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LIMIT SETS OF PROJECTIVELY FLAT MANIFOLDS

Kyeongsu Park

Abstract. In this paper, we discuss various limit sets of projectively flat manifolds and relationship between them.

1. Introduction

Let M be a projectively flat manifold with or without boundary. We fix a developing map $D : \tilde{M} \to \mathbb{RP}^n$ and a holonomy homomorphism $\rho : \pi_1(M) \to \mathrm{PGL}(n+1, \mathbb{R})$. Let $\Omega = D(\tilde{M})$ the developing image and $\Gamma = \rho(\pi_1(M))$ the holonomy group.

We endow a Riemannian metric on $\mathbb{RP}^n$. Since D is a local diffeomorphism, $\tilde{M}$ admits the pull-back metric. Let $\bar{M}$ be the metric completion of $\tilde{M}$. Let $\tilde{M}_\infty$ be the ideal boundary of $\tilde{M}$, i.e., $\tilde{M}_\infty = \bar{M} - \tilde{M}$. Since D is uniformly continuous, it has unique extension $\bar{D} : \bar{M} \to \mathbb{RP}^n$. Let $L_\infty(M)$ be the image of $\tilde{M}_\infty$ under $\bar{D}$, which is our first limit set (see [1]).

Next two limit sets are topological. Let $L_E(M)$ be the set of those points y which is the end point $c(1)$ of a continuous curve c in $\mathbb{RP}^n$ such that there exists a curve $\tilde{c}(t) \in \tilde{M}$ $(0 \le t < 1)$, $D(\tilde{c}(t)) = c(t)$ and $\tilde{c}(1)$ can not be defined continuously in $\tilde{M}$ (see [3]).

Let $L_O(M)$ be the set of points y such that the inverse image of any compact neighborhood of y under D has a nonempty and noncompact component ([6] and [7]).

Now we discuss singular projective transformations. Let $\mathrm{P}\mathfrak{gl}(n+1, \mathbb{R})$ be the projectivization of the vector space $\mathfrak{gl}(n+1, \mathbb{R})$. The projectivization of an element $A \in \mathfrak{gl}(n+1, \mathbb{R})$ is denoted by $[A]$. We define the kernel $K[A]$ and the range $R[A]$ of $[A]$ by the projectivization of $\ker A$ and $\mathrm{im} A$, respectively. As a map $[A]$ from $\mathbb{RP}^n - K[A]$ to $R[A]$, we define

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$[A][v] = [Av]$. If $K[A]$ is not empty, i.e., A is singular, then we call $[A]$ a singular projective transformation. Since $\dim \ker A + \dim \mathrm{im} A = n+1$, $\dim K[A] = \dim \ker A - 1$ and $\dim R[A] = \dim \mathrm{im} A - 1$, we have

$$\dim K[A] + \dim R[A] = n - 1.$$

Let Γ be a subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ and $\bar{\Gamma}$ the closure of Γ in $\mathrm{Pgl}(n+1, \mathbb{R})$. We denote by $K(\Gamma)$ the union of kernels of all singular projective transformations in $\bar{\Gamma}$. If Γ is a holonomy group of M, then we define $L_K(M) = K(\Gamma) \cap \bar{\Omega}$.

To define the last limit set we denote by $L_J(\Gamma)$ the set of those points where Γ is not equicontinuous. We define $L_J(M) = L_J(\Gamma) \cap \bar{\Omega}$.

Our main Theorem is relationship between above five limit sets.

THEOREM. *Let M be a closed projectively flat manifold. Then*

$$L_\infty(M) = L_E(M) \subset L_O(M) \subset L_K(M) = L_J(M).$$

Over all this paper, $|x - y|$ means the distance between x and y for the given metric in context.

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2. Relationship between limit sets

It is easy to see that the limit set $L_\infty(M)$ is independent of the choice of a Riemannian metric on the projective space $\mathbb{RP}^n$. The following Lemma is the first part of main Theorem.

LEMMA 1. *Let M be a projectively flat manifold. Then*

$$L_\infty(M) = L_E(M) \subset L_O(M).$$

PROOF. Suppose that $y \in L_\infty(M)$. There exists $x \in \tilde{M}_\infty$ such that $\bar{D}(x) = y$. Let $x_m \in \tilde{M}$ be a sequence converging to x. Let $c(t)$, $(0 \le t \le 1)$ be a curve of finite length through x_m. Then $c(1) = x$. Since $(D \circ c)(1) = y$ and $c(1) \notin \tilde{M}$, $y \in L_E(M)$.

Conversely, let y be any point in $L_E(M)$. There is a curve $\bar{c}(t)$ $(0 \le t < 1)$ in $\tilde{M}$ which can not be extended continuously on $\tilde{M}$ and $(D \circ$

$\tilde{c})(1) = y$. But the curve $\tilde{c}$ has the same length with $D \circ \tilde{c}$. This means that $\tilde{c}(1)$ is contained in $\tilde{M}_\infty$ and y is contained in $L_\infty(M)$.

Suppose that $y \in L_E(M)$. Let $c(t)$, $(0 \leq t \leq 1)$ be a curve in $\mathbb{RP}^n$ with $c(1) = y$ and $\tilde{c}$ a lifting of c so that $\tilde{c}(1)$ can not be defined continuously on $\tilde{M}$. Let U be any compact neighborhood of y. There exists α such that the curve segment $\tilde{c}(t)$, $(\alpha < t < 1)$ is contained in a component of $D^{-1}(U)$. This component is not compact. □

Let $\| \ \|$ be the 2-norm on $\mathbb{R}^{n+1}$ or on $\mathfrak{gl}(n+1, \mathbb{R})$. If $A_m \in \mathfrak{gl}(n+1, \mathbb{R})$ converges to A and $\|A_m\| = 1$ then $\|A\| = 1$ and $[A_m]$ converges to $[A]$ in the topology of $\mathrm{P}\mathfrak{gl}(n+1, \mathbb{R})$. Conversely, if $\gamma_m \in \mathrm{P}\mathfrak{gl}(n+1, \mathbb{R})$ converges to γ, then there are representatives A_m and A in $\mathfrak{gl}(n+1, \mathbb{R})$ of γ_m and γ, respectively, such that $\|A_m\| = 1$, $\|A\| = 1$ and $\lim A_m = A$. The following lemma about singular transformations is a generalization of a Theorem of Myrberg ([8]).

LEMMA 2. *Suppose that $\gamma_m \in \mathrm{P}\mathfrak{gl}(n+1, \mathbb{R})$ is a sequence converging to an element γ. Let C be a compact subset of $\mathbb{RP}^n - (K(\gamma) \cup K(\gamma_1) \cup K(\gamma_2) \cup \cdots)$. Then γ_m converges uniformly to γ on C.*

PROOF. We choose representatives A and A_m of γ and γ_m, respectively, so that $\|A\| = \|A_m\| = 1$ and $\lim A_m = A$. Let $[v] \in C$ and $\|v\| = 1$. It suffices to show that γ_m converges uniformly on a neighborhood of $[v]$.

Consider the continuous map

$$\begin{array}{rccc} \phi: & \mathfrak{gl}(n, \mathbb{R}) \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (B, w) & \longmapsto & Bw. \end{array}$$

Since v is not an element in $\ker A$, $\|Av\| > 2r$ for some $r > 0$, i.e., $(A, v) \notin \phi^{-1}(\bar{B}(0, r))$. There is a compact neighborhood $U \times V$ of (A, v) such that $\phi(U \times V)$ is contained in the complement of $\bar{B}(0, r)$. Let

$R = \sup\{\|Bw\| | B \in U, w \in V\}$. For $B, C \in U$ and $w \in V$, we have

$$\begin{aligned}&\left\|\frac{Bw}{\|Bw\|} - \frac{Cw}{\|Cw\|}\right\| \\ &= \frac{1}{\|Bw\|\|Cw\|}\|\|Cw\|Bw - \|Bw\|Cw\| \\ &\leq \frac{1}{r^2}\|\|Cw\|Bw - \|Bw\|Bw\| + \frac{1}{r^2}\|\|Bw\|Bw - \|Bw\|Cw\| \\ &\leq \frac{2}{r^2}\|Bw\|\|B - C\| \\ &\leq \frac{2R}{r^2}\|B - C\|\|w\|.\end{aligned}$$

For sufficiently large m and for any $w \in V \cap \mathbb{S}^n$,

$$\begin{aligned}|\gamma_m[w] - \gamma[w]| &\leq \pi \left\|\frac{Aw}{\|Aw\|} - \frac{A_m w}{\|A_m w\|}\right\| \\ &\leq \frac{2\pi R}{r^2}\|A - A_m\|.\end{aligned}$$

Hence γ_m converges uniformly to γ on $[V \cap \mathbb{S}^n]$. □

Now we compare last two limit sets, $L_J(M)$ and $L_K(M)$. Let e_i be the unit vector in $\mathbb{R}^{n+1}$ whose entries are zero but i-th one which is 1.

LEMMA 3. *For any subgroup Γ of* PGL$(n+1, \mathbb{R})$, $L_J(\Gamma) = K(\Gamma)$. *In particular, $L_J(M) = L_K(M)$ for a projectively flat manifold M.*

PROOF. Suppose that Γ is not equicontinuous at x. We can find a positive real number ϵ such that for any $\delta > 0$ there exist $\gamma \in \Gamma$ and $y \in B(x, \delta)$ with $|\gamma x - \gamma y| > \epsilon$. Let γ_m be such a projective transformation for $\delta = 1/m$. Through a subsequence, we may assume that γ_m converges to γ.

Now we prove that $x \in K(\gamma)$. Assume to contrary that γ is not singular or $x \notin K(\gamma)$. By the Lemma 2, we can choose a neighborhood U of x on which γ_m converges uniformly to γ. Thus there exists $N > 0$ such that $m > N$ implies that $|\gamma_m y - \gamma y| < \epsilon$ for all $y \in U$. There exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|\gamma x - \gamma y| < \epsilon$. For any $y \in B(x, \delta) \cap U$ and $m > N$ we have

$$|\gamma_m y - \gamma_m x| \leq |\gamma_m y - \gamma y| + |\gamma y - \gamma x| + |\gamma x - \gamma_m x| \leq 3\epsilon.$$

This contradicts our assumption. Hence γ is singular and x is an element of $K(\gamma)$.

Conversely, suppose that x is an element in $K(\Gamma)$. There is a sequence $\gamma_m \in \Gamma$ such that $\gamma = \lim \gamma_m$ and $x \in K(\gamma)$. Let $v = (v_i)$, $A_m = (a_{m;ij})$ and $A = (a_{ij})$ be representatives of x, γ_m and γ, respectively so that $\sum v_i^2 = 1$, $\sum (a_{m;ij})^2 = 1$ and $\sum (a_{ij})^2 = 1$. We may assume that $v = e_1$. Since $v \in \ker A$, the first column of A is zero. We choose a nonzero column of A, say the second column. Consider the line segment $v + t\delta e_2$, $(-1 \leq t \leq 1)$. The image of the line segment under A_m is

$$l_m = (a_{m;11}, \dots, a_{m;n+1,1}) + t\delta(a_{m;12}, \dots, a_{m;n+1,2}).$$

We emphasize that the origin of $\mathbb{R}^{n+1}$ is not contained in l_m for all m. For sufficiently large m, the radial projection of l_m almost covers a half of an equator of $\mathbb{S}^n$. Therefore $\gamma_m(B(x, \delta))$ almost covers a full line in $\mathbb{RP}^n$ for given δ. This implies that $\{\gamma_m\}$ is not equicontinuous and $x \in L_J(\Gamma)$. □

In the theory of the conformally flat structure, the following Lemma was proved by Kulkarni and Pinkall ([6]). See also [7].

LEMMA 4. *If M is a closed projectively flat manifold then $L_O(M) \subset L_J(M)$.*

PROOF. Assume to contrary that $y \in L_O - L_J$. We take a sequence of positive numbers so that $r_m \to 0$. Let V_m be the noncompact component of $D^{-1}\bar{B}(y, r_m)$. We choose any point $x_m \in V_m$. Since M is compact, there are deck transformations $g_m \in \pi_1(M)$ such that $g_m x_m$ converges to a point $x \in \tilde{M}$ through a subsequence, if necessary. Let $y_m = Dx_m$. Then $y_m \to y$ and $\rho(g_m)y_m = D(g_m x_m) \to Dx$. Since Γ is equicontinuous at y, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|z - y| < \delta$ implies $|\gamma z - \gamma y| < \epsilon$ for all $\gamma \in \Gamma$. If m is so large that $|y_m - y| < \delta$ and $|\rho(g_m)y_m - Dx| < \epsilon$ then

$$\begin{aligned} |h(g_m)y - Dx| &\leq |\rho(g_m)y - \rho(g_m)y_m| + |\rho(g_m)y_m - Dx| \\ &\leq 2\epsilon. \end{aligned}$$

Therefore $\rho(g_m)y \to Dx$.

Let U be a neighborhood of x so that $D|_U$ is an isometry onto its image. We choose a neighborhood W of x such that $\bar{W} \subset U$. Since $D(W)$ is open, $\rho(g_m)y \to Dx$ and $\{\rho(g_m)\}$ is equicontinuous at y, there is a positive integer N such that $m > N$ implies $\rho(g_m)\bar{B}(y, r_m) \subset D(W)$. We fix $m > N$. If $g_m V_m \subset W$ then $g_m V_m$ is a closed subset of the compact set $\bar{W}$. Hence V_m is compact. It is a contradiction. If $g_m V_m \not\subset W$ then

$$\begin{aligned} D(g_m V_m) &= \rho(g_m) D(V_m) \\ &\subset \rho(g_m)\bar{B}(y, r_m) \\ &\subset D(W). \end{aligned}$$

This implies that $g_m V_m \subset D^{-1}(D(W))$. Since W is a connected component of $D^{-1}(D(W))$ and $g_m V_m \cap W \neq \emptyset$, we have $g_m V_m \subset W$. It is also a contradiction. Our result follows. □

To get our main Theorem, we combine Lemma 1, Lemma 3 and Lemma 4. It is a natural question whether $L_E(M)$, $L_O(M)$, $L_K(M)$ are equal or not. The following two examples give a negative answer to the question.

EXAMPLE 5. Let σ be the 2×2 diagonal matrix with diagonal elements 3 and 2. Let M be 2-torus $\mathbb{R}^2 - \{0\}/\langle\sigma\rangle$. Through the natural imbedding from $\mathbb{R}^n$ into $\mathbb{RP}^n$, we consider M as a projectively flat manifold. Then $L_O(M) = \{[0,0,1]\} \cup \overleftrightarrow{xy}$ and $L_K(M) = \overleftrightarrow{xy} \cup \overleftrightarrow{yz}$ where $\overleftrightarrow{xy}$ is the line in $\mathbb{RP}^2$ joining $[1,0,0]$ and $[0,1,0]$ and so on.

EXAMPLE 6 ([2]). Let Γ be the subgroup of Aff(3), the group of affine transformations on $\mathbb{R}^3$, generated by the following three elements:

$$L = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1/p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$S = \begin{pmatrix} 1 & 0 & 0 & p-1 \\ 0 & 1 & 0 & 1/p-1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $p > 1$. Let M be the affine space form $\mathbb{R}^3/\Gamma$. It is easy to see that $\overleftrightarrow{yz}$ is contained in $L_K(M)$ but not in $L_O(M)$.

As above, $L_K(M) \neq \mathbb{RP}^n - \mathbb{R}^n$ even though M is an affine space form. But it is true in case of a Euclidean space form.

EXAMPLE 7. Let $\mathbb{RP}^{n-1}$ be the complement of $\mathbb{R}^n$ in $\mathbb{RP}^{n+1}$ and M a Euclidean space form. It is easy to show that $L_\infty(M) = L_K(M) = \mathbb{RP}^{n-1}$.

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Department of Mathematics
Jeonju University
Jeonju 560-759, Korea
E-mail: pine@jeonju.ac.kr