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A NOTE ON JANOWITZ'S HULLS OF GENERALIZED ORTHOMODULAR LATTICES

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ABSTRACT. If G is a strict generalized orthomodular lattice and $H = \{I | I = [0,x],\ x \in G\}$, then H is a prime ideal of the Janowitz's hull $J(G)$ of G. If f is the Janowitz's embedding, then the set of all commutators of $f(G)$ equals the set of all commutators of the Janowitz's hull $J(G)$ of G. Let L be an OML. Then $L \simeq J(G)$ for a strict GOML G if and only if there exists a proper nonprincipal prime ideal G in L.

1. Preliminaries

Let P be a bounded poset. An *orthocomplementation* on P is a *unary operation* $'$ on P which satisfies the following properties: (1) if $x \leq y$, then $y' \leq x'$; (2) $x'' = x$; (3) $x \vee x' = 1$ and $x \wedge x' = 0$ hold.

We call a bounded poset P with an orthocomplementation an *orthoposet*. Two elements x, y of an orthoposet are *orthogonal*, written $x \perp y$, in case $x \leq y'$. An *ortholattice* is an orthoposet which is also a lattice.

An *orthomodular lattice* (abbreviated by OML) is an ortholattice L which satisfies *the orthomodular law*: if $x \leq y$, then $y = x \vee (x' \wedge y)$ [1, 2 & 4]. A *Boolean algebra* B is an ortholattice satisfying the *distributive law*: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in B$.

The *commutator of* a *and* b of an OML L is denoted by $a * b$, and is defined by $a * b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. For elements a, b of an OML, we say that a *commutes with* b, in symbols $a\,\mathbf{C}\,b$, if $a = (a \wedge b) \vee (a \wedge b')$ [2, 4].

A generalized orthomodular lattice was studied by Janowitz [3]. The Janowitz's hull of a generalized orthomodular lattice and some properties

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of generalized orthomodular lattices was studied by Beran [2]. In this paper, we will study further properties of Janowitz's hulls of generalized orthomodular lattices.

DEFINITION 1.1. Let $(G, \vee, \wedge)$ be a lattice with a least element 0. For any a in G, define $^{P(a)} : [0,a] \to [0,a]$ to be a unary operation on $[0,a]$ such that $^{P(a)} : x \to x^{P(a)}$. We shall say that G is a *generalized orthomodular lattice* (abbreviated by GOML) if it satisfies the following conditions:
(1) The algebra $([0,a], \vee, \wedge, ^{P(a)}, 0, a)$ is an OML for every $a \in G$, and
(2) for any $x \leq a \leq b$, $x^{P(a)} = x^{P(b)} \wedge a$.
We shall say that a GOML G is a strict GOML if G is not an OML.

Note that any OML is a GOML and a GOML is an OML if and only if it has a greatest element [2, p. 168]. Henceforth, G will denote a GOML unless specified.

DEFINITION 1.2. Let a, b be in a GOML G. we shall write $a \perp b$ if $a \leq b^{P(a \vee b)}$.

Let a, b be elements in an OML L. If we define $x^{P(a)} = x' \wedge a$ for $x \in L[a,b]$, then $x^{P(b)} \wedge a = x' \wedge b \wedge a = x' \wedge a = x^{P(a)}$ for any $x \leq a \leq b$. Then $a \perp b$ if and only if $a \leq b'$ since $a \leq b^{P(a \vee b)} = b' \wedge (a \vee b) = b' \wedge a$.

DEFINITION 1.3. Let M be a subset of a GOML G. Define $M^{\perp} = \{x | x \in G,\ x \perp y\ \forall y \in M\}$.

Let $[0,x]$ be the *principal ideal generated by* x in G and $x \downarrow = \{y | y \in G, y \leq x\}$, $x \uparrow = \{y | y \in G, x \leq y\}$.

Beran proved the following theorem[2].

THEOREM 1.4. *Let G be a GOML. Then the following are equivalent:*
(1) *G is an OML;*
(2) *There exist x, y in G such that $[0,x]^{\perp} = [0,y]$;*
(3) *There exist x, y in G such that $[0,x]^{\perp}$ is a subset of $[0,y]$.*
If (2) is valid, then y is the orthocomplement x' of the element x.

2. Janowitz's Hull

All ideals $I(L)$ of a lattice L form a lattice $(I(L), \vee, \wedge)$ where $I \vee J =$

$\{x|x \in L, x \leq i \vee j$ for $i \in I,\ j \in J\}$ and $I \wedge J = I \cap J$ for $I, J \in I(L)$. The lattice $I(L)$ will be called the *ideal lattice* of L[7].

Janowitz proved the following theorem [2, 3].

THEOREM 2.1. *Let $I(G)$ be the ideal lattice of a* GOML G *and $J(G)$ be the set of all the subsets I of G which are of the form $I = [0, x]$ or $I = [0, x]^{\perp}$ for $x \in G$. Then $J(G)$ is a sublattice of $I(G)$ and $J(G)$ is an* OML.

The OML $J(G)$ in Theorem 2.1 will be called the *Janowitz's hull* of the GOML G.

COROLLARY 2.2. *The following hold in the* Janowitz's *hull $J(G)$ of a* GOML G*:*
(1) $[0, x] \vee [0, y] = [0, x \vee y]$,
(2) $[0, x]^{\perp} \vee [0, y] = [0, x \wedge y^{P(x \vee y)}]^{\perp}$,
(3) $[0, x]^{\perp} \vee [0, y]^{\perp} = [0, x \wedge y]^{\perp}$,
(4) $[0, x] \wedge [0, y] = [0, x \wedge y]$,
(5) $[0, x] \wedge [0, y]^{\perp} = [0, x \wedge y^{p(x \vee y)}]$,
(6) $[0, x]^{\perp} \wedge [0, y]^{\perp} = [0, x \vee y]^{\perp}$.

DEFINITION 2.3. A *prime ideal* $I \neq 0$ of a lattice L is a proper ideal of L such that $a \wedge b \in I$ implies that $a \in I$ or $b \in I$.

LEMMA 2.4. *Let I be a proper ideal of an ortholattice L. The following are equivalent:*
(1)I is a prime ideal;
(2) $a \wedge b \in I$ and $a \notin I$ imply that $b \in I$;
(3) $a \in I$ or $a' \in I$;
(4) $0 \neq a \notin I$ and $a \wedge b = 0$ imply that $b \in I$.

PROOF. Clearly, (1) holds if and only if (2) holds, (2) implies (4) and (4) implies (3). Let us prove that (3) implies (2). Suppose that $a \in I$ or $a' \in I$, and that $a \wedge b \in I$ and $a \notin I$. Then $a' \in I$. Suppose that $b \notin I$. Then $b' \in I$. Thus $(a' \vee b') \vee (a \wedge b) = (a \wedge b)' \vee (a \wedge b) = 1 \in I$. Thus $I = L$ contrary to the fact that I is a proper ideal. This completes the proof. □

THEOREM 2.5. *Let G be a strict* GOML, *and let $H = \{I|I =$

$[0, x] \forall x \in G\}$. *Then* H *is a prime ideal of the* Janowitz's *hull* $J(G)$ *of* G.

PROOF. It is sufficient to show that H is a proper ideal by Lemma 2.4 since $I \in J(G)$ or $I^{\perp} \in J(G)$ by the definition of $J(G)$. Let us show that H is an ideal of $J(G)$. If $I = [0, x]$ and $J = [0, y] \in H$, then $I \vee J = [0, x \vee y] \in H$. If $K \in J(G)$ and $K \subset I$ in H, then $K = [0, z]$ or $K = [0, u]^{\perp}$ for some z, u in G. If $K = [0, z]$, then K is in H. If $K = [0, u]^{\perp}$, then there exists $u' \in G$ such that $[0, u'] = [0, u]^{\perp}$ is a subset of $I = [0, x]$ by theorem (1.4) since $[0, x]$ is an OML. Thus $[0, u]^{\perp} = [0, u'] \in H$. Therefore H is an ideal in $J(G)$. Furthermore, H is a proper ideal since $J(G) \notin H$. Thus H is a prime ideal of $J(G)$ by lemma (2.4). This completes the proof. □

Define $f : G \to J(G)$ such that $f(x) = [0, x]$ for $x \in G$. Then f is injective, $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for all x, y in G. Thus we have the following theorem[2].

THEOREM 2.6. *There exists a lattice monomorphism* f *of the lattice* $(G, \vee, \wedge)$ *into* $(J(G), \vee, \wedge)$ *so that the image of* G *under* f *is a prime ideal of* $J(G)$, *i. e.* G *is isomorphic to a prime ideal* $\widetilde{G}$ *of* $J(G)$.

The mapping f in the above theorem is called the *Janowitz's embedding.* Note that the Janowitz's embedding preserves arbitrary suprema and infima whenever they exist in G. Let $\widetilde{G} = f(G)$. Then $\widetilde{G}$ is a sublattice of $J(G)$. We know that for each $I \in J(G)$, $I = [0, x]$ or $[0, x]^{\perp}$ for some $x \in G$. Hence, in particular, $0 \downarrow$ is the least element and G is the greatest element in $J(G)$. Now put $x = x \downarrow$, $x' = [0, x]^{\perp}$, $0 = 0 \downarrow$ and $1 = G$ in $J(G)$. As usual, denote the orthocomplement of x by x'. Then $J(G) = \{y | y = x \text{ or } y = x' \text{ for } x \in G\}$ and $(J(G), \vee, \wedge, ', 0, 1)$ is an OML, i.e. $(J(G), \vee, \wedge, ^{\perp}, 0, 1) \simeq (J(G), \vee, \wedge, ', 0, 1)$ The commutator of x and y in an OML L equals $x * y$ on $L[0, x \vee y]$ for all $x, y \in L$. Thus we define $x * y$ on a GOML G such that $x * y = (x \vee y) \wedge \left(x \vee y^{P(x \vee y)}\right)$ $\wedge \left(x^{P(x \vee y)} \vee y\right) \wedge \left(x^{P(x \vee y)} \vee y^{P(x \vee y)}\right)$.

THEOREM 2.7. *Let* f *be the* Janowitz's *embedding and* $\widetilde{G} = f(G)$. *Then* $Com\, \widetilde{G} = Com\, J(G)$ *where* $Com\, \widetilde{G}$ *and* $Com\, J(G)$ *denote the set of all commutators of* $\widetilde{G}$ *and* $J(G)$, *respectively.*

PROOF. Let $\widetilde{m}, \widetilde{n}$ be elements in $J(G)$. Then there exist x, y in G such that $\widetilde{m} = [0,x]$ or $[0,x]^{\perp}$ and $\widetilde{n} = [0,y]$ or $[0,y]^{\perp}$. We may assume $\widetilde{m} = [0,x]$ and $\widetilde{n} = [0,y]$ for some x, y in G, since $\widetilde{m} * \widetilde{n} = \widetilde{m}^{\perp} * \widetilde{n}^{\perp} = \widetilde{m}^{\perp} * \widetilde{n} = \widetilde{m} * \widetilde{n}^{\perp}$.

Using the equalities in corollary (2.2), we have the following:
$\widetilde{m} * \widetilde{n} = (\widetilde{m} \vee \widetilde{n}) \wedge (\widetilde{m} \vee \widetilde{n}^{\perp}) \wedge (\widetilde{m}^{\perp} \vee \widetilde{n}) \wedge (\widetilde{m}^{\perp} \vee \widetilde{n}^{\perp})$
$= ([0,x] \vee [0,y]) \wedge ([0,x] \vee [0,y]^{\perp}) \wedge ([0,x]^{\perp} \vee [0,y]) \wedge ([0,x]^{\perp} \vee [0,y]^{\perp})$
$= [0, x \vee y] \wedge [0, x^{P(x\vee y)} \wedge y]^{\perp} \wedge [0, x \wedge y^{P(x \vee y)}]^{\perp} \wedge [0, x \wedge y]^{\perp}$
$= [0, x\vee y] \wedge ([0, x\vee y] \wedge [0, x^{P(x\vee y)} \wedge y]^{\perp}) \wedge ([0, x \vee y] \wedge [0, x \wedge y^{P(x\vee y)}]^{\perp}) \wedge ([0, x \vee y]) \wedge [0, x \wedge y]^{\perp})$.

Now, $[0, x^{P(x\vee y)} \wedge y]^{\perp} \wedge ([0, x \vee y]$
$= [0, (x^{P(x\vee y)} \wedge y)^{P((x^{P(x\vee y)} \wedge y) \vee (x \vee y))} \wedge (x \vee y)]$
$= [0, (x^{P(x\vee y)} \wedge y)^{P(x\vee y)} \wedge (x \vee y)]$
$= [0, (x \vee y^{P(x\vee y)}) \wedge (x \vee y)]$
$= [0, x \vee y^{P(x\vee y)}]$.

Similarly, $[0, x \wedge y^{P(x\vee y)}]^{\perp} \wedge [0, x \vee y] = [0, x^{P(x\vee y)} \vee y]$ and $[0, x \wedge y] \wedge [0, x \vee y]^{\perp} = [0, x^{P(x\vee y)} \vee y^{P(x\vee y)}]$.

Thus $\widetilde{m} * \widetilde{n}$
$= [0, x \vee y] \wedge [0, x \vee y^{P(x\vee y)}] \wedge [0, x^{P(x\vee y)} \vee y] \wedge [0, x^{P(x\vee y)} \vee y^{P(x\vee y)}]$
$= [0,x] * [0,y]$ in $[0, x \vee y]$. Therefore $Com\, \widetilde{G} = Com\, J(G)$. □

THEOREM 2.8. *Let L be an* OML. *The following are equivalent:*
(1)$L \simeq J(G)$ *for a strict GOML G.*
(2)*There exists a proper nonprincipal prime ideal G in L.*

PROOF. (1) $\Longrightarrow$ (2) G is a proper nonprincipal ideal in $J(G)$ by Theorem 2.6.

(1) $\Longleftarrow$ (2) Let G be a prime ideal in L. Let us show that G is a strict GOML and $J(G) \simeq L$.

First, let us show two conditions in definition (1.1) and the strictness:
(a) $([0,a], \vee, \wedge, ^{P(a)}, 0, a]$ is an OML for all $a \in G$,
(b) for any $y \leq a \leq b$ of G, $y^{P(a)} = y^{P(a)} \wedge a$ where $y^{P(a)} = y' \wedge a$,
(c) G is not an OML; Suppose that G is an OML. Then there exists a greatest element y in G. Thus G is a principal ideal contrary to the hypothesis.

Finally, Let us show that $J(G) \simeq L$. Define $F : J(G) \to L$ such that $F([0,x]) = x$, $F([0,x]^\perp) = x'$ and $F([0,x^{P(a)}]) = x' \wedge a$. Then F is an ortho-isomorphism by the following (a), (b) and (c).

(a) F is an join-homomorphism:

$F([0,x] \vee [0,y]) = F([0,x \vee y]) = x \vee y = F([0,x] \vee F([0,y])$.

$F([0,x]^\perp \vee [0,y]^\perp) = F([0,x \wedge y]^\perp) = (x \wedge y)' = x' \vee y' = F([0,x]^\perp \vee F([0,y]^\perp)$.

$F([0,x]^\perp \vee [0,y]) = F([0, x \wedge y^{P(x \vee y)}]^\perp) = (x \wedge (y' \wedge (x \vee y)))' = x' \vee (y \vee (x' \wedge y')) = x' \vee y \vee (x' \wedge y') = x' \vee y = F([0,x]^\perp \vee F([0,y])$.

(b) F preserves the orthocomplement:

$F([0,x]^\perp) = x' = (F([0,x]))'$ and $F([0,x]^{\perp\perp}) = F([0,x]) = x = (x')'$.

(c) F is onto and one-to-one:

We know that $x = F([0,x])$ and $x' = F([0,x]^\perp)$, and $[0,x]^\perp \neq [0,y]$ for all $x, y \in G$ since G is not an OML by theorem (1.4). Moreover, $x = y$ if and only if $[0,x] = [0,y]$ and $x' = y'$ if and only if $[0,x]^\perp = [0,y]^\perp$. This completes the proof. □

We call a subset Y of a set X is *join dense*(*meet dense*) in X if every element of X is the join(meet) of a subset of Y.

LEMMA 2.9. *Let G be a* GOML, *and $J(G)$ be the* Janowitz's *hull of G and f be the* Janowitz's *embedding. Then $f(G) = \widetilde{G}$ is join dense in $J(G)$.*

PROOF. Since $J(G) = \{I | I = [0,x] \text{ or } [0,x]^\perp \text{ for } x \in G\}$, it is sufficient to show that $[0,x]^\perp$ is join dense. Indeed, $[0,x]^\perp = \bigcup_{z \geq x} \{[0, x^{P(z)}]\}$. Thus $[0,x]^\perp = \bigvee_{z \geq x} \{[0, x^{P(z)}]\}$ where $[0, x^{P(z)}] \in f(G)$. □

THEOREM 2.10. *Let G be a strict* GOML, *$J(G)$ be the* Janowitz's *hull of G and $\widetilde{G}_{\bar{x}} = \widetilde{G} \wedge [0,\bar{x}]$ for $\bar{x} \notin \widetilde{G}$ and $\bar{x} \in J(G)$. Then G is a strict* GOML, *$J(\widetilde{G}_{\bar{x}}) \simeq [0,\bar{x}]$ and $\widetilde{G}_{\bar{x}}$ is a prime ideal of $[0,\bar{x}]$.*

PROOF. Let us show that $\widetilde{G}_{\bar{x}}$ is a strict GOML.

First, we can prove that$\widetilde{G}_{\bar{x}}$ is a GOML by the following statements (a) and (b):

(a) $([0,\bar{a}], \vee, \wedge, {}^{P(\bar{a})}, 0, \bar{a})$ is an OML for all $\bar{a}$ in $\widetilde{G}_{\bar{x}}$.

(b) For any $\bar{y} \leq \bar{a} \leq \bar{b}$ in $\widetilde{G}_{\bar{x}}$, $\bar{y}^{P(a)} = \bar{y}^{P(b)} \wedge \bar{a}$. Indeed, for each $\bar{a} \in \widetilde{G}_{\bar{x}}$, $[0, \bar{a}]$ is an OML since $[0, \bar{a}] \subset [0, \bar{x}]$ and for any $\bar{y} \leq \bar{a} \leq \bar{b}$ in $\widetilde{G}_{\bar{x}}$, $\bar{y}^{P(\bar{a})} = \bar{y}^{P(\bar{b})} \wedge \bar{a}$.

Next, let us prove that $\widetilde{G}_{\bar{x}}$ is not an OML. Suppose that $\widetilde{G}_{\bar{x}}$ is an OML. Then there exists a greatest element $\bar{y} \in \widetilde{G}_{\bar{x}}$. Then $\bar{y}^{\perp} \notin \widetilde{G}$ since $\widetilde{G}$ is a prime ideal of $J(G)$. Thus $\bar{x} \wedge \bar{y}^{\perp} \notin \widetilde{G}_{\bar{x}}$. Therefore there exists $\bar{a} \in \widetilde{G}$ such that $\bar{a} \leq \bar{x} \wedge \bar{y}^{\perp}$ and $\bar{a} \not\leq \bar{y}$ since $\widetilde{G}$ is join dense in $J(G)$ by lemma (2.9), but $\bar{a} \in \widetilde{G}_{\bar{x}}$. Thus $\bar{a} \leq \bar{y}$ contrary to the fact that $\bar{a} \not\leq \bar{y}$. Therefore $\widetilde{G}_{\bar{x}}$ is a strict GOML.

Let us show that $J(\widetilde{G}_{\bar{x}}) \simeq [0, \bar{x}]$ and $\widetilde{G}_{\bar{x}}$ is a prime ideal of $[0, \bar{x}]$. First, define $H : J(\widetilde{G}_{\bar{x}}) \to [0, \bar{x}]$ such that $H([0, \bar{a}]) = \bar{a} \wedge \bar{x} = \bar{a}$ and $H([0, \bar{a}]^{\perp}) = \bar{a}^{\perp} \wedge \bar{x} = \bar{a}^{P(\bar{x})}$. Then H is an isomorphism by the following (a), (b), (c) and (d).

(a) H is a join-homomorphism:

$H([0, \bar{a}] \vee [0, \bar{b}]) = H([0, \bar{a} \vee \bar{b}]) = \bar{a} \vee \bar{b} = H([0, \bar{a}]) \vee H([0, \bar{b}])$,

$H([0, \bar{a}]^{\perp} \vee [0, \bar{b}]) = H([0, \bar{a} \vee \bar{b}^{P(\bar{a} \vee \bar{b})}]^{\perp}) = (\bar{a} \wedge \bar{b}^{P(\bar{a} \vee \bar{b})})^{\perp} \wedge \bar{x} = (\bar{a} \wedge (\bar{b}^{\perp} \wedge (\bar{a} \vee \bar{b})))^{\perp} \wedge \bar{x} = (\bar{a} \vee \bar{b}^{\perp})^{\perp} \wedge \bar{x} = (\bar{a}^{\perp} \vee \bar{b}) \wedge \bar{x} = (\bar{a}^{\perp} \wedge \bar{x}) \vee (\bar{b} \wedge \bar{x}) = (\bar{a}^{\perp} \wedge \bar{x}) \vee \bar{b} = H([0, \bar{a}]^{\perp}) \vee H([0, \bar{b}])$,

$H([0, \bar{a}]^{\perp} \vee [0, \bar{b}]^{\perp}) = H([0, \bar{a} \wedge \bar{b}]^{\perp}) = (\bar{a} \wedge \bar{b})^{\perp} \wedge \bar{x} = (\bar{a}^{\perp} \vee \bar{b}^{\perp}) \wedge \bar{x} = (\bar{a}^{\perp} \wedge \bar{x}) \vee (\bar{b}^{\perp} \wedge \bar{x}) = H([0, \bar{a}]^{\perp}) \vee H([0, \bar{b}]^{\perp})$.

(b) H preserves the orthocomplementation:

$H([0, \bar{a}]^{\perp}) = \bar{a}^{\perp} \wedge \bar{x}$ and $(H([0, \bar{a}]))^{P(\bar{x})} = \bar{a}^{P(\bar{x})} = \bar{a}^{\perp} \wedge \bar{x}$.

(c) H is onto:

Since $J(G) = \widetilde{G} \cup \widetilde{G}^{\perp}$, $\bar{a} \in [0, \bar{x}] \cap J(G)$ implies that $\bar{a} \in [0, \bar{x}] \cap \widetilde{G}$ or $\bar{a} \in [0, \bar{x}] \cap \widetilde{G}^{\perp}$. If $\bar{a} \in [0, \bar{x}] \cap \widetilde{G}$, then $H([0, \bar{a}]) = \bar{a}$. If $\bar{a} \in [0, \bar{x}] \cap \widetilde{G}^{\perp}$, then there exists $\bar{b}$ in G such that $\bar{b}^{\perp} \wedge \bar{x} = \bar{a}$. Thus $H([0, \bar{b}]^{\perp}) = \bar{b}^{\perp} \wedge \bar{x} = \bar{a}$.

(d) H is one-to-one:

Since $\widetilde{G}_{\bar{x}}$ is a strict GOML, $[0, \bar{a}] \neq [0, \bar{b}]^{\perp}$ for all $\bar{a}$, $\bar{b}$ in $\widetilde{G}_{\bar{x}}$ by Theorem 1.4. Moreover, $[0, \bar{a}] = [0, \bar{b}]$ if and only if $\bar{a} = \bar{b}$ if and only if $[0, \bar{a}]^{\perp} = [0, \bar{b}]^{\perp}$ if and only if $\bar{a}^{\perp} \wedge \bar{x} = \bar{b}^{\perp} \wedge \bar{x}$. Finally, $\widetilde{G}_{\bar{x}}$ is a prime ideal of $[0, \bar{x}]$ by Theorem 2.8. This completes the proof. □

Let $(P, \leq)$ be a poset, and let l and $u : \mathcal{P}(P) \to \mathcal{P}(P)$ by $M^l = \{x \in P | x \leq m \quad \forall m \in M\}$ and $M^u = \{x \in P | x \geq m \quad \forall m \in M\}$, for $M \subset P$. Let $M^{ul} = (M^u)^l$, $\{x\}^l = x^l$, $\{x\}^u = x^u$ $\forall x \in P$. Then

$\overline{P} = \{\overline{X} | \overline{X} = X^{ul}, \forall X \subset P\}$ is a complete lattice[1, 6] and the complete lattice $\overline{P}$ is called the *completion by cuts* of a poset P.

LEMMA 2.11. *Let P be a poset and $\overline{P}$ be the completion by cuts of P. Then P is join and meet dense in $\overline{P}$, i.e., if $M \in \overline{P}$, then $M = \bigwedge_{m \in M} m^l$ and $M = \bigvee_{y \in M^u} y^l$*[1, 6].

For any subset A of a set S is called *closed* if $A = A^{\perp\perp}$.

THEOREM 2.12. *Let S be a set with an orthogonality relation and with the partial ordering induced by the orthogonality relation. Then the closed subsets of S, partially ordered by inclusion, form a complete lattice $L(S)$. If $\{A_j\}$ is a family of closed subsets, $\bigwedge_j A_j$, the meet of A_j in $L(S)$, is just $\bigcap_j A_j$. The mapping $A \to A^{\perp}$ is an orthocomplementation in $L(S)$. Further, there exists a one-to-one mapping of S into $L(S)$ which preserves orthogonality, order, and all existing meets in S. If S is orthocomplemented, this map also preserves orthocomplements and all joins existing in S*[5].

We will use $L(S)$ to denote the lattice of closed subsets of S and call $L(S)$ the *completion* of S.

The following two theorems are well known[5].

THEOREM 2.13. *If S is an orthoposet, then $L(S) = \overline{S}$.*

THEOREM 2.14. *Let S be an orthoposet, and let I be a join dense subset of S. Let $\perp$ be the orthogonality relation in S. Then restricted to I, $\perp$ is an orthogonality relation. Further, the partial ordering induced in I by the orthogonality relation $\perp$ coincides with the partial ordering inherited from S. Finally $L(S)$ and $L(I)$ are ortho-isomorphic.*

Now we are ready to prove the following theorem.

THEOREM 2.15. *Let $J(G)$ be the* Janowitz's *hull of a* GMOL G, *and let f be the* Janowitz's *embedding from $(G, \wedge, \vee)$ into $(J(G), \wedge, \vee)$. Let $\widetilde{G} = f(G)$. Then $\overline{\widetilde{G}} \simeq \overline{J(G)}$.*

PROOF. $\widetilde{G}$ is join dense in $\overline{J(G)}$ since $\widetilde{G}$ is join dense in $J(G)$ by Lemma 2.9 and $J(G)$ is join dense in $\overline{J(G)}$ by Lemma 2.11. Thus $L(\widetilde{G}) \simeq L(J(G))$ by Theorem 2.14. Therefore $\overline{\widetilde{G}} \simeq \overline{J(G)}$ by Theorem 2.13. □

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