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# $\omega$-LIMIT SETS FOR MAPS OF THE CIRCLE

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ABSTRACT. For a continuous map of the circle to itself, we give necessary and sufficient conditions for the $\omega$-limit set of each nonwandering point to be minimal.

## 1. Introduction

Let $S^1$ be the circle. Throughout this paper $f$ will denote a continuous map of the circle to itself. For any positive integer $n$, we define $f^1 = f$ and $f^{n+1} = f \circ f^n$. Let $f^0$ be the identity map of the circle. Let $AP(f), P(f), R(f), \Gamma(f), \Lambda(f)$ and $\Omega(f)$ denote the set of almost periodic points, periodic points, recurrent points, $\gamma$-limit points, $\omega$-limit points and nonwandering points of $f$, respectively.

A subset $Y$ in $S^1$ is called invariant if $f(Y) \subset Y$, and strongly invariant if $f(Y) = Y$. Suppose $Y \subset S^1$ is non-void, closed and invariant relative to $f$.

If $Y$ has no proper subset which is non-void and invariant relative to $f$, then $Y$ is said to be a minimal set.

J. C. Xiong [4,5] proved that for any continuous map $g$ of the interval, the following conditions are equivalent.

(1) $\Gamma(g) = AP(g)$.

(2) The period of each periodic point of $g$ is a power of 2.

In this paper, we obtain the following theorem for maps of the circle.

THEOREM 5. *Suppose that $f$ is a continuous map of the circle. Then the following conditions are equivalent :*

(1) $\Gamma(f) = AP(f)$.

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(2) *For every $x \in \Omega(f)$, the $\omega$-limit set $\omega(x, f)$ of $x$ is minimal.*

In 1986, L. Block and E. M. Coven [3] proved that for a continuous map $g$ of the interval, if $x \in \Lambda(g) \setminus \overline{R(g)}$, then $\omega(x, g)$ is infinite minimal, and if $x \in \Omega(g) \setminus \overline{R(g)}$, then $\omega(x, g)$ need not be minimal. We have the following theorem for maps of the circle.

THEOREM 6. *Suppose that $f$ is a continuous map of the circle. Let $R(f)$ be closed, $x \in \Omega(f), f^{kN}(x) = p \in F(f^N)$ and $x \in int(W_i)$ for some $i$. If $x \in \Omega(f) \setminus \overline{R(f)}$, then $\omega(x, f)$ is infinite minimal.*

## 2. Preliminaries and Definitions

Let $f$ be a continuous map of the circle $S^1$ to itself. The orbit $Orb(x)$ of $x \in S^1$ is the set $\{f^k(x)|k = 1, 2, \cdots\}$. A point $x \in S^1$ is a fixed point of $f$ if $f(x) = x$ and we denote the set of fixed points by $F(f)$. A point $x \in S^1$ is a periodic point of $f$ provided that for some positive integer $n$, $f^n(x) = x$. The period of $x$ is the least such integer $n$. We denote the set of periodic point of $f$ by $P(f)$.

A point $x \in S^1$ is a recurrent point of $f$ provided that there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$, or equivalently, $f^n(x) \to x$. We denote the set of recurrent points of $f$ by $R(f)$.

A point $x \in S^1$ is called a nonwandering point of $f$ provided that for every neighborhood $U$ of $x$, there exists a positive integer $m$ such that $f^m(U) \bigcap U \neq \emptyset$. We denote the set of nonwandering points of $f$ by $\Omega(f)$.

A point $x \in S^1$ is almost periodic point of $f$ provided that for any $\epsilon > 0$ one can find an integer $n > 0$ with the following property that for any integer $q > 0$ there exists an integer $r$ with $q \leq r < q + n$ such that $d(f^r(x), x) < \epsilon$, where $d$ is the metric of $S^1$. We denote the set of almost periodic points of $f$ by $AP(f)$.

J. C. Xiong [4] investigated the set $AP(g)$ of almost periodic points of a continuous map $g$ of the interval and proved the followings.

$AP(g) = P(g)$ if and only if $\Omega(g) = P(g)$, and $AP(g)$ is closed if and only if $R(g)$ is closed. Also, if $g$ has a periodic point of period which is not a power of 2, then $AP(g) - P(g) \neq \emptyset$ and $R(g) - AP(g) \neq \emptyset$, and if the period of each periodic point of $g$ is power of 2, then $R(g) = AP(g)$. Therefore the period of each periodic point of $g$ is power of 2 if and only if $R(g) = AP(g)$.

A point $y \in S^1$ is called an $\omega$-limit point of $x \in S^1$ provided that there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. We denote the set of $\omega$-limit points of $x$ by $\omega(x, f)$. Define $\Lambda(f) = \bigcup_{x \in S^1} \omega(x, f)$.

A point $y \in S^1$ is called an $\alpha$ -limit point of $x \in S^1$ if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{x_i\}$ of points in $S^1$ with $x_i \to x$ such that $f^{n_i}(x_i) = y$ for all $i \geq 1$. We denote the set of $\alpha$-limit points of $x$ by $\alpha(x, f)$.

A point $x \in S^1$ is called an $\gamma$-limit point of $y \in S^1$ if $x \in \omega(y, f) \bigcap \alpha$ $(y, f)$. Define $\Gamma(f) = \bigcup_{x \in S^1} \{\omega(x, f) \cap \alpha\ (x, f)\ \}$.

For a fixed point $p$ of $f$ and a side $S$, the one-side unstable set of $p$ is

$$W^u(p, f, s) = \cap_u \cup_{k \geq 0 f^k(U)},$$

where the intersection is taken over all $s$-half-neighborhoods $U$ of $p$. Let $p$ be a fixed point of $f^N$ and $S_i$ a side at $f^i(p)$ for each $i$. We denote $W_i$ by $W^u(f^i(p), f^N, S_i)$ for each $i$.

## 3. Main results

The following lemmas appear in [1], [2], [4] and [6].

LEMMA 1 [1]. *Suppose that $f$ is a continuous map of the circle $S^1$ to itself. Then*

$$P(f) \subset AP(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f).$$

LEMMA 2 [4]. *Suppose that $f$ is a continuous map of the circle $S^1$ to itself. Then $x \in AP(f)$ if and only if $x \in \omega(x, f)$ and $\omega(x, f)$ is minimal.*

LEMMA 3 [6]. *Suppose that $f$ is a continuous map of the circle. Then*

$$\Lambda(\Omega(f)) = \Lambda(\Gamma(f)) = \Gamma(f).$$

LEMMA 4 [2]. *Suppose that $f$ is a continuous map of the circle. If $x \in \Omega(f)$ has a finite orbit, $f^{kN}(x) = p \in F(f^N)$ and $x \in int(W_i)$ for some $i$, then $x \in \overline{R(f)}$.*

*Proof of Theorem 5* (1) $\Rightarrow$ (2) : Suppose that $\Gamma(f) = AP(f)$. Let $x$ be any point in $\Omega(f)$, and let $y$ be arbitrary point in $\omega(x, f)$. Let $z \in \omega(y, f)$. Then there exists a sequence of positive integers $n_i \to \infty$ such that $f^{n_i}(y) \to z$. Since $y \in \omega(x, f)$, there exists a sequence of positive integers $m_i \to \infty$ such that $f^{mi}(x) \to y$. Hence $f^{m_i+n_i}(x) \to z$. Thus $z \in \omega(x, f)$. Hence $\omega(y, f) \subset \omega(x, f)$.

Since $y$ is arbiturary point in $\omega(x, f)$, it suffices to show that $y \in w(y, f)$. Since $x \in \Omega(f)$, $\omega(x, f) \subset \Lambda(\Omega(f))$. By Lemma 3, $y \in \omega(x, f) \subset \Gamma(f)$. Since $\Gamma(f) = AP(f)$, $y \in AP(f)$. By Lemma 2, $y \in \omega(y, f)$. Hence $\omega(x, f) \subset \omega(y, f)$. Therefore $\omega(x, f) = \omega(y, f)$ and $\omega(x, f)$ is minimal.

(2) $\Rightarrow$ (1) : Suppose that for any $x \in \Omega(f)$, $\omega(x, f)$ is minimal. Let $y \in \Gamma(f)$. Then by Lemma 3, $y \in \Lambda(\Omega(f))$. There is $z \in \Omega(f)$ such that $y \in \omega(z, f)$. Since $\omega(z, f)$ is minimal, $\omega(y, f) = \omega(z, f)$. Hence $y \in \omega(y, f)$. By Lemma 1, $y \in \Omega(f)$. So $\omega(y, f)$ is minimal. Thus, by Lemma 2, $y \in AP(f)$. Therefore $\Gamma(f) \subset AP(f)$.

COROLLARY 1. *Suppose that $f$ is a continuous map of the circle. Let $R(f)$ be closed. Then the following conditions are equivalent :*
(1) $R(f) = AP(f)$.
(3) *For every $x \in \Omega(f)$, the $\omega$-limit set $\omega(x, f)$ of $x$ is minimal.*

*Proof of Theorem 6* Suppose that $R(f)$ is closed. Let $x \in int(W_i)$ for some $i$ and $x \in \Omega(f) \setminus \overline{R(f)}$. Since $R(f)$ is closed, by Lemma 1, $\Gamma(f) = R(f)$. By Theorem 5, $\omega(x, f)$ is minimal. Now we show that $\omega(x, f)$ is infinite. Assume that $\omega(x, f)$ is finite. Since $\omega(x, f)$ is closed and invariant, $\omega(x, f) = \overline{Orb(x, f)}$ by definition. Then $\overline{Orb(x, f)}$ is finite. Hence $Orb(x, f)$ is finite, a contradiction.

The set $\Omega(f)$ of nonwandering points of $f$ is always closed and invariant and $P(f) = P(f^n) \subseteq \Omega(f^n) \subseteq \Omega(f)$ holds for all $n$. It is well known that $R(f) = R(f^n)$ for all $n$. Therefore we have the following corollary.

COROLLARY 2. *Suppose that $f$ is a continuous map of the circle. Let $R(f)$ be closed, $x \in \Omega(f)$, $f^{kN}(x) = p \in F(f^N)$ and $x \in int(W_i)$ for some $i$. If $x \in \Omega(f^n) \setminus R(f)$, then $\omega(x, f)$ is minimal.*

## References

[1] J. S. Bae, S. H. Cho and S. K. Yang, *Relations between nonwandering points and turning points on the circle*, Bull.Korean Math. Soc. **32** (1995), 57–66.

[2] L. Block, E. Coven, I. Mulvey and Z. Nitecki, *Homoclinic and nonwandering points for maps of the circle*, Erg. Th. & Dynam. sys. **3** (1983), 521–532.

[3] L. Block and E. M. Coven, *$\omega$-limit sets for maps of the interval*, Erg. Th. & Dynam. sys. **6** (1986), 335–344.

[4] J. C. Xiong, *Sets of almost periodic points of continuous self-map of the interval*, preprint.

[5] ———, *The attracting center of a continuous self-map of the interval*, Erg. Th. & Dynam. sys. **8** (1988), 205–213.

[6] S. K. Yang and K. J. Min, *The attracting centre of a map on the circle*, preprint.

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