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EVALUATIONS OF $\zeta(2n)$

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ABSTRACT. Since the time of Euler, there have been many proofs giving the value of $\zeta(2n)$. We also give an evaluation of $\zeta(2n)$ by analyzing the generating function of Bernoulli numbers.

Many methods evaluating the value of $\zeta(2n)$ have been developed since L. Euler (1707-1783). In general $\zeta(2n)$ could be evaluated (for example) by applying the contour integral method [3, p. 129], by using the Fourier series [7, p. 376], and by appealing to the Riemann's functional equation for $\zeta(s)$ [2, p. 266]. Recently Choi *et al.* [4, 5] employed the theory of hypergeometric series and other methods to evaluate $\zeta(2)$. For further references see [1], [6], [7, p. 237], [8], [9], and so on.

We also want to participate in evaluating $\zeta(2n)$ by analyzing the generating function of Bernoulli numbers. We begin by introducing the Riemann Zeta function and Bernoulli numbers.

The Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} (2n-1)^{-s} \qquad (\Re(s) > 1). \tag{1}$$

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The nth Bernoulli numbers B_n (see Apostol [2, p. 264]) are defined by the generating function

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n \qquad (|z|<2\pi). \tag{2}$$

It is not difficult to deduce, from (2), the recurrence formula for the Bernoulli numbers:

$$1+\sum_{k=1}^{n}\binom{n+1}{k}B_{n-k+1}=0, \tag{3}$$

which gives the following first few explicit numbers:

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\ B_5=0,$$
$$B_6=\frac{1}{42},\ B_7=0,\ B_8=-\frac{1}{30},\ B_9=0,\ B_{10}=\frac{5}{66},\ \ldots .$$

It is easy to show that all numbers B_n with odd index greater than 1 are zero. In fact, upon setting $B_1=-1/2$ in (2), we have

$$\frac{z}{e^z-1}+\frac{1}{2}z=\frac{z}{2}\coth\frac{z}{2}=1+\frac{B_2}{2!}z^2++\frac{B_3}{3!}z^3+\cdots . \tag{4}$$

But $f(z)=(z/2)\coth(z/2)$ is an even function, that is, $f(-z)=f(z)$. Hence it follows that $B_{2k+1}=0$ ($k\in\mathbb{N}:=\{1,2,3,\ldots\}$). Thus the equation (4) is rewritten in the form:

$$\frac{z}{e^z-1}=1-\frac{1}{2}z+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n} \qquad (|z|<2\pi). \tag{5}$$

Now we shall analyze the generating function for Bernoulli numbers in (5) in a different way. It is well-known that (see [7, p. 207]) the expansion in partial fractions of $\pi z\cot\pi z$ is

$$\pi z\cot(\pi z)=1+2z^2\sum_{k=1}^{\infty}\frac{1}{z^2-k^2} \qquad (z\neq 0,\pm1,\pm2,\ldots). \tag{6}$$

A number of other expansions can be derived from (6). For instance, replacing z by iz in (6), we find that

$$\pi z \coth(\pi z) = 1 + 2z^2 \sum_{k=1}^{\infty} \frac{1}{z^2 + k^2} \qquad (z \neq \pm i, \pm 2i, \ldots). \tag{7}$$

.Substituting $z/(2\pi)$ for z in (7), in view of (4), we obtain

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z + 2z^2 \sum_{k=1}^{\infty} \frac{1}{z^2 + 4k^2\pi^2} \qquad (z \neq \pm 2k\pi i). \tag{8}$$

Denote the series in (8) by $g(z)$. We observe that the series $g(z)$ converges absolutely and uniformly on every compact set K which contains none of the numbers $\pm 2k\pi i$ ($k \in \mathbb{N}$). Indeed, if $A = \max_{z \in K} |z|$ and $k > A/(\sqrt{3}\pi)$, we have

$$\left| \frac{1}{z^2 + 4k^2\pi^2} \right| \leq \frac{1}{4k^2\pi^2 - A^2} < \frac{1}{\pi^2 k^2},$$

and the stated observation follows from the Weierstrass M-test.

So, except at those points $z \neq \pm 2k\pi i$ ($k \in \mathbb{N}$), $g(z)$ is analytic. In particular, $g(z)$ is analytic in a domain including the origin $z = 0$, and is an even function. Consequently there is a Maclaurin series expansion such that

$$g(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n} \qquad (|z| < 2\pi), \tag{9}$$

where $a_{2n} = g^{(2n)}(0)/(2n)!$ ($n \in \mathbb{N} \cup \{0\}$).

To determine a_{2n}, we find that

$$g(z) = \sum_{k=1}^{\infty} \frac{1}{4k\pi i} \left\{ \frac{1}{z - 2k\pi i} - \frac{1}{z + 2k\pi i} \right\}.$$

Since the series $g(z)$ converges uniformly in every closed region contained in $|z| < 2\pi$, we can differentiate it term by term and readily arrive at

$$g^{(2n)}(0) = (2n)! \sum_{k=1}^{\infty} \frac{1}{4k\pi i} \left\{ (-2k\pi i)^{-2n-1} - (2k\pi i)^{-2n-1} \right\}$$
$$= \frac{(-1)^n (2n)!}{(2\pi)^{2n+2}} \zeta(2n+2),$$

from which, with the help of (8) and (9), we get the following Maclaurin series for $z/(e^z - 1)$:

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2} z + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2\zeta(2n)}{(2\pi)^{2n}} z^{2n} \qquad (|z| < 2\pi). \tag{10}$$

Finally equating the coefficients of z^{2n} in equations (5) and (10), we obtain our desired evaluation of $\zeta(2n)$:

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!} \qquad (n \in \mathbb{N}), \tag{11}$$

which is well-known (*cf.* [2, p. 266]).

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References

[1] T. M. Apostol, *Another elementary proof of the formula for $\zeta(2n)$*, Amer. Math. Monthly **80** (1973), 425–431.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, 1976.

[3] J. Bak and D. J. Newman, *Complex Analysis*, Springer-Verlag, 1982.

[4] J. Choi and A. K. Rathie, *An evaluation of $\zeta(2)$*, Far East J. Math. Sci. **5** (1997), 393–398.

[5] J. Choi, A. K. Rathie, and H. M. Srivastava, *Some hypergeometric and other evaluations of $\zeta(2)$ and allied series*, Appl. Math. Comput. **104** (1999), 101–108.

[6] L. Euler, *Introduction to Analysis of the Infinite*, Book I, Springer-Verlag, 1988.

[7] K. Knopp, *Theory and Application of Infinite Series*, Dover Publications, Inc., 1990.

[8] I. Papadimitriou, *A simple proof the formula $\Sigma_{k=1}^{\infty} k^{-2} = \pi^2/6$*, Amer. Math. Monthly **80** (1973), 424–425.

[9] G. T. Williams, *A new method of evaluating $\zeta(2n)$*, Amer. Math. Monthly **60** (1953), 19–25.

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