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ON THE STUDY OF SOLUTION UNIQUENESS TO THE TASK OF DETERMINING UNKNOWN PARAMETERS OF MATHEMATICAL MODELS

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ABSTRACT. The problem of solution uniqueness to the task of determining unknown parameters of mathematical models from input-output observations is studied. This problem is known as structural identifiability problem. We offer a new approach for testing structural identifiability of linear state space models. The approach compares favorably with numerous methods proposed by other authors for two main reasons. First, it is formulated in obvious mathematical form. Secondly, the method does not involve unfeasible symbolic computations and thus allows to test identifiability of large-scale models.

In case of non-identifiability, when there is a set of solutions to the task, we offer a method of computing functions of the unknown parameters which can be determined uniquely from input-output observations and later used as new parameters of the model. Such functions are called parametric functions capable of estimation. To develop the method of computation of these functions we use Lie group transformation theory.

Illustrative example is given to demonstrate applicability of presented methods.

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1. Introduction and preliminaries

Generally accepted way of studying natural-science phenomena is mathematical modelling. On the basis of physical, biological or social laws, scientist formulates mathematical model that often is the system of differential equations and contains unknown parameters. Let we have mathematical model $M(\theta)$ in which θ is vector of unknown parameters. By model structure we mean parameterized set of models $M = \{M(\theta) : \theta \in \Omega \subset R^p\}$, where Ω is called admissible parametric space. The next step after construction of the model structure is determining "the best" model in the set M corresponding to the observed phenomenon in the best way. On this step the situation may take place when there are several such "best" models equally well describing the process under study. In other words the problem of determining the unknown parameters of the model has non-unique solution. If that's the case, then the model structure is called nonidentifiable. Thus, the property of model structure to permit unique solution to the task of determining unknown parameters is called structural identifiability.

Let us introduce stronger definitions borrowed from [4]. Denote the equality of the model input/output maps obtained for two values θ and θ^* of the parameter vector by

$$M(\theta) \approx M(\theta^*).$$

This property is also called indistinguishability of the models from input/output observations. The parameter θ_i is called structurally globally identifiable (s.g.i.) if for almost any $\theta^* \in \Omega$ (with the exception of subsets of Ω of measure zero)

$$M(\theta) \approx M(\theta^*) \Rightarrow \theta_i = \theta_i^*.$$

It is structurally locally identifiable (s.l.i.) if for almost any θ^* there exists a neighborhood $v(\theta^*)$ such that if $\theta \in v(\theta^*)$, then

$$M(\theta) \approx M(\theta^*) \Rightarrow \theta_i = \theta_i^*.$$

Local identifiability is of course a necessary condition for global identifiability. A parameter that is not s.l.i. is structurally nonidentifiable (s.n.i.). A model $M(.)$ is s.g.i.(s.l.i.) if all its parameters are s.g.i.(s.l.i.). A model is s.n.i. if any of its parameters is s.n.i.

As we can see structural properties are defined for all points of parametric space simultaneously (with exception of points belonging to subsets of measure zero). In practice when we construct a model structure, we have no information about exact values of the parameter vector. So we have to test identifiability of model structure in almost all points of parametric space. To do this, we must use analytic (symbolic) computations. Testing structural identifiability of sufficiently simple models leads rapidly to very complicated algebraic manipulations. Computer algebra is a good tool to obtain an answer. However, to rely only on the computing power could be disappointing because most algorithms generate very complicated computations.

We suggest a new approach to testing structural local identifiability which do not generate such complicated computations. The approach was recently described in [1]. In this paper we evolve the method and consider a question what to do in case of non-identifiability.

2. Main results

Consider the following model structure in state space form:

$$M(\theta) : \begin{cases} \dfrac{\partial}{\partial t}x(t) = A(\theta)x(t) + B(\theta)u(t), \quad x(0) = 0, \\ y(t) = C(\theta)x(t), \end{cases} \tag{2.1}$$

where t is time variable, $x \in R^n$, $u \in R^k$, $y \in R^m$ are state, control and observation vectors, $\theta \in \Omega \subset R^p$ is vector of unknown parameters of the model, $A \in R^{n\times n}, B \in R^{n\times k}, C \in R^{m\times n}$.

The first order differential equation in (2.1) is state equation. It is constructed on the basis of physical laws (for example, conservation laws) and includes vector of input variables to take into account possible control of the process under study. The second equation in (2.1) is observation equation. It describes connection between variables which we can observe (measure) and state variables that are not directly observed in general. Models in the state space allow to take into account apriori information about observed phenomenon in the most organic way. Unknown parameters θ in such models have clear physical meaning. But the problem of uniqueness of the task of determining these parameters from input/output observations is very typical.

In order to derive conditions for identifiability of model structure (2.1), we impose limitations on model structure.

We propose the following convenient form of representing linear restrictions on the elements of matrix A:

$$\psi\bar{a} = \psi_0, \tag{2.2}$$

where ψ is $r \times n^2$ - matrix of restrictions and ψ_0 is $r \times 1$ - vector, both with numerical elements, vector $\bar{a}$ consists of elements of matrix A:

$$\bar{a} = (A_{11}; A_{12}; \cdots; A_{1n}; A_{21}; \cdots; A_{nn})^T,$$

and r is the number of restrictions.

Let U and V be upper triangular matrices of transformation of matrices B^T and C to the column echelon forms $\tilde{B}^T$ and $\tilde{C}$:

$$\tilde{B}^T = B^T U; \quad \tilde{C} = CV. \tag{2.3}$$

Assuming without loss of generality that $\operatorname{rank} B = k$ and $\operatorname{rank} C = m$, matrix $\tilde{B}^T$ has $(n-k)$ columns consisting of zeros, and matrix $\tilde{C}$ has $(n-m)$ such columns. Let the numbers of zero columns of $\tilde{B}^T$ and $\tilde{C}$ form sets J^1 and J^2. Define $\tilde{U} = U(J^1)$, $\tilde{V} = V(J^2)$ - submatrices of matrices U and V consisting of columns with numbers from J^1 and J^2.

Thus, only matrix A depend on the unknown parameters. For linear restrictions (2.2) it is convenient to assume natural parameterization of this matrix when vector θ consists of elements of matrix A, for example $\theta = \bar{a}$.

Our method is based on similarity transformation approach for linear models. In accord with this approach we search for the set of all state-space models with the same model structure and the same input/output map as (2.1).

$$M(\theta^*): \begin{cases} \dfrac{\partial}{\partial t}x^*(t) = A(\theta^*)x^*(t) + B(\theta^*)u(t), \quad x^*(0) = 0, \\ y(t) = C(\theta^*)x^*(t). \end{cases} \tag{2.4}$$

It is known[3] that under assumption of controllability and observability of model structures (2.1) and (2.4) the state vectors x and x^* are related by nonsingular similarity transformation $x^* = Tx$, where $T \in GL(n) = \{T : det T \neq 0\}$. Transformation matrix T is determined from the following system of matrix equation (taking into account restrictions (2)):

$$\begin{cases} C(\theta^*)T - C(\theta) = 0; \\ B(\theta^*) - TB(\theta) = 0, \\ A(\theta^*)T - TA(\theta) = 0, \\ \psi(\bar{a}(\theta^*) - \bar{a}(\theta)) = 0. \end{cases} \tag{2.5}$$

The system (2.5) is that of nonlinear algebraic equations with respect to the pair (T, θ^*). The number of solutions to this system determines the number of models $M(\theta^*)$ indistinguishable by input/output map with the initial model $M(\theta)$, and therefore determines the number of solutions to the task of determining unknown parameters of the model. If the system has only one solution ($T = I,\ \theta^* = \theta$) the model structure is globally identifiable. If we have continuous set of solutions, the model structure is non-identifiable. Otherwise, if several solutions are isolated points in the parametric space, the model structure is locally identifiable.

In present paper we only concern a question of local identifiability. So we are interested in local solvability of the system (2.5). Write down the system (2.5) in general form:

$$F(T, \theta^*; \theta) = 0. \tag{2.6}$$

System (2.6) means indistinguishability of models $M(\theta)$ and $M(\theta^*)$ by input/output observations. It gives transformation $F : GL(n) \times \Omega \times \Omega \longrightarrow R^N$, where $N = n(n + k + m) + r$. It is clear that F is a C' function. Let us fix an arbitrary point θ in Ω. Note that the point $(I, \theta; \theta)$ is solution to the equation (2.6) (it follows from the fact that substitution $(T = I, \theta^* = \theta)$ into the system (2.5) gives identity).

Consider Jacobian of the system (2.6) for variables (T, θ^*):

$$F' = \left.\frac{\partial F(T, \theta^*; \theta)}{\partial (T, \theta^*)}\right|_{(T=I;\, \theta^*=\theta)}$$

Suppose that all columns of matrix F' are linearly independent, i.e.

$$\text{rank}\,(F') = \text{rank}\,(F'_T \mid F'_{\theta^*}) = n^2 + p, \tag{2.7}$$

where F'_T and F'_{θ^*} are $N \times n^2$ and $N \times p$ submatrices of matrix F'. Then by the implicit function theorem there are neighborhoods $N_1 \subset \Omega$ of point θ and $N_2 \subset GL(n) \times \Omega$ of point (I, θ), and also single-valued transformation $N_1 \longrightarrow N_2$. Therefore if the condition (2.7) is satisfied at every point $\theta \in \Omega$ except at points of a set of measure zero, this is a necessary and sufficient condition for structural local identifiability of the model.

Suppose now that the condition (2.7) does not satisfied for sets of nonzero measure. If that is the case, then we have

$$\text{rank}\,(F') = n^2 + p - \nu, \quad \nu > 0 \tag{2.8}$$

for almost all points of parametric space. So there exist $n^2 + p - \nu$ linearly independent columns of matrix F'. Let these linearly independent columns form submatrix $\tilde{F}'$ of matrix F'. Generate also vectors $\tilde{t}$ and $\tilde{\theta}^*$ consisting of elements T_{ij} and elements θ_i^* corresponding to the columns entering the matrix $\tilde{F}'$. Remaining elements of matrix T and vector θ^* we unite into subvectors $\bar{t}$ and $\bar{\theta}^*$. Introduce a vector $s = (\bar{t}, \bar{\theta}^*)$, $dim\, s = \nu$. Denote similar partition of vector θ by $\tilde{\theta}$ and $\bar{\theta}$. By analogy, divide elements of the identity matrix I into two vectors $\tilde{e}$ and $\bar{e}$. In new notations the system (2.6) has the form:

$$F(\tilde{t}, \tilde{\theta}^*; s, \theta) = 0. \tag{2.9}$$

Note that the system (2.9) becomes identity for $\tilde{t} = \tilde{e}$, $\tilde{\theta}^* = \tilde{\theta}$, $s = s_0 = (\bar{e}, \bar{\theta})$, i.e. $F(\tilde{e}, \tilde{\theta}; s_0, \theta) = 0$. Taking into account condition (2.8), by the implicit function theorem there are neighborhoods $N_1 \subset R^{p+\nu}$ of point

(s_0, θ), $N_2 \subset R^{n^2+p-\nu}$ of point $(\tilde{e}, \tilde{\theta})$, and single-valued transformation $N_1 \longrightarrow N_2$:

$$\tilde{\theta}^* = \tilde{f}(\theta, s), \, \tilde{t} = \tilde{g}(\theta, s), \quad \textit{where} \quad \tilde{f}(\theta, s_0) = \tilde{\theta}, \, \tilde{g}(\theta, s_0) = \tilde{e}.$$

Adding fictitious equations $\bar{t} = \bar{t}$ and $\bar{\theta}^* = \bar{\theta}^*$ to the above system, we obtain

$$\theta^* = f(\theta, s), \, t = g(\theta, s), \quad \textit{where} \quad f(\theta, s_0) = \theta, \, g(\theta, s_0) = e.$$

The last equations points to existence of continuous transformation of the parameters and elements of matrix T under which the initial system (2.6) is invariant. We are interested only in the parameter transformations and want to find invariants of this transformation being functions of the parameters θ only. Such invariants can be used as new parameters of the model structure. Under new reparameterization the model structure becomes locally identifiable.

Thus, consider the question of determining invariants $\rho(\theta)$ of the parameter transformation

$$\theta^* = f(\theta, s), \quad f(\theta, s_0) = \theta. \tag{2.10}$$

Let us expound the function $f(\theta, s)$ in Taylor series over s in the neighborhood of $s = s_0$:

$$\theta^* = \theta + \left.\frac{\partial f(\theta, s)}{\partial s}\right|_{s=s_0} (s - s_0) + o(s - s_0).$$

Analogously expound the function $\rho(\theta^*)$:

$$\begin{aligned}\rho(\theta^*) &= \rho(\theta + \left.\frac{\partial f(\theta, s)}{\partial s}\right|_{s=s_0} (s - s_0) + o(s - s_0)) \\ &= \rho(\theta) + \left.\frac{\partial \rho(\theta^*)}{\partial s}\right|_{s=s_0} (s - s_0) + o(s - s_0).\end{aligned}$$

Taking into account that invariants must not change under parameter's transformation, i.e., $\rho(\theta^*) = \rho(\theta)$, we obtain the condition of invariance:

$$\frac{\partial\rho(\theta^*)}{\partial s}\bigg|_{s=s_0} = \left(\frac{\partial\rho(\theta^*)}{\partial\theta^*}\cdot\frac{\partial f(\theta,s)}{\partial s}\right)\bigg|_{s=s_0} = \frac{\partial\rho(\theta)}{\partial\theta}\cdot\frac{\partial f(\theta,s)}{\partial s}\bigg|_{s=s_0} = 0. \tag{2.11}$$

To evaluate tangent vector fields $\partial f(\theta,s)/\partial s|_{s=s_0}$ we use expounding $F(T,\theta^*;\theta)$ in Taylor series over s in the neighborhood of $s = s_0$:

$$F(T,\theta^*;\theta) = F(I,\theta;\theta) + \frac{\partial F(T,\theta^*;\theta)}{\partial s}\bigg|_{s=s_0}(s-s_0) + o(s-s_0).$$

Since $F(T,\theta^*;\theta) = F(I,\theta;\theta) = 0$, we have

$$\frac{\partial F(T,\theta^*;\theta)}{\partial s}\bigg|_{s=s_0} = F'_T\cdot\frac{\partial g(\theta,s)}{\partial s}\bigg|_{s=s_0} + F'_{\theta^*}\cdot\frac{\partial f(\theta,s)}{\partial s}\bigg|_{s=s_0} = 0. \tag{2.12}$$

On the other hand, the condition (2.8) means that dimension of null space of matrix F' is equal to ν, i.e., there exist matrices L and Λ of sizes $n^2\times\nu$ and $p\times\nu$ such that

$$F'_T\cdot L(\theta) + F'_{\theta^*}\cdot\Lambda(\theta) = 0, \tag{2.13}$$

where $L(\theta) = (l_1, l_2, \dots, l_\nu)$ and $\Lambda(\theta) = (\lambda_1, \lambda_2, \dots, \lambda_\nu)$, vectors $({l_j}^T, \lambda_j^T)^T$, $j = 1,\dots,\nu$, being linearly independent. Comparing equations (2.12) and (2.13), we have

$$L(\theta) = \frac{\partial g(\theta,s)}{\partial s}\bigg|_{s=s_0}, \quad \Lambda(\theta) = \frac{\partial f(\theta,s)}{\partial s}\bigg|_{s=s_0}.$$

Now the condition of invariance (2.11) can be rewritten as the following homogeneous system of partial differential equations of the first order:

$$\frac{\partial\rho(\theta)}{\partial\theta}\cdot\lambda_j(\theta) = 0, \quad j = 1,\dots,\nu, \tag{2.14}$$

where vectors $\lambda_j(\theta)$ can be obtained from (2.13). Solving the system (2.14), we obtain functions $\rho(\theta)$ being invariant to the parameter's transformations. It is well known from the theory of partial differential equations that such system has an unbounded set of solutions. Functional basis of the set includes $p-\nu$ independent solutions; $\rho_B(\theta) = [\rho_1(\theta), \rho_2(\theta), \ldots, \rho_{p-\nu}(\theta)]^T$.

The evaluation technique for determining $\rho_B(\theta)$ is explained in a series of publications and is not easy. But we think that heuristic approach offered in [2] is very interesting and appropriate for symbolic computations. The idea of the approach is based on Lie group transformation theory and uses the observation that for each λ_j, $j = 1, \ldots, \nu$, the equation (2.14) is a condition of invariance of vector $\rho(\theta)$ to one-parameter group of continuous transformations of $\boldsymbol{\theta}$. To determine such groups we have to solve the following Lie differential equations for each of linearly independent tangent vector field λ_j, obtained from the equation (2.13):

$$\frac{d\check{f}_j}{d\check{s}_j} = \lambda_j(\check{f}_j), \quad \check{f}_j\big|_{\check{s}=0} = \theta, \quad j = 1, \ldots, \nu. \tag{2.15}$$

After solving the equations (2.15) we obtain ν expressions each describing one-parameter group of continuous transformations connected with given tangent vector field

$$\theta^* = \check{f}_j(\theta, \check{s}_j), \quad \check{f}_j(\theta, 0) = \theta, \quad j = 1, \ldots, \nu.$$

Superposition of ν such transformations will give us ν-parameter group of continuous transformations of the form

$$\theta^* = \check{f}(\theta, \check{s}), \quad \check{f}(\theta, 0) = \theta, \quad \check{s} = (\check{s}_1, \check{s}_2, \ldots, \check{s}_\nu)^T, \tag{2.16}$$

to which the system (2.6) is invariant. Note that equations (2.16) are equivalent to the equations (2.10) when $\check{s} = s - s_0$. Eliminating parameters of transformation $\check{s}_j$, $j = 1, \ldots, \nu$ from p equations (2.16) and representing $p - \nu$ remaining equations in the form

$$\rho_B(\theta^*) = \rho_B(\theta), \tag{2.17}$$

one can determine basis of invariants $\rho_B(\theta)$.

Now let us return to the system (2.5) to see if one can simplify computation of $\operatorname{rank}(F')$ using specific structure of the system. Computation of matrix F' for the system (2.5) gives

$$F' = (F'_T | F'_{\theta^*}) = \begin{pmatrix} -I_n & \otimes & B^T(\theta) & | & \partial\bar{b}/\partial\theta \\ C(\theta) & \otimes & I_n & | & \partial\bar{c}/\partial\theta \\ A(\theta) & \otimes & I_n - I_n \otimes A^T(\theta) & | & \partial\bar{a}/\partial\theta \\ 0 & & & | & \psi \end{pmatrix},$$

where $\bar{b}$ and $\bar{c}$ are as $\bar{a}$ vectors composed from elements of matrices B and C arranged along rows, symbol $\otimes$ means Kronecker product of matrices. Since by our supposition elements of matrices B and C do not depend on θ and $\theta = \bar{a}$, we get $\partial\bar{b}/\partial\theta = \partial\bar{c}/\partial\theta = 0$, $\partial\bar{a}/\partial\theta = I_{n^2}$. Taking into account reductions (2.3), postmultiply matrix F' by nonsingular matrix;

$$\begin{aligned}\tilde{F}' &= \begin{pmatrix} -I_n & \otimes & B^T & | & 0 \\ C(\theta) & \otimes & I_n & | & 0 \\ A & \otimes & I_n - I_n \otimes A^T & | & I_{n^2} \\ 0 & & & | & \psi \end{pmatrix} \cdot \begin{pmatrix} V & \otimes & U & 0 \\ 0 & & & I_{n^2} \end{pmatrix} \\ &= \begin{pmatrix} -I_n & \otimes & \tilde{B}^T & | & 0 \\ \tilde{C} & \otimes & I_n & | & 0 \\ AV & \otimes & U - V \otimes A^T U & | & I_{n^2} \\ 0 & & & | & \psi \end{pmatrix}.\end{aligned}$$

It is evident that $\operatorname{rank}(F') = \operatorname{rank}(\tilde{F}')$. From the structure of matrices $\tilde{B}^T$ and $\tilde{C}$ one can see

$$\operatorname{rank}(\tilde{F}') = n^2 + (n-k)(n-m) + \operatorname{rank}\begin{pmatrix} R & I_{n^2} \\ 0 & \psi \end{pmatrix}, \tag{2.18}$$

where $R = A\tilde{V} \otimes \tilde{U} - \tilde{V} \otimes A^T\tilde{U}$.

Further considerations give us

$$\begin{aligned}\operatorname{rank}\begin{pmatrix} R & I_{n^2} \\ 0 & \psi \end{pmatrix} &= \operatorname{rank}\begin{pmatrix} I_{n^2} & 0 \\ -\psi & I_r \end{pmatrix} \cdot \begin{pmatrix} R & I_{n^2} \\ 0 & \psi \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} R & I_{n^2} \\ -\psi R & 0 \end{pmatrix} = n^2 + \operatorname{rank}(\psi R).\end{aligned}$$

Thus we obtain that rank of matrix F' is completely determined by rank of matrix ψR. If one of these matrices has full column rank then the other matrix also has full column rank, being condition of structural local identifiability.

Suppose now, as before, that there are linear dependent columns of matrix F'. Matrices L and Λ can be evaluated from (2.13). It follows from (2.18) that $n^2-(n-k)(n-m)$ columns of matrix F'_T are linearly independent. Appropriate rows of matrix L are zero. Let matrix $\hat{L}$ is obtained by deleting such zero rows from matrix L. For $\hat{L}$ from (2.18) we have

$$\begin{cases} R\hat{L}+\Lambda=0, \\ \psi\Lambda=0; \end{cases} \quad \Longrightarrow \quad \begin{cases} \psi R\hat{L}=0, \\ \Lambda=-R\hat{L}. \end{cases}$$

Thus, to formulate the system of homogeneous partial differential equations (2.14) we must determine matrix $\hat{L}$ of linear connections between the columns of matrix ψR. Then we calculate matrix $\Lambda(\theta)=-R(\theta)\hat{L}(\theta)$ with columns $\lambda_j(\theta)$ forming equations of system (2.14). To obtain invariants $\rho_B(\theta)$ we have to solve this system using, for example, heuristic method described earlier.

Note that matrix Λ has the number of rows equal to the number of parameters θ, i.e., n^2. We can reduce this number by using restrictions (2.2). Assume that rank $\psi=r$. Then we can suppose without loss of generality that first r columns of matrix ψ are linearly independent, i.e.,

$$\psi=(\psi_1|\psi_2), \quad det\ \psi_1\neq 0.$$

Now condition (2.2) can be written as

$$\psi_1\bar{a}_1+\psi_2\bar{a}_2=\psi_0,$$

from which

$$\bar{a}_1=\psi_0-\psi_1^{-1}\psi_2\bar{a}_2.$$

Thus, r elements of vector $\bar{a}$ can be expressed through basic (n^2-r) elements. Let $\theta=\bar{a}_2$, where $\bar{a}_2$ consists of (n^2-r) basic elements of vector $\bar{a}$.

Therefore there is no need to find n^2 rows of matrix Λ. Note that each row of matrix $R(\theta)$ corresponds to element of vector $\bar{a}$. Let $\hat{R}(\theta)$ is submatrix of matrix $R(\theta)$ consisting of rows corresponding to elements of subvector $\bar{a}_2$. Now define $\hat{\Lambda}(\theta) = -\hat{R}(\theta)\hat{L}(\theta)$ and use $\hat{\Lambda}$ instead of Λ in the system (2.14) for searching functions capable of estimation $\rho(\theta)$.

Now we can formulate our main result.

THEOREM 2.1. (*Rank condition*) *A necessary and sufficient condition for structural local identifiability of the models (2.1), (2.2) with vector of unknown parameters θ, consisting of $(n^2 - r)$ basic elements of vector $\bar{a}$, under assumption of controllability and observability is that for almost all θ excluding sets of measure zero*

$$\text{rank}\, \psi R(\theta) = (n-k)(n-m), \tag{2.19}$$

where $R(\theta) = A(\theta)\tilde{V} \otimes \tilde{U} - \tilde{V} \otimes A(\theta)^T\tilde{U}$.

In this case the condition (2.19) does not satisfied and

$$\text{rank}\, \psi R(\theta) = (n-k)(n-m) - \nu, \quad \nu > 0.$$

We can find $p-\nu = n^2-r-\nu$ parametric functions capable of estimation from the following system of homogeneous partial differential equations of the first order

$$\frac{\partial \rho(\theta)}{\partial \theta} \cdot \hat{\lambda}_j(\theta) = 0, \quad j = 1, \dots, \nu,$$

where $\hat{\lambda}_j$ are columns of matrix

$$\hat{\Lambda}(\theta) = -\hat{R}(\theta)\hat{L}(\theta),$$

where $\hat{R}(\theta)$ consists of rows of matrix $R(\theta)$ corresponding to elements of vector θ, $\hat{L}(\theta)$ is determined from the condition

$$\psi\, R(\theta)\hat{L}(\theta) = 0.$$

COROLLARY 2.1. (*Order condition*) *A necessary condition for structural local identifiability of the models (2.1), (2.2) with vector of unknown parameters θ, consisting of $(n^2 - r)$ basic elements of vector $\bar{a}$, under assumption of controllability and observability is that*

$$r \geq (n-k)(n-m).$$

3. Examples

Let us consider model structure (2.1) with state, control and observation matrices of the following form:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}, \quad B^T = (1,\ 0,\ 0)^T, \ C = (0,\ 0,\ 1).$$

For this model we have four restrictions on the elements of matrix A: $a_{12} = a_{13} = a_{31} = a_{12} + a_{22} + a_{32} = 0$. Thus we can form vector θ consisting of $n^2 - r = 5$ elements

$$\theta = (a_{11}, a_{21}, a_{23}, a_{32}, a_{33})^T.$$

Matrix of the restrictions ψ is as follows :

$$\psi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Matrices B^T and C are already in the column echelon form, so $U = V = I$ and

$$\tilde{U} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T, \quad \tilde{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^T.$$

According to our approach, in order to answer the question about local identifiability of this model structure we evaluate matrix ψR :

$$\psi R = \begin{pmatrix} a_{11} + a_{32} & -a_{32} & 0 & 0 \\ -a_{23} & a_{11} - a_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{11} + a_{32} + a_{21} & -a_{32} & a_{32} & -a_{32} \end{pmatrix}.$$

It is easy to test that $\operatorname{rank}(\psi R) = 3 < 4$, therefore the model structure is non-identifiable. In order to investigate it more carefully and determine parametric functions capable of estimation we obtain basis of null space of matrix ψR, dimension of which is equal to $\nu = 1$. So basis contains one vector $\hat{L} = (0,\ 0,\ 1,\ 1)^T$. Therefore matrix $\hat{\Lambda}$ also contains only one column

$$\hat{\Lambda} = -\hat{R}\hat{L} = (0,\ -a_{21},\ -a_{23} - a_{32} - a_{33},\ a_{32},\ a_{32})^{\mathrm{T}}.$$

Note that coefficient of $\hat{\Lambda}$ corresponding to a_{11} is equal to zero. It means that this parameter itself is invariant under parameters transformation, and hence a_{11} is structurally globally identifiable. As regards other parameters we have one equation for determining invariants of transformation

$$-a_{21}\frac{\partial \rho}{\partial a_{21}} + a_{32}\frac{\partial \rho}{\partial a_{32}} + a_{32}\frac{\partial \rho}{\partial a_{33}} - (a_{23} + a_{32} + a_{33})\frac{\partial \rho}{\partial a_{23}} = 0.$$

In order to solve this equation we use approach considered earlier. Write down Lie differential equations (2.15) for determining one-parameter groups of transformation:

$$\frac{da_{21}^*}{ds} = -a_{21}^*; \quad \frac{da_{32}^*}{ds} = a_{32}^*; \quad \frac{da_{33}^*}{ds} = a_{32}^*;$$
$$\frac{da_{23}^*}{ds} = -a_{23}^* - a_{33}^* - a_{32}^*; \quad a_{ij}|_{s=0} = a_{ij}.$$

Solving the equations in series (after determining a_{32}^* and a_{33}^* we substitute them to subsequent equations) with use of initial conditions we obtain the following transformations:

$$a_{21}^* = a_{21}e^{-s}; \quad a_{32}^* = a_{32}e^{s}; \quad a_{33}^* = a_{32}e^{s} + a_{33} - a_{32};$$
$$a_{23}^* = -a_{32}e^{s} + (a_{33} + a_{23})e^{-s} + a_{32} - a_{33}.$$

Eliminating variable s from the system we can write it in the form

$$a_{21}^* a_{32}^* = a_{21}a_{32}; \quad a_{33}^* - a_{32}^* = a_{33} - a_{32}; \quad \frac{a_{33}^* + a_{23}^*}{a_{21}^*} = \frac{a_{33} + a_{23}}{a_{21}}.$$

Thus, we have the following invariants of transformations of parameter vector θ,

$$\rho_1(\theta) = a_{11}; \quad \rho_2(\theta) = a_{21}a_{32}; \quad \rho_3(\theta) = a_{33} - a_{32}; \quad \rho_4(\theta) = \frac{a_{33} + a_{23}}{a_{21}}.$$

If we now reparameterize the model structure using these functions as new parameters, it will be structurally locally identifiable.

4. Conclusion

The approach based on applying rank condition offered in this paper has the following advantages. First, rank condition has an obvious mathematical form and is easy for programming without knowing fundamentals of the theory. Secondly, matrix ψR in (2.19) has relatively small dimension $r \times (n-k)(n-m)$ and , that is extremely important, its elements are linear in parameters. As a result, with growth of the dimension of the system symbolic computations do not become unfeasible as computations produced by other methods Investigations carried out with the help of Maple program revealed effectiveness of our approach in cases when other methods failed.

The method has good prospects for further development. Research on extending the method to global identifiability and to the case when control and observation matrices depend on the unknown parameters are being carried out.

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