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# A CLASS OF SERIES INVOLVING THE ZETA FUNCTION

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Abstract. The authors apply the theory of multiple Gamma functions, which was recently revived in the study of the determinants of the Laplacians, in order to present a class of closed-form evaluations of series involving the Zeta function by appealing only to the definitions of the double and triple Gamma functions.

## 1. Introduction and Preliminaries

The multiple Gamma functions were defined and studied by Barnes (*cf.* [3] and [4]) and others in about 1900. Although these functions did not appear in the tables of the most well-known special functions, yet the double Gamma function was cited in the exercises by Whittaker and Watson [26, p. 264] and recorded also by Gradshteyn and Ryzhik [14, p. 661, Entry 6.441(4); p. 937, Entry 8.333]. Recently, these functions were revived in the study of the determinants of the Laplacians on the $n$-dimensional unit sphere $\mathbf{S}^n$ (see [7], [10], [11], [18], [23], and [25]). More recently, Choi *et al.* ([8], [9], [11], and [12]) used these functions to evaluate the sums of several classes of series involving the Riemann Zeta function, the subject of which can be traced back to an over two-century old theorem of Christian Goldbach (1690-1764) noted in the

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work of Srivastava [20, p. 1] who investigated this subject in a rather systematic and unified manner among many other authors. Before their investigation by Barnes, these functions had been introduced in a different form by several authors. The recent uses besides the above-mentioned subjects have been made of the theory of the double Gamma function by many authors for their own purposes.

In this note we aim at providing a class of closed-form evaluations of series involving the Zeta function by appealing only to the definitions of the double and triple Gamma functions.

To do so, we start with recalling the Barnes $G$-function ($1/G = \Gamma_2$ is so-called the double Gamma function) which has several equivalent forms one of which is

$$\begin{aligned}\{\Gamma_2(z+1)\}^{-1} &= G(z+1)\\ &= (2\pi)^{\frac{z}{2}} e^{-\frac{1}{2}[(1+\gamma)z^2+z]} \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}} \right\},\end{aligned} \tag{1}$$

where $\gamma$ denotes the Euler-Mascheroni constant given by

$$\gamma = \lim_{n\to\infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577\,215\,664\,901\,532\,5\ldots. \tag{2}$$

For sufficiently large real $x$ and $a \in \mathbb{C}$, we have the Stirling formula for the $G$-function:

(3)

$$\begin{aligned}\log G(x+a+1) &= \frac{x+a}{2}\log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax\\ &\quad + \left(\frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax\right)\log x + O(1/x) \ \ (x\to\infty),\end{aligned}$$

where $A$ is the Glaisher-Kinkelin constant defined by

$$\begin{aligned}\log A &= \lim_{n\to\infty} \left[ \sum_{k=1}^{n} k\log k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log n + \frac{n^2}{4} \right]\\ &\cong 1.282427130\cdots.\end{aligned} \tag{4}$$

The $G$-function satisfies the following fundamental functional relationships:

$$G(1) = 1 \quad \text{and} \quad G(z+1) = \Gamma(z)\,G(z) \quad (z \in \mathbb{C}), \tag{5}$$

where $\Gamma$ denotes the familiar Gamma function which also satisfies the basic relationships:

$$\Gamma(1) = 1 \quad \text{and} \quad \Gamma(z+1) = z\,\Gamma(z) \quad (z \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}). \tag{6}$$

Vignéras [24] obtained a recurrence formula of the multiple Gamma functions $\Gamma_n$ ($n \in \mathbb{N}$) which, for $n = 3$, readily yields the explicit Weierstrass canonical product form of the triple Gamma function $\Gamma_3$ as follows (see Choi and Srivastava [11]):

$$\begin{aligned}\Gamma_3(1+z) = G_3(1+z) = \exp\bigg[&-\frac{1}{6}\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right) z^3 \\ &+\frac{1}{4}\left(\gamma + \log(2\pi) + \frac{1}{2}\right) z^2 + \left(\frac{3}{8} - \frac{1}{4}\log(2\pi) - \log A\right) z\bigg] \\ &\times \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{-\frac{1}{2}k(k+1)} \exp\left[\frac{1}{2}(k+1)z \right.\right. \\ &\left.\left. -\frac{1}{4}\left(1 + \frac{1}{k}\right) z^2 + \frac{1}{6k}\left(1 + \frac{1}{k}\right) z^3\right]\right\}.\end{aligned} \tag{7}$$

The triple Gamma function $\Gamma_3$ also satisfies the following fundamental functional relationships:

$$\Gamma_3(1) = 1 \quad \text{and} \quad \Gamma_3(z+1) = G(z)\,\Gamma_3(z) \quad (z \in \mathbb{C}). \tag{8}$$

Note that another form of the triple Gamma function $\Gamma_3$ appeared in the work of Choi [7] who expressed, in terms of the double and triple Gamma functions, the analogous Weierstrass canonical product of the shifted sequence of the eigenvalues of the Laplacian on the unit sphere $\mathbf{S}^3$ with standard metric which was indispensable to evaluate the determinant of the Laplacian on $\mathbf{S}^3$ there.

The Riemann Zeta function $\zeta(s)$ defined by

(9)
$$\zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \left(1-2^{1-s}\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} & (\Re(s) > 0;\ s \neq 1), \end{cases}$$

which can, except for a simple pole only at $s = 1$ with its residue 1, be continued analytically to the whole $s$-plane by the contour integral representation (*cf.* Whittaker and Watson [26, p. 266]) or many other integral representations (*cf.* Erdélyi *et al.* [13, p. 33]) and satisfies the functional equation (see [22, p. 13]):

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \zeta(1-s) \sin \frac{\pi s}{2}. \tag{10}$$

## 2. Review of some known mathematical constants

In addition to the Glaisher-Kinkelin constant $A$ defined by (4), by means of the Euler-Maclaurin summation formula (*cf.* Hardy [16, p. 318]), Choi and Srivastava [11] introduced two mathematical constants:

$$\log B = \lim_{n\to\infty} \left[ \sum_{k=1}^{n} k^2 \log k - \left( \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \log n + \frac{n^3}{9} - \frac{n}{12} \right] \tag{11}$$

and

(12)
$$\log C = \lim_{n\to\infty} \left[ \sum_{k=1}^{n} k^3 \log k - \left( \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \log n + \frac{n^4}{16} - \frac{n^2}{12} \right],$$

where the approximate numerical values of $B$ and $C$ are given by

$$B \cong 1.030\,888\,116\,106\,913 \ldots \tag{13}$$

and

$$C \cong 301\,393\,392\,412\,467\,84 \times 10^{-2714341} \dots . \tag{14}$$

The constant $B$ was also considered by Choi and Srivastava [9, p. 102]. Just as the constant $A$, the strangely-looking constants $B$ and $C$ are readily seen to be involved naturally in the study of the multiple Gamma functions (*cf.* Choi and Srivastava [9, 11]).

Voros [25] gave a connection between the Glaisher-Kinkelin constant $A$ and the Riemann Zeta function $\zeta(s)$:

$$\log A = \tfrac{1}{12} - \zeta'(-1), \tag{15}$$

which was also proved in a different way by Vardi [23]. By making use of the Voros's technique, Choi and Srivastava [11] showed that

$$\log B = -\zeta'(-2) \tag{16}$$

and

$$\log C = -\zeta'(-3) - \tfrac{11}{720}. \tag{17}$$

We also obtain the relationship:

$$\log B = \frac{\zeta(3)}{4\pi^2}, \tag{18}$$

which immediately follows by applying the special case ($n = 1$) of the general identity which is derivable from (10) (see Srivastava [21, p. 387, Eq. (1.15)]):

$$\zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n)!} \zeta'(-2n) \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}) \tag{19}$$

to (16). Choi and Srivastava [11] also showed the relationship (18) in a markedly different way by recalling a result of Choi and Srivastava [9, p. 111, Eq. (4.34)] in the following *corrected* form:

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}} = \frac{1}{2} - \log 2 + 14 \log B \tag{20}$$

and comparing (20) with another known result (*cf.*, *e.g.*, [6, p. 191, Eq. (3.19)]):

$$\zeta(3) = \frac{2\pi^2}{7}\left(\log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}}\right). \tag{21}$$

Just as the well-known result

$$\Gamma\left(\tfrac{1}{2}\right) = \pi^{\frac{1}{2}}, \tag{22}$$

it is known from the work of Barnes [3, p. 288, Section 17] that

$$G\left(\frac{1}{2}\right) = 2^{\frac{1}{24}} \cdot \pi^{-\frac{1}{4}} \cdot e^{\frac{1}{8}} \cdot A^{-\frac{3}{2}}. \tag{23}$$

Choi and Srivastava [11] also expressed the value of $\Gamma_3\left(\frac{1}{2}\right)$ in terms of the mathematical constants $\pi$, $e$, $A$, and $B$:

$$\Gamma_3\left(\tfrac{1}{2}\right) = 2^{-\frac{1}{24}} \cdot \pi^{\frac{3}{16}} \cdot e^{-\frac{1}{8}} \cdot A^{\frac{3}{2}} \cdot B^{\frac{7}{8}}. \tag{24}$$

The Catalan's constant $\mathbf{G}$ is defined by

$$\mathbf{G} = \frac{1}{2}\int_0^1 \mathbf{K}(k)dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \cong 0.915965\ldots, \tag{25}$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind, given by

$$\mathbf{K}(k) := \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}. \tag{26}$$

The constant $\mathbf{G}$ becomes a special case of many other functions one of which will be considered in the next section and has, among other things, been used to evaluate integrals, for example (see [14, p. 526, Entry 4.224]):

$$\int_0^{\pi/4} \log(\sin t)\, dt = -\frac{\pi}{4}\log 2 - \frac{1}{2}\mathbf{G}, \tag{27}$$

and to give closed-form evlauations of a certain class of series involving the Zeta function (*cf.* Choi and Srivastava [9]).

By recalling the special value of the Gamma function (see Spiegel [19, p. 1]):

$$\Gamma\left(\tfrac{1}{4}\right) \cong 3.62560\,99082\,21908\,\ldots \tag{28}$$

and making use of a duplication formula for the $G$-function (*cf.* Barnes [3, p. 291] for the general case), Choi and Srivastava [9] showed that

$$G\left(\tfrac{1}{4}\right) = e^{\frac{3}{32}-\frac{G}{4\pi}} \cdot A^{-\frac{9}{8}} \left\{\Gamma\left(\tfrac{1}{4}\right)\right\}^{-\frac{3}{4}} \cong 0.293756\ldots\,; \tag{29}$$

or, equivalently,

$$G\left(\tfrac{3}{4}\right) = 2^{-\frac{1}{8}} \cdot \pi^{-\frac{1}{4}} \cdot e^{\frac{3}{32}+\frac{G}{4\pi}} \cdot A^{-\frac{9}{8}} \left\{\Gamma\left(\tfrac{1}{4}\right)\right\}^{\frac{1}{4}} \cong 0.848718\ldots. \tag{30}$$

**Remark.** While writing this note, we have found that the constants $A$, $B$, and $C$ were considered by Bendersky [5] and Adamckik [1] including their Riemann Zeta function representations (15), (16), and (17).

## 3. Series involving the Zeta function

Taking logarithms on both sides of (1) with the Maclaurin series expansion of $\log(1+z)$ yields

$$\begin{aligned}&\sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{k+1} z^{k+1} \\ &\quad = [1-\log(2\pi)]\frac{z}{2} + (1+\gamma)\frac{z^2}{2} + \log G(z+1) \quad (|z|<1),\end{aligned} \tag{31}$$

which, upon replacing $z$ by $-z$, gives

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{\zeta(k)}{k+1} z^{k+1} \\ &\quad = [1-\log(2\pi)]\frac{z}{2} - (1+\gamma)\frac{z^2}{2} - \log G(1-z) \quad (|z|<1).\end{aligned} \tag{32}$$

Adding and subtracting (31) and (32), we obtain

$$(33)\quad \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1} z^{2k+1} = [1-\log(2\pi)]\frac{z}{2} + \frac{1}{2}\log\left[\frac{G(1+z)}{G(1-z)}\right] \quad (|z|<1)$$

and

$$(34)\quad \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{k+1} z^{2k+2} = -(1+\gamma)z^2 - \log[G(1+z)G(1-z)] \quad (|z|<1).$$

If we take the limit on both sides of (31) as $z \to 1$ with (5) and (6), we find that

$$(35)\quad \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k+1} = 1 + \frac{\gamma}{2} - \frac{1}{2}\log(2\pi),$$

which was considered by several earlier authors (*cf.* Srivastava [20]).

Now setting $z = \frac{1}{2}$, $z = \frac{1}{4}$, and $z = \frac{3}{4}$ in (31) through (34) and making use of (5), (6), (23), (29), and (30), we readily obtain a class of closed-form evaluations of series involving the Riemann Zeta function:

$$(36)\quad \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{(k+1)\,2^k} = 1 + \frac{\gamma}{4} - \frac{5}{12}\log 2 - 3\log A;$$

$$(37)\quad \sum_{k=2}^{\infty} \frac{\zeta(k)}{(k+1)\,2^k} = -\frac{\gamma}{4} - \frac{7}{12}\log 2 + 3\log A;$$

$$(38)\quad \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1)\,2^{2k}} = \frac{1}{2} - \frac{1}{2}\log 2;$$

$$(39)\quad \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)\,2^{2k}} = -2 - \gamma - \frac{1}{3}\log 2 + 12\log A;$$

$$\sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{(k+1)2^{2k}} = 1+\frac{\gamma}{8}-\frac{\mathbf{G}}{\pi}+\log\left[2^{-\frac{1}{2}}\cdot\pi^{-\frac{1}{2}}\cdot A^{-\frac{9}{2}}\cdot\Gamma\left(\tfrac{1}{4}\right)\right]; \tag{40}$$

$$\sum_{k=2}^{\infty}\frac{\zeta(k)}{(k+1)2^{2k}} = -\frac{\gamma}{8}-\frac{\mathbf{G}}{\pi}+\log\left[\pi^{\frac{1}{2}}\cdot A^{\frac{9}{2}}\cdot\left\{\Gamma\left(\tfrac{1}{4}\right)\right\}^{-1}\right]; \tag{41}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k+1)2^{4k}} = \frac{1}{2}-\frac{\mathbf{G}}{\pi}-\frac{1}{4}\log 2; \tag{42}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(k+1)2^{4k}} = -4-\gamma+\log\left[2^2\cdot\pi^4\cdot A^{36}\cdot\left\{\Gamma\left(\tfrac{1}{4}\right)\right\}^{-8}\right]; \tag{43}$$

$$\begin{aligned}\sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)}{k+1}\left(\frac{3}{4}\right)^k \\ = 1+\frac{3\gamma}{8}+\frac{\mathbf{G}}{3\pi}+\log\left[\pi^{\frac{1}{2}}\cdot A^{-\frac{3}{2}}\cdot\left\{\Gamma\left(\tfrac{1}{4}\right)\right\}^{-1}\right];\end{aligned} \tag{44}$$

$$\sum_{k=2}^{\infty}\frac{\zeta(k)}{k+1}\left(\frac{3}{4}\right)^k = -\frac{3\gamma}{8}+\frac{\mathbf{G}}{3\pi}+\log\left[2^{-\frac{1}{2}}\cdot\pi^{-\frac{1}{2}}\cdot A^{\frac{3}{2}}\cdot\Gamma\left(\tfrac{1}{4}\right)\right]; \tag{45}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k+1}\left(\frac{3}{4}\right)^{2k} = \frac{1}{2}+\frac{\mathbf{G}}{3\pi}-\frac{1}{4}\log 2; \tag{46}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{k+1}\left(\frac{3}{4}\right)^{2k} \\ = -\frac{4}{3}-\gamma+\log\left[2^{-\frac{2}{3}}\cdot\pi^{-\frac{4}{3}}\cdot A^4\cdot\left\{\Gamma\left(\tfrac{1}{4}\right)\right\}^{\frac{8}{3}}\right].\end{aligned} \tag{47}$$

Taking logarithms on both sides of (7), we readily have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{(-1)^k}{k+2}\left[\zeta(k)+\zeta(k+1)\right]z^{k+2}\\
&=\left[-\frac{3}{4}+\frac{1}{2}\log(2\pi)+2\log A\right]z\\
&-\frac{1}{2}\left[\gamma+\log(2\pi)+\frac{1}{2}\right]z^2+\frac{1}{3}\left(\gamma+\frac{\pi^2}{6}+\frac{3}{2}\right)z^3\\
&+2\log\Gamma_3(1+z)\quad(|z|<1),
\end{aligned}\tag{48}$$

which, upon letting $z\to 1$, yields

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{k+2}\left[\zeta(k)+\zeta(k+1)\right]=-\frac{1}{2}+\frac{\pi^2}{18}-\frac{\gamma}{6}+2\log A.\tag{49}$$

In view of $\zeta(2)=\frac{\pi^2}{6}$ and (35), we find that

$$\sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)}{k+2}=-1-\frac{\gamma}{2}+\frac{\pi^2}{18}+\frac{1}{2}\log(2\pi),\tag{50}$$

which, upon combining with (49), yields

$$\sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)}{k+2}=\frac{1}{2}+\frac{\gamma}{3}-\frac{1}{2}\log(2\pi)+2\log A.\tag{51}$$

Replace $z$ by $-z$ in (48) and we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{\zeta(k)+\zeta(k+1)}{k+2}z^{k+2}\\
&=\left[\frac{3}{4}-\frac{1}{2}\log(2\pi)-2\log A\right]z\\
&-\frac{1}{2}\left[\gamma+\log(2\pi)+\frac{1}{2}\right]z^2-\frac{1}{3}\left(\gamma+\frac{\pi^2}{6}+\frac{3}{2}\right)z^3\\
&+2\log\Gamma_3(1-z)\quad(|z|<1).
\end{aligned}\tag{52}$$

Adding and subtracting (48) and (52), we obtain

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)+\zeta(2k+1)}{k+1} z^{2k+2} = -\left[\gamma + \log(2\pi) + \frac{1}{2}\right] z^2 + 2\log\left[\Gamma_3(1+z)\,\Gamma_3(1-z)\right] \quad (|z|<1) \tag{53}$$

and

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)+\zeta(2k+2)}{2k+3} z^{2k+3} = \left[\frac{3}{4} - \frac{1}{2}\log(2\pi) - 2\log A\right] z - \frac{1}{3}\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right) z^3 + \log\left[\frac{\Gamma_3(1-z)}{\Gamma_3(1+z)}\right] \quad (|z|<1). \tag{54}$$

Setting $z = \frac{1}{2}$ in (48) with the aid of (23) and (24), and combining (36) with the resulting identity, we readily obtain

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{(k+2)2^k} = \frac{1}{2} + \frac{\gamma}{6} + \log\left(2^{-\frac{1}{3}} \cdot A^{-2} \cdot B^7\right). \tag{55}$$

Similarly if we set $z = \frac{1}{2}$ in (52), (53), and (54), we readily find that

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{(k+2)2^k} = -\frac{\gamma}{6} + \log\left(2^{-\frac{2}{3}} \cdot A^2 \cdot B^7\right); \tag{56}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}} = \frac{1}{2} + \log\left(2^{-1} \cdot B^{14}\right); \tag{57}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+3)2^{2k}} = -\frac{1}{2} - \frac{\gamma}{3} + \log\left(2^{-\frac{1}{3}} \cdot A^4\right). \tag{58}$$

We conclude this note by remarking that these closed-form evaluations of series associated with the Riemann Zeta function may have a capability of diverse applications to compute (or evaluate) other mathematical subjects (*cf.* [10] and [11]).

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