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# FUZZY $\mathcal{I}$-IDEALS IN IS-ALGEBRAS

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Abstract. In [9], the concept of fuzzy sets is applied to the theory of $\mathcal{I}$-ideals in a BCI-semigroup (it was renamed as an **IS**-algebra for the convenience of study), and a characterization of fuzzy $\mathcal{I}$-ideals by their level $\mathcal{I}$-ideals was discussed. In this paper, we study further properties of fuzzy $\mathcal{I}$-ideals. We prove that the homomorphic image and preimage of a fuzzy $\mathcal{I}$-ideal are also fuzzy $\mathcal{I}$-ideals.

## 1. Introduction

The concept of a fuzzy set is applied to generalize some of the basic concepts of general topology ([1]). Rosenfeld [12] applied it to the elementary theory of groupoids and groups. Xi [13] applied the notion of fuzzy sets to $BCK$-algebras. Jun [6 - 7] solved the problem of classifying fuzzy ideals by their family of level ideals in $BCK(BCI)$-algebras, and introduced the notion of closed fuzzy ideals of $BCI$-algebras and studied their properties. In [8], Jun et al. introduced the concept of **IS**-algebras and $\mathcal{I}$-ideals. Moreover, Jun et al. [9] applied the notion of fuzzy sets to $BCI$-semigroups (it was renamed as an **IS**-*algebra* for the convenience of study), and introduced the concept of fuzzy $\mathcal{I}$-ideals. This paper is a continuation of [9]. We study further properties on fuzzy $\mathcal{I}$-ideals, and prove that the homomorphic image and preimage of a fuzzy $\mathcal{I}$-ideal are also fuzzy $\mathcal{I}$-ideals.

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## 2. Preliminaries

In this section we include some elementary aspects of $BCI$-algebras, $BCI$-semi-groups, and fuzzy theories which are necessary for our discussion.

Recall that a *BCI-algebra* is an algebra $(X, *, 0)$ of type (2, 0) satisfying the following axioms for every $x, y, z \in X$,

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A partial ordering $\leq$ on $X$ can be defined by $x \leq y$ if and only if $x * y = 0$. A non-empty subset $I$ of a $BCI$-algebra $X$ is called an *ideal* of $X$ if

(i) $0 \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

In [8], Jun et al. introduced a new class of algebras related to $BCI$-algebras and semigroups, called a $BCI$-semigroup, and Jun et al. [10] renamed it as an **IS**-algebra for the convenience of study.

Definition 2.1. (Jun et al. [10]) An **IS**-*algebra* is a non-empty set $X$ with two binary operations "$*$" and "$\cdot$" and constant 0 satisfying the axioms:

(i) $(X, *, 0)$ is a $BCI$-algebra,

(ii) $(X, \cdot)$ is a semigroup,

(iii) the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

In what follows, for convenience, we shall write the multiplication $x \cdot y$ by $xy$, and $X$ would mean an **IS**-algebra unless otherwise specified.

Definition 2.2. (Jun et al. [9]) A non-empty subset $I$ of $X$ is called a *left* (resp. *right*) *$\mathcal{I}$-ideal* of $X$ if

(i) $I$ is an ideal of a $BCI$-algebra $X$,

(ii) $a \in X$ and $x \in I$ imply that $ax \in I$ (resp. $xa \in I$).

We now review some fuzzy logic concepts. We refer the reader to [1], [2], [12], [13] and [14] for complete details. A *fuzzy set* in a set $S$ is a function $\mu : S \to [0,1]$. For $\alpha \in [0,1]$, the set $\mu_\alpha := \{x \in S : \mu(x) \geq \alpha\}$ is called a *level subset* of $\mu$.

Let $S$ and $S'$ be two sets and let $f$ be a function of $S$ into $S'$. Let $\mu$ and $\nu$ be fuzzy sets in $S$ and $S'$, respectively. Then $f(\mu)$, the *image* of $\mu$ under $f$, is a fuzzy set in $S'$:

$$f(\mu)(y') := \begin{cases} \sup\limits_{f(x)=y'} \mu(x) & \text{if } f^{-1}(y') \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

for all $y' \in S'$. $f^{-1}(\nu)$, the *preimage* of $\nu$ under $f$, is a fuzzy set in $S$:

$$f^{-1}(\nu)(x) = \nu(f(x))$$

for all $x \in S$.

Let $S$ and $S'$ be two sets, $\mu$ be a fuzzy set in $S$ and $f : S \to S'$ be a function. Then $\mu$ is said to be *$f$-invariant* if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in S$.

Clearly, if $\mu$ is $f$-invariant, then $f^{-1}(f(\mu)) = \mu$.

DEFINITION 2.3. (Xi [13]) A fuzzy set $\mu$ of a $BCI$-algebra $X$ is called a *fuzzy ideal* of $X$ if for any $x, y \in X$,

(i) $\mu(0) \geq \mu(x)$,

(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$.

DEFINITION 2.4. (Jun et al. [9]) A fuzzy set $\mu$ in $X$ is called a *fuzzy left* (resp. *right*) *$\mathcal{I}$-ideal* of $X$ if

(i) $\mu$ is a fuzzy ideal of a $BCI$-algebra $X$,

(ii) $\mu(xy) \geq \mu(y)$ (resp. $\mu(xy) \geq \mu(x)$) for all $x, y \in X$.

From now on, a (fuzzy) $\mathcal{I}$-ideal shall mean a (fuzzy) left $\mathcal{I}$-ideal.

PROPOSITION 2.5. (Jun et al. [9]) *A fuzzy set* $\mu$ *in* $X$ *is a fuzzy* $\mathcal{I}$*-ideal of* $X$ *if and only if it satisfies for any* $x, y \in X$,

(i) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$,

(ii) $\mu(xy) \geq \mu(y)$.

PROPOSITION 2.6. (Jun et al. [9]) *Let* $\mu$ *be a fuzzy set in* $X$.

(i) *If* $\mu$ *is a fuzzy* $\mathcal{I}$*-ideal of* $X$, *then* $\mu_\alpha$ *is an* $\mathcal{I}$*-ideal of* $X$ *for all* $\alpha \in [0, \mu(0)]$ *which is called the level* $\mathcal{I}$*-ideal of* $\mu$.

(ii) *If* $\mu_\alpha$ *is an* $\mathcal{I}$*-ideal of* $X$ *for all* $\alpha \in \mathrm{Im}(\mu)$, *then* $\mu$ *is a fuzzy* $\mathcal{I}$*-ideal of* $X$, *where* $\mathrm{Im}(\mu)$ *is the image set of* $\mu$.

PROPOSITION 2.7. (Jun et al. [9]) *Let* $\mu$ *be a fuzzy* $\mathcal{I}$*-ideal of* $X$. *If* $\mathrm{Im}(\mu) := \{\alpha_1, ..., \alpha_n\}$, *then the family of* $\mathcal{I}$*-ideals* $\mu_{\alpha_i}$, $1 \leq i \leq n$, *constitutes all the level* $\mathcal{I}$*-ideals of* $\mu$.

PROPOSITION 2.8. (Jun et al. [9]) *If a fuzzy set* $\mu$ *in* $X$ *is a fuzzy* $\mathcal{I}$*-ideal of* $X$, *then the set* $X_\mu := \{x \in X : \mu(x) = \mu(0)\}$ *is an* $\mathcal{I}$*-ideal of* $X$.

PROPOSITION 2.9. (Jun et al. [9]) *Let* $I$ *be a non-empty subset of* $X$ *and let* $\mu$ *be a fuzzy set in* $X$ *such that* $\mu$ *is into* $\{0, 1\}$, *so that* $\mu$ *is the characteristic function of* $I$. *Then* $\mu$ *is a fuzzy* $\mathcal{I}$*-ideal of* $X$ *if and only if* $I$ *is an* $\mathcal{I}$*-ideal of* $X$.

PROPOSITION 2.10. (Jun et al. [9]) *Let* $\mu$ *be a fuzzy* $\mathcal{I}$*-ideal of* $X$ *and let* $\mu_\alpha, \mu_\beta$ *be level* $\mathcal{I}$*-ideals of* $\mu$, *where* $\alpha < \beta$. *Then the following are equivalent:*

(i) $\mu_\alpha = \mu_\beta$.

(ii) *There is no* $x \in X$ *such that* $\alpha \leq \mu(x) < \beta$.

## 3. Main Results

THEOREM 3.1. *Let* $\mu$ *be a fuzzy set in* $X$ *and let* $\mathrm{Im}(\mu) = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$, *where* $\alpha_i < \alpha_j$ *whenever* $i > j$. *Suppose that there exists a* *chain of $\mathcal{I}$-ideals of $X$:*

$$I_0 \subset I_1 \subset \ldots \subset I_k = X$$

*such that $\mu(I_n^*) = \alpha_n$, where $I_n^* = I_n \setminus I_{n-1}$, and $I_{-1} = \emptyset$, for $n = 0, 1, \ldots, k$. Then $\mu$ is a fuzzy $\mathcal{I}$-ideal of $X$.*

PROOF. Let $x, y \in X$. If $x$ and $y$ belong to the same $I_n^*$, then $\mu(x) = \mu(y) = \alpha_n$, and so

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}.$$

Assume that $x \in I_i^*$ and $y \in I_j^*$ for every $i \neq j$. Without loss of generality, we may assume that $i > j$. Then $\mu(x) = \alpha_i < \alpha_j = \mu(y)$, and so

$$\min\{\mu(y * x), \mu(x)\} \leq \mu(x) < \mu(y).$$

Since $y \in I_j^*$, we have $y \in I_j$. It follows that $y \in I_{i-1}$ as $j \leq i - 1$. Now we assert that $x * y \notin I_{i-1}$. In fact, if not, then $x * y \in I_{i-1}$ and $y \in I_{i-1}$ implies $x \in I_{i-1}$, which contradicts to $x \in I_i^* = I_i \setminus I_{i-1}$. Hence $\mu(x * y) \leq \alpha_i$, and so

$$\mu(x) = \alpha_i \geq \min\{\mu(x * y), \mu(y)\}.$$

Summarizing the above results, we obtain that

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$$

for all $x, y \in X$. For any $x, y \in X$ there exist indices $i$ and $j$ such that $x \in I_i^*$ and $y \in I_j^*$. Since $I_j$ is an $\mathcal{I}$-ideal of $X$, it follows that $xy \in I_j$ so that $\mu(xy) \geq \alpha_j = \mu(y)$. Thus, by Proposition 2.5, $\mu$ is a fuzzy $\mathcal{I}$-ideal of $X$. □

THEOREM 3.2. *Let $\mu$ be a fuzzy $\mathcal{I}$-ideal of $X$. If $\operatorname{Im}(\mu) = \{\alpha_i \mid i = 0, 1, \ldots, k\}$ with $\alpha_i < \alpha_j$ whenever $i > j$, then $I_n = \mu_{\alpha_n}$, $n = 0, 1, \ldots, k$, are $\mathcal{I}$-ideals of $X$ and $\mu(I_n^*) = \alpha_n$, $n = 0, 1, \ldots, k$, where $I_n^* = I_n \setminus I_{n-1}$ and $I_{-1} = \emptyset$.*

PROOF. By Proposition 2.7, $I_n = \mu_{\alpha_n}$ $(n = 0, 1, \ldots, k)$ is an $\mathcal{I}$-ideal of $X$. Obviously, $\mu(I_0) = \alpha_0$. Since $\mu(I_1) = \{\alpha_0, \alpha_1\}$, for $x \in I_1^*$ we have $\mu(x) = \alpha_1$, namely $\mu(I_1^*) = \alpha_1$. Repeating this process, we conclude that $\mu(I_n^*) = \alpha_n$ for $n = 0, 1, \ldots, k$, ending the proof. □

THEOREM 3.3. *Let $\mu$ be a fuzzy $\mathcal{I}$-ideal of $X$ with* $\mathrm{Im}(\mu) = \{\alpha_i | i \in \Lambda\}$ *and* $\mathcal{H} = \{\mu_{\alpha_i} | i \in \Lambda\}$ *where $\Lambda$ is an arbitrary index set. Then*

(i) *there exists a unique $i_0 \in \Lambda$ such that $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Lambda$,*
(ii) *$X_\mu$ is represented by the intersection of $\mu_{\alpha_i}$, $i \in \Lambda$,*
(iii) *$X$ is represented by the union of $\mu_{\alpha_i}$, $i \in \Lambda$,*
(iv) *the members of $\mathcal{H}$ form a chain, and*
(v) *$\mathcal{H}$ contains all level $\mathcal{I}$-ideals of $\mu$ if and only if $\mu$ attains its infimum on all $\mathcal{I}$-ideals of $X$.*

PROOF. (i) Since $\mu(0) \in \mathrm{Im}(\mu)$, there exists a unique $i_0 \in \Lambda$ such that $\mu(0) = \alpha_{i_0} \geq \alpha_i$ for all $i \in \Lambda$.

(ii) Clearly, $X_\mu = \mu_{\mu(0)} = \mu_{\alpha_{i_0}}$. Since $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Lambda$, therefore $\mu_{\alpha_{i_0}} \subseteq \mu_{\alpha_i}$ for all $i \in \Lambda$. Hence $\mu_{\alpha_{i_0}} \subseteq \bigcap_{i \in \Lambda} \mu_{\alpha_i}$. The reverse inclusion is obvious, and so $X_\mu = \bigcap_{i \in \Lambda} \mu_{\alpha_i}$.

(iii) Let $x \in X$. Then $\mu(x) \in \mathrm{Im}(\mu)$ and so there exists $i(x) \in \Lambda$ such that $\mu(x) = \alpha_{i(x)}$. This implies $x \in \mu_{\alpha_{i(x)}} \subset \bigcup_{i \in \Lambda} \mu_{\alpha_i}$. This proves (iii).

(iv) Noticing that $\alpha_i \geq \alpha_j \Leftrightarrow \mu_{\alpha_i} \subseteq \mu_{\alpha_j}$ for any $i, j \in \Lambda$, (iv) is clear.

(v) Suppose that $\mathcal{H}$ contains all level $\mathcal{I}$-ideals of $\mu$. Let $I$ be an $\mathcal{I}$-ideal of $X$. If $\mu$ is a constant on $I$, then we are done. Assume that $\mu$ is not a constant on $I$. We discuss the following two cases: (1) $I = X$ and (2) $I \subsetneq X$. For the case (1), we let $\beta = \inf\{\alpha_i | i \in \Lambda\}$. Then $\beta \leq \alpha_i$ for all $i \in \Lambda$, and so $\mu_\beta \supseteq \mu_{\alpha_i}$ for all $i \in \Lambda$. Note that $\mu_0 = X \in \mathcal{H}$ because $\mathcal{H}$ contains all level $\mathcal{I}$-ideals of $\mu$. Hence there exists $j \in \Lambda$ such that $\alpha_j \in \mathrm{Im}(\mu)$ and $\mu_{\alpha_j} = X$. It follows that $\mu_\beta \supseteq \mu_{\alpha_j} = X$ so that $\mu_\beta = \mu_{\alpha_j} = X$ because every level $\mathcal{I}$-ideal of $\mu$ is an $\mathcal{I}$-ideal of $X$. Now it is sufficient to show that $\beta = \alpha_j$. If $\beta < \alpha_j$, then there exists $k \in \Lambda$ such that $\alpha_k \in \mathrm{Im}(\mu)$ and $\beta \leq \alpha_k < \alpha_j$. This implies that $\mu_{\alpha_k} \supsetneq \mu_{\alpha_j} = X$, a contradiction. Therefore $\beta = \alpha_j$. If case (2) holds, consider the restriction $\mu_I$ of $\mu$ to $I$. By Proposition 2.9, $\mu_I$ is a fuzzy $\mathcal{I}$-ideal of $X$. Let $\Lambda_I := \{i \in \Lambda | \mu(y) = \alpha_i \text{ for some } y \in I\}$ and $\mathcal{H}_I := \{(\mu_I)_{\alpha_i} | i \in \Lambda_I\}$. Notice that $\mathcal{H}_I$ contains all level $\mathcal{I}$-ideals of $\mu_I$. Then there exists $z \in I$ such that $\mu_I(z) = \inf\{\mu_I(x) | x \in I\}$, which implies that $\mu(z) = \inf\{\mu(x) | x \in I\}$. Conversely assume that $\mu$ attains its infimum on all $\mathcal{I}$-ideals of $X$. Let $\mu_\alpha$ be a level $\mathcal{I}$-ideal of $\mu$. If $\alpha = \alpha_i$ for some $i \in \Lambda$, then clearly $\mu_\alpha \in \mathcal{H}$. Assume that $\alpha \neq \alpha_i$ for all $i \in \Lambda$. Then there does not exist $x \in X$ such that $\mu(x) = \alpha$. Let $I := \{x \in X | \mu(x) > \alpha\}$. Obviously $0 \in I$. Let $x, y \in X$ be such that $x * y \in I$ and $y \in I$. Then $\mu(x * y) > \alpha$ and $\mu(y) > \alpha$. It follows that

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\} > \alpha$$

so that $x \in I$. Hence $I$ is an ideal of a $BCI$-algebra $X$. Assume that $a \in X$ and $x \in I$. Then $\mu(ax) \geq \mu(x) > \alpha$, and so $ax \in I$. Therefore $I$ is an $\mathcal{I}$-ideal of $X$. By hypothesis, there exists $y \in I$ such that $\mu(y) = \inf\{\mu(x) | x \in I\}$. Now $\mu(y) \in \mathrm{Im}(\mu)$ implies $\mu(y) = \alpha_j$ for some $j \in \Lambda$; hence $\inf\{\mu(x) | x \in I\} = \alpha_j > \alpha$. Note that there does not exist $z \in X$ such that $\alpha \leq \mu(z) < \alpha_j$. It follows from Proposition 2.10 that $\mu_\alpha = \mu_{\alpha_j}$. Hence $\mu_\alpha \in \mathcal{H}$. This completes the proof. □

DEFINITION 3.4. Let $X$ and $X'$ be **IS**-algebras. A mapping $f : X \to X'$ is called a *homomorphism* if it preserves the "$*$" and "$\cdot$"-operations.

The following proposition will be used in the sequel.

PROPOSITION 3.5. *Let $f$ be a mapping from a set $X$ to a set $X'$, and let $\mu$ be a fuzzy set in $X$. Then for every $\alpha \in (0,1]$,*

$$f(\mu)_\alpha = \bigcap_{0<\beta<\alpha} f(\mu_{\alpha-\beta}).$$

PROOF. Let $\alpha \in (0,1]$. For $y = f(x) \in X'$, assume that $y \in f(\mu)_\alpha$. Then

$$\alpha \leq f(\mu)(y) = f(\mu)(f(x)) = \sup_{z \in f^{-1}(f(x))} \mu(z).$$

Hence for every real number $\beta$ with $0 < \beta < \alpha$, there exists $x_0 \in f^{-1}(y)$ such that $\mu(x_0) > \alpha - \beta$, and so $y = f(x_0) \in f(\mu_{\alpha-\beta})$. Therefore $y \in \bigcap_{0<\beta<\alpha} f(\mu_{\alpha-\beta})$. Conversely let $y \in \bigcap_{0<\beta<\alpha} f(\mu_{\alpha-\beta})$. Then $y \in f(\mu_{\alpha-\beta})$ for every $\beta$ with $0 < \beta < \alpha$, which implies that there exists $x_0 \in \mu_{\alpha-\beta}$ such that $y = f(x_0)$. It follows that $\mu(x_0) \geq \alpha - \beta$ and $x_0 \in f^{-1}(y)$, so that

$$f(\mu)(y) = \sup_{z \in f^{-1}(y)} \mu(z) \geq \sup_{0<\beta<\alpha} \{\alpha - \beta\} = \alpha.$$

Hence $y \in f(\mu)_\alpha$, and the proof is complete. □

In the following theorems, $f$ will denote a homomorphism from $X$ onto $X'$, where $X$ and $X'$ are **IS**-algebras.

THEOREM 3.6. (i) *If $\mu$ is a fuzzy $\mathcal{I}$-ideal of $X$, then the homomorphic image $f(\mu)$ of $\mu$ under $f$ is a fuzzy $\mathcal{I}$-ideal of $X'$.*

(ii) *If $\nu$ is a fuzzy $\mathcal{I}$-ideal of $X'$, then the homomorphic preimage $f^{-1}(\nu)$ of $\nu$ under $f$ is a fuzzy $\mathcal{I}$-ideal of $X$.*

PROOF. (i) In view of Proposition 2.6, it is sufficient to show that each nonempty level subset of $f(\mu)$ is an $\mathcal{I}$-ideal of $X'$. Let $f(\mu)_\alpha$ be a nonempty level subset of $f(\mu)$ for every $\alpha \in [0, 1]$. If $\alpha = 0$ then $f(\mu)_\alpha = X'$. If $\alpha \in (0, 1]$ then, by Proposition 3.5, $f(\mu)_\alpha = \bigcap_{0<\beta<\alpha} f(\mu_{\alpha-\beta})$. Hence $f(\mu_{\alpha-\beta}) \neq \emptyset$ for each $0 < \beta < \alpha$, and so $\mu_{\alpha-\beta}$ is a nonempty level subset of $\mu$ for every $0 < \beta < \alpha$. Since $\mu$ is a fuzzy $\mathcal{I}$-ideal of $X$, it follows from Proposition 2.6 that $\mu_{\alpha-\beta}$ is an $\mathcal{I}$-ideal of $X$ so that $f(\mu_{\alpha-\beta})$ is an $\mathcal{I}$-ideal of $X'$ because $f$ is onto. Hence $f(\mu)_\alpha$ being an intersection of a family of $\mathcal{I}$-ideals is also an $\mathcal{I}$-ideal of $X'$.

(ii) For any $x \in X$, we have

$$f^{-1}(\nu)(x) = \nu(f(x)) \geq \min\{\nu(f(x) * y'), \nu(y')\}$$

for any $y' \in X'$. Since $f$ is onto, there exists $y \in X$ such that $f(y) = y'$.

Hence

$$\begin{aligned} f^{-1}(\nu)(x) &\geq \min\{\nu(f(x) * y'), \nu(y')\} \\ &= \min\{\nu(f(x) * f(y)), \nu(f(y))\} \\ &= \min\{\nu(f(x * y)), \nu(f(y))\} \\ &= \min\{f^{-1}(\nu)(x * y), f^{-1}(\nu)(y)\}, \end{aligned}$$

and this result is true for all $x, y \in X$ because $y'$ is an arbitrary element of $X'$. Now for any $x, y \in X$, we have

$$f^{-1}(\nu)(xy) = \nu(f(xy)) = \nu(f(x)f(y)) \geq \nu(f(y)) = f^{-1}(\nu)(y).$$

By Proposition 2.5, $f^{-1}(\nu)$ is a fuzzy $\mathcal{I}$-ideal of $X$. □

THEOREM 3.7. *Let $\mu$ be a fuzzy $\mathcal{I}$-ideal of $X$. The mapping $\mu \mapsto f(\mu)$ defines a one-to-one correspondence between the set of all $f$-invariant fuzzy $\mathcal{I}$-ideals of $X$ and the set of all fuzzy $\mathcal{I}$-ideals of $X'$.*

PROOF. The proof is straightforward in view of Theorem 3.6 and the following results:

(i) $f^{-1}(f(\mu)) = \mu$, where $\mu$ is any $f$-invariant fuzzy $\mathcal{I}$-ideal of $X$;
(ii) $f(f^{-1}(\nu)) = \nu$, where $\nu$ is any fuzzy $\mathcal{I}$-ideal of $X'$. □

THEOREM 3.8. *Let $\mu$ and $\nu$ be fuzzy $\mathcal{I}$-ideals of $X$ and $X'$, respectively such that*

$$\mathrm{Im}(\mu) = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \text{ with } \alpha_0 > \alpha_1 > \ldots > \alpha_n, \text{ and}$$

$$\mathrm{Im}(\nu) = \{\beta_0, \beta_1, \ldots, \beta_m\} \text{ with } \beta_0 > \beta_1 > \ldots > \beta_m.$$

*Then*

(i) $\mathrm{Im}(f(\mu)) \subset \mathrm{Im}(\mu)$ *and the chain of level $\mathcal{I}$-ideals of $f(\mu)$ is*

$$f(\mu_{\alpha_0}) \subset f(\mu_{\alpha_1}) \subset \ldots \subset f(\mu_{\alpha_n}) = X'.$$

(ii) $\mathrm{Im}(f^{-1}(\nu)) = \mathrm{Im}(\nu)$ *and the chain of level $\mathcal{I}$-ideals of $f^{-1}(\nu)$ is*

$$f^{-1}(\nu_{\beta_0}) \subset f^{-1}(\nu_{\beta_1}) \subset \ldots \subset f^{-1}(\nu_{\beta_m}) = X.$$

PROOF. (i) Since $f(\mu)(y') = \sup_{f(x)=y'} \mu(x)$ for all $y' \in X'$, obviously $\mathrm{Im}(f(\mu)) \subset \mathrm{Im}(\mu)$. Note that for any $y' \in X'$,

$$\begin{aligned} y' \in f(\mu_{\alpha_i}) &\Leftrightarrow \text{there exists } x \in f^{-1}(y') \text{ such that } \mu(x) \geq \alpha_i \\ &\Leftrightarrow \sup_{f(z)=y'} \mu(z) \geq \alpha_i \\ &\Leftrightarrow f(\mu)(y') \geq \alpha_i \\ &\Leftrightarrow y' \in (f(\mu))_{\alpha_i}. \end{aligned}$$

Hence $f(\mu_{\alpha_i}) = (f(\mu))_{\alpha_i}$ for $i = 0, 1, \ldots, n$, and therefore the chain of level $\mathcal{I}$-ideals of $f(\mu)$ is

$$f(\mu_{\alpha_0}) \subset f(\mu_{\alpha_1}) \subset \ldots \subset f(\mu_{\alpha_n}) = X'.$$

(ii) Since $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$ and since $f$ is onto, we have the equality $\mathrm{Im}(f^{-1}(\nu)) = \mathrm{Im}(\nu)$. Note that for all $x \in X$,

$$\begin{aligned} x \in f^{-1}(\nu_{\beta_i}) &\Leftrightarrow f(x) \in \nu_{\beta_i} \\ &\Leftrightarrow \nu(f(x)) \geq \beta_i \\ &\Leftrightarrow f^{-1}(\nu)(x) \geq \beta_i \\ &\Leftrightarrow x \in f^{-1}(\nu)_{\beta_i}, \end{aligned}$$

so that $f^{-1}(\nu_{\beta_i}) = f^{-1}(\nu)_{\beta_i}$ for all $i = 0, 1, \ldots, m$. Hence the chain of level $\mathcal{I}$-ideals of $f^{-1}(\nu)$ is

$$f^{-1}(\nu_{\beta_0}) \subset f^{-1}(\nu_{\beta_1}) \subset \ldots \subset f^{-1}(\nu_{\beta_m}) = X.$$

This completes the proof. □

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