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RELATIVE PROJECTIVE MONOMIAL GROUPS

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ABSTRACT. As an application of Clifford theory, we are interested in a situation in which every irreducible projective character of a finite group G is an induced character of an irreducible linear character of some subgroup H of G. For this purpose, we study relative projective monomial groups with respect to subgroups.

1. Introduction

Let χ be an irreducible projective character of a finite group G over a field F. Then χ need not be induced from a character of a normal subgroup N of G. However Clifford theorem says that χ is always induced from a character θ of a subgroup H of G containing N. Thus $\theta^G = \chi$ and $\theta_N = e\chi$ for some $e \geq 1$. But because of the troublesome factor e, the situation was studied where $\theta^G = \chi$ with a character θ of a subgroup $H < G$ containing N and θ_N is irreducible.

An irreducible projective representation ρ on G is said to be relative projective monomial over a normal subgroup N if there is a subgroup $H < G$ and there is an irreducible projective representation ψ on H such that $N < H$, $\psi^G = \rho$ and ψ_N irreducible. If every irreducible projective representation of G is relative projective monomial over N then G is called a relative projective monomial group over N. For the relative projective monomial representation ρ of G over N, ρ is said to be induced over N, and such H and ψ are said to induce ρ over N. Due to relationships between representations and group characters, an irreducible

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projective character of G can be referred as relative projective monomial by substituting representation by character.

In this paper we study relative projective monomial groups. We prove that direct products and images of relative projective monomial groups are relative projective monomial, which were studied in [1] for monomial groups. We find relationships between monomial and projective monomial groups. Since every projective representation of G is related to a 2-cocycle $\alpha \in Z^2(G, F^*)$, a different situation here is to calculate the corresponding 2-cocycles.

2. Preliminary

Let F be a field, N a normal subgroup of a finite group G and $\alpha \in Z^2(G, F^*)$. An irreducible (projective) α-character χ is called α-monomial if there is a subgroup H of G and a linear α-character θ of H such that $\rho = \theta^G$. Thus a relative α-monomial character over 1 is an α-monomial character.

Projective monomial groups share many properties with monomial groups, however there are differences that if F is an algebraically closed field and if G/N is supersolvable then G is relative projective monomial over N. But G can be projective α-monomial if G/N is supersolvable, N is abelian and α_N is coboundary. Moreover in contrast to monomial groups, metabelian group need not be projective monomial (for example $G = A_4$ the alternating group of degree 4) while it is monomial as well as relative monomial over every normal subgroups.

As one of the fundamental distinctions of a projective character with an ordinary character, a projective character need not be a class function and so the resulting formula is more complicate than the corresponding one for character.

An element $x \in G$ is said to be α-regular if $\alpha(x, g) = \alpha(g, x)$ for all $g \in G$ such that $g = g^x$, where $g^x = x^{-1}gx$. If $\alpha(x, y) = \alpha(y, x^y)$ for any α-regular $x \in G$ and $y \in G$ then α is said to be normal. And α is a class function cocycle if every α-character of G is a class function.

LEMMA 1. ([5, (1.6.2)]) *Let $\alpha \in Z^2(G, F^*)$ and χ be an α-character of G. Then α is cohomologous to a normal cocycle and $\chi(x) = \alpha(x, y)\ \alpha^{-1}$*

(y, x^y) $\chi(x^y)$ for $x, y \in G$. Thus if α is normal then χ is a class function. If F splits G with $charF \nmid |G|$ then α is normal if and only if α is a class function cocycle.

Let χ be a character of $H < G$. Then the induced character χ^G of G is given by $\chi^G(g) = \frac{1}{|H|}\sum_{x\in G}\chi^\circ(g^x)$ where χ° satisfies $\chi^\circ(y) = \chi(y)$ if $y \in H$ and $\chi^\circ(y) = 0$ otherwise. For a projective α-character, we have the next.

LEMMA 2. ([5, (1.9.1)]) *For $\alpha \in Z^2(G, F^*)$, let χ be an α-character of H with $\chi(x) = 0$ for all $x \in G - H$. Let $\{g_1, .., g_t\}$ be a left transversal of H in G. Then for any $g \in G$, $\chi^G(g) = \sum_{i=1}^t \alpha(g, g_i)\alpha^{-1}(g_i, g^{g_i})\chi(g^{g_i})$. If G_0 is the set of α-regular elements of G and α is normal, then $\chi^G(g) = \sum_{i=1}^t \chi(g^{g_i})$ if $g \in G_0$, and zero otherwise. Furthermore $\chi^G(g) = 1/|H| \sum_{x\in G}\chi(g^x)$ if $g \in G_0$.*

3. Relative projective monomial groups

Let $\mathrm{Irr}(G)_\alpha$ be the set of irreducible α-representations of G over an algebraically closed field F of characteristic 0. And the same notation will be used for the set of irreducible α-characters abusively. If $\alpha = 1$, then $\mathrm{Irr}(G)_\alpha = \mathbf{Irr}(G)$ is the set of irreducible characters of G.

THEOREM 3. *Let G and K be two isomorphic groups. If G is a projective monomial group then so is K.*

PROOF. Let $f : G \to K$ be an isomorphism and for some $\beta \in Z^2(K, F^*)$, χ be any irreducible β-character afforded by an irreducible β-representation $\rho : K \to GL(V)$ for some simple K-module V.

If we define α by $\alpha(g, x) = \beta(f(g), f(x))$ for $g, x \in G$, then clearly α is a 2-cocycle in $Z^2(G, F^*)$. Since V can be a simple G-module by setting $gv = f(g)v$ for $v \in V$, the composition $\rho f : G \to GL(V)$ satisfies the following.

$$\rho f(g)\rho f(x) = \beta(f(g), f(x))\rho(f(g)f(x)) = \alpha(g, x)\rho f(gx) \quad (g, x \in G).$$

Hence ρf is an irreducible α-representation on G that induces a character χf because $\mathrm{tr}(\rho f)(g) = \mathrm{tr}\rho(f(g)) = \chi f(g)$ for all $g \in G$. We write $\theta = \chi f$.

For $\alpha \in Z^2(G, F^*)$, we may regard G as an α-monomial group. Hence there is a subgroup $H < G$ and an irreducible α-representation $\psi_1 \in$

$\mathrm{Irr}(H)_\alpha$ which affords a linear α-character θ_1 such that $\psi_1^G = \rho f$ and $\theta_1^G = \theta$.

Now let $f(H) = T < K$ and $\rho_1 = \psi_1 f_H^{-1}$. Then for any $s, t \in T$, since

$$\begin{aligned}\rho_1(s)\rho_1(t) &= \psi_1 f^{-1}(s)\psi_1 f^{-1}(t)\\ &= \alpha(f^{-1}(s), f^{-1}(t))\psi_1(f^{-1}(st))\\ &= \beta(s,t)\rho_1(st),\end{aligned}$$

ρ_1 is an irreducible β-representation on T. And by computing

$$\mathrm{tr}(\rho_1(t)) = \mathrm{tr}\rho_1(f(h)) = \mathrm{tr}\psi_1(h) = \theta_1(h) = \theta_1 f_H^{-1}(t), \quad (t \in T, f(h) = t),$$

it follows that ρ_1 affords the character $\chi_1 = \theta_1 f_H^{-1}$.

We claim that $x \in G$ is α-regular if and only if $f(x)$ is β-regular. Indeed, if $k \in K$ with $f(x)k = kf(x)$ and if $k = f(g)$ $(g \in G)$, then since $f(xg) = f(x)k = kf(x) = f(gx)$ it follows that $xg = gx$ and $\alpha(x,g) = \alpha(g,x)$. This shows that $\beta(f(x),k) = \alpha(x,g) = \alpha(g,x) = \beta(k, f(x))$, as required.

Therefore since $\chi_1^\circ(g) = (\theta_1 f_H^{-1})^\circ(g) = \theta_1^\circ f_H^{-1}(g)$ for all $g \in G$, whether $g \in H$ or $g \notin H$, we have, for any $w \in K$ with $f(x) = w$ $(x \in G)$

$$\begin{aligned}\chi_1^K(w) &= \frac{1}{|T|}\sum_{u\in K}\chi_1^\circ(w^u) = \frac{1}{|f(H)|}\sum_{u\in K}\theta_1^\circ f_H^{-1}(w^u) \qquad (\text{for } f(y) = u)\\ &= \frac{1}{|f(H)||\mathrm{Ker}f_H|}\sum_{y\in G}\theta_1^\circ(x^y) = \frac{1}{|H|}\sum_{y\in G}\theta_1^\circ(x^y)\\ &= \theta_1^G(x) = \chi f(x) = \chi(w).\end{aligned}$$

Moreover since $\chi_1(1) = 1$, χ_1 is linear and this completes the proof. □

For $\alpha, \beta \in Z^2(G, F^*)$, the theory of α-characters of G can be different from that of β-characters as the ordinary character theories of two different groups.

THEOREM 4. *Let α and β be in $Z^2(G, F^*)$. If they are cohomologous then G is an α-monomial group if and only if G is a β-monomial group.*

PROOF. Suppose that G is a β-monomial group. Let $\chi \in \mathrm{Irr}(G)_\alpha$ be afforded by an α-representation ρ on G. Since α is cohomologous to β, there is a map $t : G \to F^*$, $t(1) = 1$ such that $\beta = \alpha(\delta t)$. If

we define ρ_1 on G by $\rho_1(g) = t(g)\rho(g)$ for $g \in G$ then ρ_1 is an irreducible β-representation because $\rho_1(g)\rho_1(x) = t(g)t(x)\alpha(g,x)\rho(gx) = t(gx)\beta(g,x)\rho(gx) = \beta(g,x)\rho_1(gx)$ for $g, x \in G$. And since $\mathrm{tr}\rho_1(g) = t(g)\chi(g)$ for all $g \in G$, by letting χ_1 on G by $\chi_1(g) = t(g)\chi(g)$, χ_1 is an irreducible β-character of G afforded by ρ_1.

There is $H < G$ and an irreducible β-character θ_1 of H such that $\theta_1^G = \chi_1$ and θ_1 linear. Write ψ_1 for the β-representation on H which affords θ_1.

Let $\psi = t^{-1}\psi_1$ and let θ be defined by $\theta(h) = t^{-1}(h)\theta_1(h)$ for all $h \in H$. It is easy to see that ψ is an α-representation on H, which affords the character θ because $\mathrm{tr}(\psi(h)) = t^{-1}(h)\mathrm{tr}(\psi_1(h)) = t^{-1}(h)\theta_1(h) = \theta(h)$. Clearly $\theta(1) = t^{-1}(1)\theta_1(1) = 1$ so that θ is linear. Moreover, due to Lemma 2, for a left transversal $\{g_1, \cdots, g_t\}$ of H in G, we have

$$\chi(g) = t^{-1}(g)\theta_1^G(g) = t^{-1}(g)\sum_{i=1}^{t}\beta(g,g_i)\beta^{-1}(g_i, g^{g_i})\theta_1(g^{g_i})$$

$$= t^{-1}(g)\sum_{i=1}^{t} t(g)t(g_i)t^{-1}(gg_i)\alpha(g,g_i)t^{-1}(g_i)t^{-1}(g^{g_i})t(gg_i)\alpha^{-1}(g_i,g^{g_i})\theta(g^{g_i})$$

$$= \sum_{i=1}^{t}\alpha(g,g_i)\alpha^{-1}(g_i,g^{g_i})\theta(g^{g_i}) = \theta^G(g).$$

Thus G is an α-monomial group. This completes the proof. □

We remark that Theorem 3 and Theorem 4 are true for relative projective monomial groups over some normal subgroups.

By Lemma 1, any $\alpha \in Z^2(G, F^*)$ is cohomologous to a normal cocycle, and any α-character χ of G is a class function provided α is normal. Hence due to Theorem 4, if χ is an α-monomial character of G then we may regard α as normal and we can use the form in Lemma 2 for induced characters.

Let G be a direct product of groups G_i $(i = 1, 2)$ and χ_i be class functions on G_i. If we define $\chi = \chi_1 \times \chi_2$ by $\chi(g_1g_2) = \chi_1(g_1)\chi_2(g_2)$ for $g_i \in G_i$, then χ is a class function of G. Moreover if χ_i is a character of G_i, then under the isomorphism $G_i \cong G/G_j$ $(i \neq j)$, there is a character

$\hat{\chi}_i$ of G with $G_j < \mathrm{Ker}\hat{\chi}_i$ and $\hat{\chi}_i(g_1g_2) = \chi_i(g_i)$. Thus $\chi_1 \times \chi_2 = \hat{\chi}_1\hat{\chi}_2$ is a character of G.

For $\chi_i \in \mathrm{Irr}(G_i)$, $\chi_1 \times \chi_2$ are irreducible characters of $G_1 \times G_2$ ([3, (4.21)]). The similar situation holds for projective characters; if F is algebraically closed and if $\alpha_i \in Z^2(G_i, F^*)$ then the set of all irreducible $(\alpha_1 \times \alpha_2)$-characters of $G_1 \times G_2$ is $\{\chi_1 \times \chi_2 | \chi_i \in \mathrm{Irr}(G_i)_{\alpha_i}\}$ ([5, (1.5.1)]). On the other hand, all irreducible α-characters of $G_1 \times G_2$ for $\alpha \in Z^2(G_1 \times G_2, F^*)$ are as follows. Let G_i' be the commutator subgroup of G_i.

LEMMA 5. ([5, (1.5.2)]) *Let F be an algebraically closed field and G_1, G_2 be groups with $(|G_1/G_1'|, |G_2/G_2'|) = 1$. If $\alpha \in Z^2(G_1 \times G_2, F^*)$ then there is $\alpha_i \in Z^2(G_i, F^*)$ such that α is cohomologous to $(\alpha_1 \times \alpha_2)t$ for some $t : G_1 \times G_2 \to F^*$ with $t(1,1) = 1$. And $\{(\chi_1 \times \chi_2)t | \chi_i \in \mathrm{Irr}(G_i)_{\alpha_i}\}$ is the set of all distinct irreducible α-characters of $G_1 \times G_2$.*

THEOREM 6. *Let F be an algebraically closed field. A finite direct product of (relative) projective monomial groups is (relative) projective monomial. That is, for $\alpha_i \in Z^2(G_i, F^*)$ $(i = 1, .., n)$, if G_i are (relative) α_i-monomial groups (over $N_i \lhd G_i$), then $\prod G_i$ is (relative) $\prod \alpha_i$-monomial (over $\prod N_i$).*

PROOF. We show this for $i = 2$. Choose any irreducible $(\alpha_1 \times \alpha_2)$-character χ on $G_1 \times G_2$. Due to the statement above Lemma 5, we may write $\chi = \chi_1 \times \chi_2$ for some $\chi_i \in \mathrm{Irr}(G_i)_{\alpha_i}$. If G_i is α_i-monomial, there is $H_i < G_i$ and $\theta_i \in \mathrm{Irr}(H_i)_{\alpha_i}$ such that $\theta_i^{G_i} = \chi_i$ and θ_i linear. Then it is clear that $\theta_1 \times \theta_2 \in \mathrm{Irr}(H_1 \times H_2)_{\alpha_1 \times \alpha_2}$, and $(g_1, g_2) \in G_1 \times G_2$ is $(\alpha_1 \times \alpha_2)$-regular if and only if $g_i \in G_i$ are α_i-regular $(i = 1, 2)$. Hence, by computing we have

$$
\begin{aligned}
&(\theta_1 \times \theta_2)^{G_1 \times G_2}(g_1, g_2) \\
&= \frac{1}{|H_1 \times H_2|} \sum_{(x_1,x_2) \in G_1 \times G_2} (\theta_1 \times \theta_2)^\circ((g_1, g_2)^{(x_1,x_2)}) \\
&= \frac{1}{|H_1||H_2|} \sum_{x_1 \in G_1} \sum_{x_2 \in G_2} \theta_1^\circ(g_1^{x_1}) \theta_2^\circ(g_2^{x_2}) \\
&= \theta_1^{G_1} \times \theta_2^{G_2}(g_1, g_2),
\end{aligned}
$$

which shows that $(\theta_1 \times \theta_2)^{G_1 \times G_2} = \theta_1^{G_1} \times \theta_2^{G_2} = \chi_1 \times \chi_2 = \chi$. Since $1 = \theta_1(1)\theta_2(1)$, $\theta_1 \times \theta_2$ is linear, and so $G_1 \times G_2$ is $(\alpha_1 \times \alpha_2)$-monomial.

For the second assertion, if $\chi \in \mathrm{Irr}(G_1 \times G_2)_{\alpha_1 \times \alpha_2}$ then $\chi = \chi_1 \times \chi_2$ for some $\chi_i \in \mathrm{Irr}(G_i)_{\alpha_i}$ as before. Thus there is H_i with $N_i < H_i < G_i$ and $\theta_i \in \mathrm{Irr}(H_i)_{\alpha_i}$ such that $\theta_i^{G_i} = \chi_i$ and $\theta_{iN_i} \in \mathrm{Irr}(N_i)_{\alpha_i}$. Therefore we have $\theta_1 \times \theta_2 \in \mathrm{Irr}(H_1 \times H_2)_{\alpha_1 \times \alpha_2}$ and $(\theta_1 \times \theta_2)^{G_1 \times G_2} = \chi$. Since $(\theta_1 \times \theta_2)_{N_1 \times N_2} = \theta_{1N_1} \times \theta_{2N_2}$ which is an irreducible $(\alpha_1 \times \alpha_2)$-character of $N_1 \times N_2$. □

In next theorem, we study a converse of Theorem 6.

THEOREM 7. *Let $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$. If G is a projective α-monomial group for $\alpha \in Z^2(G, F^*)$ then G_i are projective α_i-monomial groups for some 2-cocycles $\alpha_i \in Z^2(G_i, F^*)$. This statement is true for relative projective monomial groups over normal subgroup N of G.*

PROOF. For $\alpha \in Z^2(G_1 \times G_2, F^*)$, there exists $\alpha_i \in Z^2(G_i, F^*)$ such that α is cohomologous to $(\alpha_1 \times \alpha_2) \cdot t$ for some $t : G_1 \times G_2 \to F^*$ with $t(1,1) = 1$.

Choose any $\chi_i \in \mathrm{Irr}(G_i)_{\alpha_i}$. Then $(\chi_1 \times \chi_2)t$ is an irreducible α-character of G by Lemma 5. Thus if we denote $\chi = (\chi_1 \times \chi_2)t \in \mathrm{Irr}(G)_\alpha$ then $\chi(g_1, g_2) = \chi_1(g_1)\chi_2(g_1)t(g_1, g_2)$ for $g_i \in G_i$. Since G is α-monomial, there is $H < G$ and there is $\theta \in \mathrm{Irr}(H)_\alpha$ such that $\theta^G = \chi$ and θ linear. As a subgroup of $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$, we may write $H = H_1 \times H_2$ with $H_i < G_i$. For $\alpha \in Z^2(H, F^*)$ and $\alpha_i \in Z^2(H_i, F^*)$, α is cohomologous to $(\alpha_1 \times \alpha_2) \cdot t_{H_1 \times H_2}$ where $t_{H_1 \times H_2} : H_1 \times H_2 \to F^*$ with $t_{H_1 \times H_2}(1,1) = 1$. Lemma 5 tells us that the irreducible α-character θ of $H = H_1 \times H_2$ forms $\theta = (\theta_1 \times \theta_2)t_{H_1 \times H_2}$ for some $\theta_i \in \mathrm{Irr}(H_i)_{\alpha_i}$. Now

$$(\chi_1 \times \chi_2)t = \chi = \theta^G = ((\theta_1 \times \theta_2)t_{H_1 \times H_2})^G = (\theta_1 \times \theta_2)^G t = (\theta_1^{G_1} \times \theta_2^{G_2})t,$$

which shows that $\chi_i = \theta_i^{G_i}$ $(i = 1, 2)$. Since $1 = \theta(1) = \theta_1(1)\theta_2(1)t(1,1) = \theta_1(1)\theta_2(1)$ and θ_i are linear, G_i are α_i-monomial.

For relative projective monomial case, we write $\alpha = (\alpha_1 \times \alpha_2) \cdot \delta t$ for some t and for some $\alpha_i \in Z^2(G_i, F^*)$ as before. And for any $\chi_i \in \mathrm{Irr}(G_i)_{\alpha_i}$, $(\chi_1 \times \chi_2)t$ is an irreducible α-character of G, which we denote by χ.

Since G is relative α-monomial over a normal subgroup N, there is H with $N < H < G$ and $\theta \in \mathrm{Irr}(H)_\alpha$ such that $\theta^G = \chi$ and $\theta_N \in \mathrm{Irr}(N)$.

Write $H = H_1 \times H_2$ and $N = N_1 \times N_2$ with $N_i < H_i < G_i$. Then the α-character θ forms $\theta = (\theta_1 \times \theta_2)t$ for some $\theta_i \in \mathrm{Irr}(H_i)_{\alpha_i}$ and clearly $\chi_i = \theta_i^{G_i}$. Moreover since $\theta_N = ((\theta_1 \times \theta_2)t)_N = (\theta_{1N_1} \times \theta_{2N_2})t_{N_1 \times N_2} \in \mathrm{Irr}(N)_\alpha$ with $t_{N_1 \times N_2}(1,1) = 1$ and θ_N is irreducible, each θ_{iN_i} is irreducible. □

As for ordinary monomial case, it is known that G_i $(i = 1, .., n)$ are monomial groups if and only if $\prod_{i=1}^n G_i$ is a monomial group.

COROLLARY 8. *If G_i $(i = 1, .., n)$ is relative monomial over $N_i \lhd G_i$ then $\prod G_i$ is relative monomial over $\prod N_i$. Conversely, if $G = \prod G_i$ is relative monomial over $N \lhd G$ then G_i are relative monomial over some normal subgroups.*

4. Factor groups of monomial groups

It was proved in Theorem 3 that if $G \cong K$ and G is projective monomial then so is K. We study more general situation in next theorem.

THEOREM 9. *If $f : G \to K$ is a homomorphism and G is relative monomial over N, then so is $f(G)$ over $f(N)$.*

PROOF. Let χ be a character afforded by an irreducible representation $\rho \in \mathrm{Irr}(f(G))$. As was done in Theorem 3, we have an irreducible representation ρf on G which affords character χf.

Since G is relative monomial over N, there is H with $N < H < G$ and an irreducible representation ψ_1 with character $\theta_1 \in \mathrm{Irr}(H)$ such that

$$\psi_1^G = \rho f, \quad \theta_1^G = \chi f \quad \text{and} \quad \theta_{1N} \in \mathrm{Irr}(N).$$

We remark that $\mathrm{Ker}f < \mathrm{Ker}\rho f = \mathrm{Ker}\psi_1^G = \bigcap_{g \in G}(\mathrm{Ker}\psi_1)^g < \mathrm{Ker}\psi_1$, using [3, (5.11)]. Consider $f(H)$. Then $f(N) < f(H) < f(G)$ and we may consider $f(H) \cong H/(\mathrm{Ker}f \cap H)$. Since $\mathrm{Ker}f \cap H < \mathrm{Ker}\psi_1$, there is a homomorphism ψ on $f(H)$ satisfying $\psi f = \psi_1$, which is clearly an irreducible representation on $f(H)$. Let θ be a map defined by $\theta(k) = \theta_1(h)$ for $k \in f(H)$ with $f(h) = k$ $(h \in H)$. Since $\mathrm{tr}\psi(k) = \mathrm{tr}\psi_1(h) = \theta_1(h) = \theta(k)$, θ is the irreducible character of $f(H)$ afforded by ψ. Now,

for any $y \in f(G)$ with $y = f(x)$ $(x \in G)$,

$$\begin{aligned}\theta^{f(G)}(y) &= \frac{1}{|f(H)|}\sum_{b\in f(G)}\theta^\circ(y^b) = \frac{1}{|f(H)|}\frac{1}{|\mathrm{Ker}f\cap H|}\sum_{a\in G}\theta^\circ f_H(x^a)\\ &= \frac{1}{|H|}\sum_{a\in G}(\theta f_H)^\circ(x^a) = \frac{1}{|H|}\sum_{a\in G}\theta_1^\circ(x^a)\\ &= \theta_1^G(x) = \chi f(x) = \chi(y)\end{aligned}$$

for $b = f(a)$. Moreover, since

$$\begin{aligned}[\theta_{f(N)}, \theta_{f(N)}] &= \frac{1}{|f(N)|}\sum_{t\in f(N)}\theta_{f(N)}(t)\theta_{f(N)}(t^{-1}) \quad \text{for} \quad t = f(n)\\ &= \frac{1}{|f(N)||\mathrm{Ker}f_N|}\sum_{n\in N}\theta_N(n)\theta_N(n^{-1})\\ &= \frac{1}{|N|}\sum_{n\in N}\theta f(n)\theta f(n^{-1}) = [\theta_{1N}, \theta_{1N}],\end{aligned}$$

we have $\theta_{f(N)} \in \mathrm{Irr}(f(N))$. This completes the proof. □

We note that it is known that every homomorphic image of monomial group is monomial. In monomial case, an irreducible character can be induced from a linear character of a subgroup which is in fact a homomorphism. However in relative monomial case the inducing character need not be linear. The next corollary follows immediately from Theorem 9.

COROLLARY 10. *If G is a monomial group then so is G/A for any normal subgroup A of G. If G is a relative monomial group over N then for any normal A of G contained in N, G/A is a relative monomial group over N/A.*

In what follows, we study the situation for projective monomial case.

THEOREM 11. *If G is a projective α-monomial group then G/A is a projective β-monomial group for some $\alpha \in Z^2(G, F^*)$ and $\beta \in Z^2(G/A, F^*)$.*

PROOF. Let χ be an irreducible β-character on G/A which is afforded by a representation ρ. Then there is a map ρ_1 on G satisfying $\rho_1(x) = \rho(xA)$ and $A < \mathrm{Ker}\rho_1$ for $x \in G$. Let χ_1 be a map such that $\chi_1(x) = \chi(xA)$.

Under the inflation map inf : $Z^2(G/A, F^*) \to Z^2(G, F^*)$, if we denote $\alpha = \inf\beta$ then for all $x, y \in G$, $\alpha(x, y) = \inf\beta(x, y) = \beta(xA, yA)$ and we have

$$\rho_1(x)\rho_1(y) = \rho(xA)\rho(yA) = \beta(xA, yA)\rho(xyA) = \alpha(x, y)\rho_1(xy).$$

This shows that ρ_1 is an irreducible α-representation on G, and χ_1 is the corresponding α-character on G. Since G is α-monomial, there is a subgroup $H < G$ and an irreducible α-character $\theta_1 \in \mathrm{Irr}(H)_\alpha$ afforded by an irreducible α-representation ψ_1 on H such that $\psi_1^G = \rho_1$, $\theta_1^G = \chi_1$ and θ_1 linear.

Now for any $a \in G$, if a is α-regular then $a \in \mathrm{Ker}\theta_1^G$ if and only if $\theta_1^G(a) = \theta_1^G(1)$, that is, $\sum_{g\in G}\theta_1^\circ(a^g) = \sum_{g\in G}\theta_1(1)$. Since $|\theta_1(a^g)| \le \theta_1(1)$ for all $g \in G$, the above condition happens only if $|\theta_1(a^g)| = \theta_1(1)$ for all $a^g \in H$. Thus $a^g \in \mathrm{Ker}\theta_1$, and $a \in (\mathrm{Ker}\theta_1)^g$ for all $g \in G$. On the other hand if a is not α-regular then $\theta_1^G(a) = 0$ which implies $a \notin \mathrm{Ker}\theta_1^G$.

Since $A < \mathrm{Ker}\rho_1 = \mathrm{Ker}\chi_1 = \mathrm{Ker}\theta_1^G = \bigcap_{g\in G}(\mathrm{Ker}\theta_1)^g < \mathrm{Ker}\theta_1$, and since $H/A < G/A$, we have mappings θ and ψ on H/A such that $\theta(hA) = \theta_1(h)$ and $\psi(hA) = \psi_1(h)$ for all $h \in H$. For any $h, k \in H$, we have

$$\psi(hA)\psi(kA) = \psi_1(h)\psi_1(k) = \alpha(h, k)\psi_1(hk) = \beta(hA, kA)\psi(hkA),$$

and so ψ is an irreducible β-representation that affords a character θ. Now for any $gA \in G/A$ ($g \in G$), we have

$$\theta^{G/A}(gA) = \frac{|A|}{|H|}\sum_{xA\in G/A}\theta^\circ(g^xA) = \frac{1}{|H|}\sum_{x\in G}\theta_1^\circ(g^x) = \theta_1^G(g) = \chi(gA)$$

for $g^xA \in H/A$. Hence $\theta^{G/A} = \chi$. Clearly θ is linear because $\theta(1A) = \theta_1(1) = 1$. This completes the proof. □

COROLLARY 12. *If G is a relative projective α-monomial group over N then G/A is relative projective β-monomial over N/A for some $\alpha \in Z^2(G, F^*)$ and $\beta \in Z^2(G/A, F^*)$.*

PROOF. We use the same notations β, χ, $\alpha = \inf\beta$ and H, θ_1 as in Theorem 11. Since G is a relative α-monomial group, we may assume H contains N and $\theta_{1N} \in \mathrm{Irr}(N)_\alpha$ instead of θ_1 linear. By letting θ on H/A by $\theta(hA) = \theta_1(h)$ for $h \in H$, it is clear that $\theta^{G/A} = \chi$ and $\theta_{N/A}$ is an

irreducible β-representation on N/A since

$$[\theta_{N/A}, \theta_{N/A}] = \frac{1}{|N/A|} \sum_{nA \in N/A} \theta_{N/A}(nA)\theta_{N/A}(n^{-1}A) = [\theta_{1N}, \theta_{1N}] = 1. \qquad \square$$

In general even if G/A is monomial, G may not be a monomial group. We have partial answers for this situation.

COROLLARY 13. (1) *If G/A is monomial, then any irreducible character χ of G with $A < Ker\chi$ is a monomial character.*
(2) *If G is an α-monomial group, then G is a relative α-monomial character with respect to any abelian normal subgroup N such that α_N is coboundary.*

PROOF. If ρ is an irreducible representation of G affording the character χ and A is contained in $\mathrm{Ker}\chi = \mathrm{Ker}\rho$, then there is a representation ρ_1 on G/A such that $\rho_1(gA) = \rho(g)$ $(g \in G)$. Write χ_1 for the character afforded by ρ_1. Since G/A is monomial, there exist $H/A < G/A$ and $\theta_1 \in \mathrm{Irr}(H/A)$ such that θ_1 is linear and $\theta_1^{G/A} = \chi_1$. Define θ on H by $\theta(h) = \theta_1(hA)$ for $h \in H$. Then

$$\chi(g) = \chi_1(gA) = \theta_1^{G/A}(gA) = \frac{|A|}{|H|} \sum_{xA \in G/A} \theta_1^{\circ}(g^x A) = \frac{1}{|H|} \sum_{x \in G} \theta^{\circ}(g^x) = \theta^G(g).$$

Clearly θ is linear and this proves (1).

For any $\chi \in \mathrm{Irr}(G)_\alpha$, there is $H < G$ and $\theta \in \mathrm{Irr}(H)_\alpha$ such that $\theta^G = \chi$ and θ linear. Considering $NH < G$, we have $(\theta^{NH})^G = \theta^G = \chi$ which is irreducible on G. Thus it follows from [3, (5.11)] that θ^{NH} is an irreducible character on NH. Because N is abelian such that α_N is coboundary, $(\theta^{NH})_N = (\theta_{N \cap H})^N$ has multiplicity 1. Thus $(\theta^{NH})_N$ is irreducible on N. $\square$

Note that the similar result for monomial case has been mentioned in [3, (6.10)].

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