gamma/2) \\ &= \frac{e^{-\pi i\gamma}(H(\gamma/2) + H(\gamma/2+1/2))}{H(\gamma/2) - H(\gamma/2+1/2)}. \end{aligned}$$

It is now easy to see that $A \in L^\infty(\mathbb{T})$, that A is 1-periodic, and that A satisfies (3.7). □

EXAMPLE 3.21. We first give an example of an OFMRA such that W_0 has no frames of translates, and then generalize it to an FMRA. Let $\hat{\varphi}(\gamma) = \chi_{[-a,a]}(\gamma)$, $1/4 < a < 1/3$. By Proposition 3.4 and Theorem A.3 it is easy to see that $(\{V_j\}, \varphi)$ forms an OFMRA, with $V_j := \overline{\text{span}}\{D^j T^n \varphi\}_n$. For example, H can be chosen as $H(\gamma) = \sqrt{2}\chi_{[-a/2,a/2]}(\gamma) + \alpha(\gamma)\chi_{[-1/2,-a]\cup[a,1/2]}(\gamma)$, where α is an arbitrary bounded function. Then $V_0 = PW_{[-a,a]}$ and $V_1 = PW_{[-2a,2a]}$, where PW denotes a Paley-Wiener space such that $PW_S := \{f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subset S \subset \mathbb{R}\}$. Hence $W_0 = V_1 \ominus V_0 = PW_{[-2a,-a]\cup[a,2a]}$. Suppose there exists $\psi \in W_0$ whose integer translates form a frame for W_0. Then clearly $\text{supp}(\hat{\psi}) = [-2a,-a] \cup [a,2a]$, and $W_0 = \{f : \hat{f}(\gamma) = c(\gamma)\hat{\psi}(\gamma), c \in L^2([-1/2,1/2]),\ \text{1-periodic}\}$. Since $\check{\chi}_{[-2a,-a]\cup[a,2a]} \in PW_{[-2a,-a]\cup[a,2a]}$, there exists a 1-periodic c such that $\chi_{[-2a,-a]\cup[a,2a]}(\gamma) = c(\gamma)\hat{\psi}(\gamma)$. Notice that $[-2a, 2a-1] \subset [-2a,-a]\cup[a,2a]$ and $[-2a, 2a-1]+1 = [-2a+1, 2a] \subset [-2a,-a]\cup[a,2a]$, since $a < 1/3$. Hence, on $[-2a, 2a-1]$, $c(\gamma) = 1/\hat{\psi}(\gamma) = c(\gamma+1) = 1/\hat{\psi}(\gamma+1)$. Let $\hat{f}(\gamma) := \chi_{[-2a,2a-1]}(\gamma) - \chi_{[-2a+1,2a]}(\gamma)$. Since $f \in W_0$, there exists 1-periodic b such that $\hat{f}(\gamma) = b(\gamma)\hat{\psi}(\gamma)$. If $\gamma \in [-2a, 2a-1]$, then $1 = \hat{f}(\gamma) = b(\gamma)\hat{\psi}(\gamma) = b(\gamma+1)\hat{\psi}(\gamma+1) = \hat{f}(\gamma+1) = -1$. This contradiction shows that W_0 has no frames of translates of a single function.

Actually we have a stronger result: If $(\{A_j\}, \varphi)$ is an FMRA such that $\text{ran}\, A_j = V_j := PW_{[-2^j a, 2^j a]}$ with $1/4 < a < 1/3$, then W_0 has no frames of translates of a single function.

First notice that $W_0 = V_1 \cap \ker A_0$ by definition. Then $V_1 = PW_{[-2a,2a]} = V_0 \dot{+} W_0 = PW_{[-a,a]} \dot{+} W_0$ by Proposition 3.9. Since Fourier transform is unitary, we have $\widehat{PW}_{[-2a,2a]} = \widehat{PW}_{[-a,a]} \dot{+} \widehat{W_0}$, where $\widehat{X} := \{\hat{f} : f \in X\}$. Suppose there exists $\psi \in W_0$ such that $\{T^n\psi\}$ is a frame for W_0. Then $\widehat{W_0} = \{c(\gamma)\hat{\psi}(\gamma) : c \in L^2([-1/2,1/2]),\ \text{1-periodic}\}$. Obviously $[-2a,-a] \cup [a,2a] \subset \text{supp}(\hat{\psi})$. Since $\check{\chi}_{[-2a,-a]\cup[a,2a]} \in PW_{[-2a,2a]}$, there exist 1-periodic $b, c \in L^2([-1/2,1/2])$ such that $\chi_{[-2a,-a]\cup[a,2a]}(\gamma) = b(\gamma)\chi_{[-a,a]}(\gamma) + c(\gamma)\hat{\psi}(\gamma)$. This implies that $c(\gamma)\hat{\psi}(\gamma) = 1$ on $[-2a,-a]\cup[a,2a]$. Notice that, since $a < 1/3$, $[-2a, 2a-1] \subset [-2a,-a]$ and $[-2a,2a-1]+1 = [-2a+1, 2a] \subset [a, 2a]$. Hence $c(\gamma)\hat{\psi}(\gamma) = 1 = c(\gamma+1)\hat{\psi}(\gamma+1) = c(\gamma)\hat{\psi}(\gamma+1)$ on $[-2a, 2a-1]$. So $\hat{\psi}(\gamma) = \psi(\gamma+1)$ on $[-2a, 2a-1]$. Let $\hat{f}(\gamma) := \chi_{[-2a,-a]} - \chi_{[a,2a]} \in \widehat{PW}_{[-2a,2a]}$. Then there exist 1-periodic $d, e \in L^2([-1/2,1/2])$ such that $\hat{f}(\gamma) = d(\gamma)\chi_{[-a,a]}(\gamma) + e(\gamma)\hat{\psi}(\gamma)$. Then, for $\gamma \in [-2a, 2a-1] \subset [-2a,-a]$, $1 = \hat{f}(\gamma) = e(\gamma)\hat{\psi}(\gamma) = e(\gamma+1)\hat{\psi}(\gamma+1) = d(\gamma+1)\chi_{[-a,a]}(\gamma+1) + e(\gamma+1)\hat{\psi}(\gamma+1) = \hat{f}(\gamma+1) = -1$. This contradiction proves our assertion.

Appendix A. Frames of integer translates

In this section we give a simple proof of Lemma 3.3 of [13] (1-dimensional version of Lemma 3.53 of [3]), present a new proof of a stronger version of Theorem 3.4 of [1] (1-dimensional version of Theorem 3.56 of [3]), and finally prove Corollary A.4 which was used in Section 3. Theorem A.3 was proved independently by several authors with varying methods. Bibliographic information on Theorem A.3, including this manuscript, can be found in [2].

Let $\varphi \in L^2(\mathbb{R})$ and $V_0 := \overline{\operatorname{span}}\{T^k\varphi\}_{k\in\mathbb{Z}}$. It is curious to know when $\{T^k\varphi\}_{k\in\mathbb{Z}}$ is a frame of V_0. Let $\Phi(\gamma) := \sum_k |\hat{\varphi}(\gamma + k)|^2$. Then it is easy to see that $\Phi \in L^1(\mathbb{T})$.

LEMMA A.1. *If $\{T^n\varphi\}$ is a Bessel sequence (for V_0), then $\Phi \in L^2(\mathbb{T})$.*

PROOF. Note that $\{\langle\varphi, T^k\varphi\rangle\} \in \ell^2(\mathbb{Z})$. The result follows from the Plancherel theorem for Fourier series, since the k^{th} Fourier coefficient of Φ is $\{\langle\varphi, T^k\varphi\rangle\}$. □

Compare the above Lemma with Corollary A.4.

The proof of the following lemma in [13], [1] and [3] is quite technical. We present an elementary proof. Notice also that we slightly weaken the hypothesis in the sense that unlike as in Lemma 3.3 of [1] $\Phi \in L^2(\mathbb{T})$ is established in the course of the proof.

LEMMA A.2. *$\{T^k\varphi\}$ is a frame of V_0 with bounds A and B if and only if $\Phi \in L^2(\mathbb{T})$ and*

$$A \int_{\mathbb{T}} |\Theta(\gamma)|^2\Phi(\gamma)\,d\gamma \le \int_{\mathbb{T}} |\Theta(\gamma)|^2\Phi(\gamma)^2\,d\gamma \le B \int_{\mathbb{T}} |\Theta(\gamma)|^2\Phi(\gamma)\,d\gamma \tag{A.1}$$

for each trigonometric polynomial $\Theta(\gamma) := \sum c_k e^{-2\pi i k\gamma}$.

PROOF. By a standard density argument (Lemma 3.52 in [3]) $\{T^k\varphi\}$ is a frame for V_0 if and only if $A\|f\|^2 \le \sum_k |\langle f, T^k\varphi\rangle|^2 \le B\|f\|^2$ holds for each $f \in \operatorname{span}\{T^k\varphi\}$. Let $f := \sum_{k\in F} c_k T^k\varphi$, where $F \subset \mathbb{Z}$ is finite. Let

$\Theta(\gamma) := \sum_{k\in F} c_k e^{-2\pi i k\gamma}$. Then

$$\begin{aligned}
\|f\|^2 &= \int_{\mathbb{R}} |\sum_{k\in F} c_k T^k\varphi(t)|^2\, dt \\
&= \int_{\mathbb{R}} |\sum_{k\in F} c_k e^{-2\pi i k\gamma}\hat{\varphi}(\gamma)|^2\, d\gamma \\
&= \sum_{l\in\mathbb{Z}} \int_l^{l+1} |\Theta(\gamma)|^2|\hat{\varphi}(\gamma)|^2\, d\gamma \\
&= \sum_{l\in\mathbb{Z}} \int_0^1 |\Theta(\gamma)|^2|\hat{\varphi}(\gamma+l)|^2\, d\gamma \\
&= \int_{\mathbb{T}} |\Theta(\gamma)|^2\Phi(\gamma)\, d\gamma, \qquad \text{(A.2)}
\end{aligned}$$

where the last equality holds by the monotone convergence theorem.

$$\begin{aligned}
\sum_k |\langle f, T^k\varphi\rangle|^2 &= \sum_k |\langle \hat{f}, \hat{\varphi}e^{-2\pi i k\cdot}\rangle|^2 \\
&= \sum_k \Big| \int_{\mathbb{R}} \hat{\varphi}(\gamma)\Theta(\gamma)\overline{\hat{\varphi}(\gamma)}e^{2\pi i k\gamma}\, d\gamma\Big|^2 \\
&= \sum_k \Big|\sum_l \int_l^{l+1} |\hat{\varphi}(\gamma)|^2\Theta(\gamma)e^{2\pi i k\gamma}\, d\gamma\Big|^2 \\
&= \sum_k \Big|\sum_l \int_0^1 |\hat{\varphi}(\gamma+l)|^2\Theta(\gamma)e^{2\pi i k\gamma}\cdot d\gamma\Big|^2 \\
&= \sum_k \Big| \int_{\mathbb{T}} \Phi(\gamma)\Theta(\gamma)e^{2\pi i k\gamma}\, d\gamma\Big|^2 \\
&= \int_{\mathbb{T}} \Phi(\gamma)^2|\Theta(\gamma)|^2\, d\gamma, \qquad \text{(A.3)}
\end{aligned}$$

where the dominated convergence theorem is used in the next-to-last equality, and the Plancherel theorem is used in the last equality.

If $\{T^k\varphi\}$ is a frame for V_0, then by Lemma A.1 $\Phi \in L^2(\mathbb{T})$. Hence (A.3) is finite for each $f \in \text{span}\{T^k\varphi\}$. Therefore (A.1) follows from (A.2) and (A.3). On the other hand, suppose (A.1) holds and $\Phi \in L^2(\mathbb{T})$. Then again by (A.2) and (A.3) $\{T^k\varphi\}$ is a frame. □

The following theorem is proved in [1] under the additional hypothesis that $\Phi \in L^\infty(\mathbb{T})$.

THEOREM A.3. *$\{T^k\varphi\}$ is a frame for V_0 if and only if there are positive constants A and B such that*

$$A \le \Phi \le B \qquad \text{a.e. on } \mathbb{T} \setminus N, \tag{A.4}$$

where N is the zero set (modulo measure zero sets) of Φ. In this case A and B are frame bounds of $\{T^k\varphi\}$.

PROOF. The 'if' part is as in Theorem 3.4 of [1] (or Theorem 3.56 of [3]), since (A.4) implies that $\Phi \in L^\infty(\mathbb{T})$.

We prove the 'only if' part. Notice that $\Phi \in L^2(\mathbb{T})$ by Lemma A.2. Suppose that $\Phi < A$ on $E \subset \mathbb{T} \setminus N$, where E is a set of positive measure. Then we have

$$A \int_{\mathbb{T}} |\chi_E(t)|^2 \Phi(t)\, dt > \int_{\mathbb{T}} |\chi_E(t)|^2 \Phi(t)^2\, dt.$$

We show that this is impossible. Let $\sigma_n(\chi_E)$ be the n^{th} Cesaro sum of the Fourier series of χ_E. Then obviously it is a trigonometric polynomial and $\sigma_n(\chi_E)(t) = \int_{\mathbb{T}} \chi_E(x) K_n(t-x)\, dx$, where K_n is the n^{th} Fejer kernel. Recall that $\int_{\mathbb{T}} K_n(x)\, dx = 1$ and that K_n is positive. Hence each $\sigma_n(\chi_E)(t)$ is bounded by 1 on $\mathbb{T}$. Moreover, $\{\sigma_n(\chi_E)\}$ converges to χ_E as $n \to \infty$ in $L^2(\mathbb{T})$ [10]. Hence there exists a subsequence $\{\sigma_{n_j}(\chi_E)\}$ which converges to χ_E pointwise almost everywhere. Hence by (A.1)

$$A \int_{\mathbb{T}} |\sigma_{n_j}(\chi_E)(t)|^2 \Phi(t)\, dt \le \int_{\mathbb{T}} |\sigma_{n_j}(\chi_E)(t)|^2 \Phi(t)^2\, dt.$$

Since $\Phi \in L^1(\mathbb{T}) \cap L^2(\mathbb{T})$, the dominated convergence theorem implies that

$$A \int_{\mathbb{T}} |\chi_E(t)|^2 \Phi(t)\, dt \le \int_{\mathbb{T}} |\chi_E(t)|^2 \Phi(t)^2\, dt.$$

This contradicts our hypothesis. Similarly, there is no set of positive measure on which Φ is strictly larger than B. □

COROLLARY A.4. *$\{T^k\varphi\}$ is a Bessel sequence for V_0 if and only if $\Phi \in L^\infty(\mathbb{T})$.*

PROOF. This is implicitly contained in Lemma A.2 and Theorem A.3. □

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