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SUBMANIFOLDS OF CODIMENSION 2 OF ODD-DIMENSIONAL SPHERES

Yong Ho Shin

Abstract. This paper is to show that a submanifold of codimension 2 of an odd-dimensional sphere with an almost contact metric structure is an intersection of a complex cone with generator as a normal vector and a sphere.

0. Introduction

A pseudo-umbilical submanifold of an even-dimensional Euclidean space E^{2n+4} with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$ is an intersection of a complex cone with a generator and a sphere.

Also, the $(f, g, u, v, w, \lambda_1, \mu_1, \nu)$-structure is naturally induced on submanifolds of codimension 2 of an odd-dimensional sphere. In a previous paper [1], the author have studied the submanifold of codimension 2 of an odd-dimensional unit sphere and determined the global form of them by using minimal condition.

The purpose of the present paper is devoted to study some intrinsic properties of submanifolds of codimension 2 of an odd-dimensional sphere and determine the global form of them admitting an almost contact metric structure.

In Section 1, we discuss the differential geometry of $S^{2n+1}(1)$.

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In Section 2, we find some algebraic relations and structure equations of codimension 2 submanifolds of an odd-dimensional sphere.

In Section 3, we determine the global form of submanifolds of codimension 2 of an odd-dimensional sphere admitting an almost contact metric structure. We introduce the following Theorem for later use.

THEOREM A ([1]). *Let M^{2n-1} be a $(2n-1)$-dimensional submanifold of codimension 2 of an odd-dimensional sphere $S^{2n+1}(1)$. If M^{2n-1} is minimal, then M^{2n-1} as a submanifold of codimension 3 of a Euclidean space E^{2n+2} is pseudo-umbilical.*

THEOREM B ([2]). *Let M^{2n+1} be a pseudo-umbilical submanifold of an even-dimensional Euclidean space E^{2n+4} with $(f,g,u,v,w,\lambda,\mu,\nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. Then M^{2n+1} is an intersection of a complex cone with a generator as a normal vector and a sphere.*

1. Differential geometry of $S^{2n+1}(1)$

Let E^{2n+2} be a $(2n+2)$-dimensional Euclidean space and O the origin of the cartesian coordinate system in E^{2n+2}, and denote by X the position vector of a point P in E^{2n+2} with respect to the origin.

We consider a sphere $S^{2n+1}(1)$ with center at O and radius 1, and suppose that $S^{2n+1}(1)$ is covered by a system of coordinate neighborhoods $\{U; x^h\}$. Here and in the sequel, the indices $h,i,j,k,\cdots$ run over the range $\{1,2,\cdots,2n+1\}$. Then $X\cdot X=1$ for the position vector X at each point in $S^{2n+1}(1)$, where the dot denotes the inner product of two vectors in a Euclidean space. Now we put $X_i=\partial_i X$, $C=-X$, $g_{ij}=X_i\cdot X_j$ where $\partial_i=\partial/\partial x^i$, and denote by ∇_i the operator of covariant differentiation form with the metric tensor g_{ij}. Then the equations of Gauss and Weingarten are respectively given by

$$(1.1)\quad \nabla_j X_i = g_{ji}C,\quad \nabla_i C = -X_i,$$

from which we can easily derive

$$(1.2)\quad K_{kji}{}^h = \delta_k^h g_{ji} - \delta_j^h g_{ki},$$

which are the equations of Gauss, ${K_{kji}}^h$ being the component of the curvature tensor of $S^{2n+1}(1)$.

In E^{2n+2}, there exists a natural Kaehlerian structure $F = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$, E being the unit matrix of degree $n+1$. It follows that

$$F^2 = -I, \; FU \cdot FV = U \cdot V$$

for arbitrary vectors U and V in E^{2n+2}, where I denotes the identity transformation in E^{2n+2}. Transformating X_j, C by F, we get

$$(1.3) \quad FX_j = f_j^t X_t + u_j C, \quad FC = -u^t X_t,$$

where f_i^h are the component of a tensor field of type (1.1), u_i is 1-form on $S^{2n+1}(1)$, u^h is given by $u^h = u_t g^{th}$.

Applying F to (1.3), respectively, we get an *almost contact metric structure*

$$(1.4) \quad \begin{cases} f_t^h f_i^t = -\delta_i^h + u_i u^h, \\ f_i^t u_t = 0, u^t u_t = 1, \\ f_i^t f_j^h g_{th} = g_{ij} - u_i u_j. \end{cases}$$

It is easily verified that $f_{ji} = f_j^h g_{ih}$ is skew-symmetric in j and i. Differentiating (1.3) covariantly and using $\nabla F = 0$, we have

$$(1.5) \quad \begin{cases} \nabla_j f_i^h = -g_{ji} u^h + \delta_j^h u_i, \\ \nabla_j u_i = f_{ji}. \end{cases}$$

Thus, $S^{2n+1}(1)$ admits the *Sasakian structure.*

2. Structure equations of codimension 2 submanifolds

Let M^{2n-1} be a $(2n-1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and immersed isometrically in $S^{2n+1}(1)$ by an immersion $i : M^{2n-1} \to S^{2n+1}(1)$, where here and in the sequel the indices $a, b, c, d, \cdots$ run over the range $\{1, 2, \cdots, 2n-1\}$. We identify $i(M^{2n-1})$ with M^{2n-1} itself and represent the immersion by

$$x^h = x^h(y^a).$$

We put $B_c^h = \partial_c x^h$, $\partial_c = \partial/\partial y^c$. Then each B_c^h is a $2n-1$ linearly independent vector of $S^{2n+1}(1)$ tangent to M^{2n-1}. And let D^h and E^h be mutually orthogonal unit normals to $S^{2n+1}(1)$. Then, denoting by g_{cb} the components of the induced metric tensor of M^{2n-1}, we have $g_{cb} = g_{ji} B_c^j B_b^i$ since the immersion is isometric.

As to transforms of B_c^j, D^j and E^j by f_j^h, we have respectively

$$(2.1)\quad \begin{cases} f_j^h B_c^j = f_c^a B_a^h + v_c D^h + w_c E^h, \\ f_j^h D^j = -v^a B_a^h - \lambda E^h, \\ f_j^h E^j = -w^a B_a^h + \lambda D^h, \end{cases}$$

where f_c^a denote the components of a tensor field of type (1.1), v_c, w_c 1-forms and λ a function in M^{2n-1}, v^a and w^a being vector fields associated with v_a and w_a, respectively.

We may put the vector field u^h as follows:

$$(2.2)\quad u^h = u^a B_a^h + \mu D^h + \nu E^h,$$

where u^a is a vector field, μ and ν are functions in M^{2n-1}.

Applying the operator f_h^k to (2.1) and (2.2) respectively, and using (1.4), we obtain the so-called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure as follows:

$$(2.3)\quad f_b^e f_e^a = -\delta_b^a + u_b u^a + v_b v^a + w_b w^c,$$

$$(2.4)\quad \begin{cases} f_e^a u^e = \mu v^a + \nu w^a, \\ f_e^a v^e = -\mu u^a + \lambda w^a, \\ f_e^a w^e = -\nu u^a - \lambda v^a, \end{cases}$$

or

$$u_e f_c^e = -\mu v_c - \nu w_c, \quad v_e f_c^e = \mu u_c - \lambda w_c, \quad w_e f_c^e = \nu u_c + \lambda v_c,$$

$$(2.5)\quad \begin{cases} u_e u^e = 1 - \mu^2 - \nu^2,\ u_e v^e = -\lambda\nu, \\ u_e w^e = \lambda\mu,\ v_e v^e = 1 - \lambda^2 - \mu^2, \\ v_e w^e = -\mu\nu,\ w_e w^e = 1 - \lambda^2 - \nu^2, \end{cases}$$

where u_c, v_c and w_c are 1-forms associated with u^a, v^a and w^a respectively. We can easily verify that $f_{cb} = f_c^a g_{ba}$ is skew-symmetric because f_{ji} is skew-symmetric. Transvecting the last equation of (1.4) with $B_c^j B_b^i$, making use of (2.2) and the definition of the induced metric tensor g_{cb}, we have

$$f_c^e f_b^d g_{ed} = g_{cb} - u_c u_b - v_c v_b - w_c w_b.$$

Denoting by ∇_c the operator of the van der Waerden-Bortotti covariant differentiation, we can write the equation of Gauss for M^{2n-1} as

$$\text{(2.6)} \quad \nabla_c B_b^h = k_{cb} D^h + l_{cb} E^h,$$

where k_{cb} and l_{cb} are the second fundamental tensors with respect to D^h and E^h, respectively.

The equations of Weingarten are given by

$$\text{(2.7)} \quad \nabla_c D^h = -k_c^b B_b^h + l_c E^h,$$

$$\text{(2.8)} \quad \nabla_c E^h = -l_c^b B_b^h - l_c D^h,$$

where $k_c^b = k_{ca} g^{ab}$, $l_c^b = l_{ca} g^{ab}$, $(g^{ab}) = (g_{ab})^{-1}$, l_c being the third fundamental tensor. The normal vector field D^h is said to be *parallel* in the normal bundle if l_c vanishes identically. Then, from (1.2), the equation of Gauss is given by

$$K_{dcb}{}^a = \delta_d^a g_{cb} - \delta_c^a g_{db} + k_d^a k_{cb} - k_c^a k_{db} + l_d^a l_{cb} - l_c^a l_{db}.$$

By differentiating (2.1) and (2.2) covariantly along M^{2n-1} and taking account of (1 5), (2.2), (2.6), (2.7) and (2.8), we find

$$\text{(2. 9)} \quad \nabla_b f_c^a = -g_{bc} u^a + \delta_b^a u_c + k_b^a v_c - k_{bc} v^a + l_b^a w_c - l_{bc} w^a,$$

$$\text{(2.10)} \quad \nabla_b v_c = -\mu g_{bc} + \lambda l_{bc} - k_{ba} f_c^a + l_b w_c,$$

$$\text{(2.11)} \quad \nabla_b w_c = -\nu g_{bc} - \lambda k_{bc} - l_{ba} f_c^a - l_b v_c,$$

$$\text{(2.12)} \quad \nabla_b u^a = f_b^a + \mu k_b^a + \nu l_b^a,$$

$$\nabla_b \lambda = k_b^a w_a - l_{ba} v^a,$$

$$\nabla_b \mu = v_b - k_{ba} u^a + \nu l_b,$$

$$\nabla_b \nu = w_b - l_{ba} u^a - \mu l_b.$$

3. Almost contact metric structure

In this section, we assume that M^{2n-1} admits an almost contact metric structure $(f_b^a,\ g_{cb},\ u^a)$, that is, M^{2n-1} satisfies

(3.1) $\quad f_b^a f_e^a = -\delta_b^a + u_b u^a,$

(3.2) $\quad u_a u^a = 1.$

From (2.3), (2.5), and (3.1) , we see

(3.3) $\quad (1-\lambda^2-\mu^2)+(1-\lambda^2-\nu^2)=0.$

On the other hand, (2.5) and (3.2) implies that

(3.4) $\quad \mu = 0, \quad \nu = 0.$

Substituting (3.4) into (3.3), we find

(3.5) $\quad \lambda^2 = 1.$

Thus we have:

LEMMA 1. *Let M^{2n-1} be a (2n-1)-dimensional submanifold of codimension 2 of an odd-dimensional unit sphere $S^{2n+1}(1)$ with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure. If M^{2n-1} admits an almost contact metric structure $(f_c^a,\ g_{cb},\ u^a)$, then we have $\lambda^2+\mu^2+\nu^2 = 1$ on M^{2n-1}.*

From (2.5) we have, with the aid of (3.4) and (3.5),

(3.6) $\quad v^a = 0, \quad w^a = 0.$

(2.9) and (2.12) implies that

$$\nabla_b f_c^a = -g_{bc} u^a + \delta_b^a u_c, \quad \nabla_b u^a = f_b^a,$$

with aid of (3.6).
Thus the aggregate $(f_c^a,\ g_{cb},\ u^a)$ defines a Sasakian structure.
Hence, we obtain:

THEOREM 2. *Under the same assumptions as those stated in Lemma 1, M^{2n-1} admits a Sasakian structure.*

Substituting (3.4) and (3.6) into (2.10), we find

$$\lambda l_{bc} - k_{ba} f_c^a = 0,$$

from which, contracting this equation with respect to b and c taking account of (3.5), we get

(3.7) $\quad l_b^b = 0,$

because k_{cb} are symmetric and f_{cb} is skew-symmetric with respect to b and c.

From (2.11), (3.4), (3.6) and (3.5), we have

(3.8) $\quad k_b^b = 0.$

The mean curvature vector is given by

$$H^h = \frac{1}{2n-1} g^{ba} \nabla_b B_a^h = \frac{1}{2n-1}(k_a^a D^h + l_a^a E^h)$$

This equation together with (3.7) and (3.8) becomes

$$H^h = 0,$$

which shows that M^{2n-1} is a minimal submanifold.

Therefore, we obtain:

THEOREM 3. *Under the same assumptions as those stated in Lemma 1, M^{2n-1} is a minimal submanifold.*

Combining Theorem A and Theorem 3 , we find

THEOREM 4. *Under the same assumptions as those stated in Lemma 1, M^{2n-1} as a submanifold of codimension 3 of the Euclidean space E^{2n+2} is pseudo-umblical.*

Hence, by making use of Theorem B, Lemma 1 and Theorem 4, we conclude :

THEOREM 5. *Let M^{2n-1} be a (2n-1)-dimensional submanifold of codimension 2 of an odd-dimensional spheres $S^{2n+1}(1)$ with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure. If M^{2n-1} admits an almost contact metric structure $(f_c^a,\ g_{cb},\ u^a)$ and $n > 1$, then M^{2n-1} as a submanifold of codimension 3 of Euclidean space E^{2n+2} is an intersection of a complex cone with a generator as a normal vector C and a (2n+1)-dimensional unit sphere $S^{2n+1}(1)$.*

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Department of Mathematics
University of Ulsan
Ulsan 680–749, Korea
E-mail: yhshin@uou.ulsan.ac.kr