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A NOTE ON PERIODIC POINT THEOREMS

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1. Introduction

The concept of w-distance on a metric space was introduced by Kada, Suzuki and Takahashi[3] and has provided a tool in generalized fixed point theorems and minimization theorems[3,4,8]. Consider the following fixed point theorem of Kada et al.;

Let T be a mapping from a complete metric space X into itself and p a ω-distance on X. If for some $r \in [0, 1)$

$$p(Tx, T^2x) \leq r \cdot p(x, Tx) \quad \text{for each} \quad x \in X$$

and

$$\inf\{p(x, y) + p(x, Tx) : \ x \in X\} > 0 \qquad (*)$$

for every $y \in X$ with $y \neq Ty$, then T has a fixed point. If either a ω-distance p is a metric or T is continuous then the inequality $(*)$ holds and thus this theorem generalizes the fixed point theorems of Subrahmanyam[6], Kannan[5] and Ćirić[1]. Ume[8] improved the above fixed point theorem for more general contractive mapping. Ćirić[2] obtained a periodic and fixed point theorem in a quasi-metric space using the concept of T-orbital weak lower semicontinuity.

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In this paper, using the concept of ω-distance, we obtain periodic point theorems and a common fixed point theorem for two mappings in a complete metric space, which generalizes fixed point theorems of Subrahmanyam[6] and Kada et al.[3].

2. Preliminaries

DEFINITION 2.1. Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a ω-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x,\ y,\ z \in X$;
(2) for any $x \in X$, $p(x, \cdot)\ :\ X \to [0, \infty)$ is lower semicontinuous ;
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The following example shows that the concept of ω-distances is a generalization of metrics.

EXAMPLE 1. [3] Let X be a metric space with a metric d. Then $p = d$ is a symmetric (*i.e.*, $p(x, y) = p(y, x)$ for $x, y \in X$) ω-distance on X.

Generally, a ω-distance $p : X \times X \to [0, \infty)$ is not symmetric. Let's consider a symmetric ω-distance.

EXAMPLE 2. [4] Let X be a metric space and p a ω-distance on X. Then the function $q : X \times X \to [0, \infty)$ defined by

$$q(x, y) = \max\{p(x, y),\ p(y, x)\} \text{ for } x, y \in X$$

is a symmetric ω-distance on X, provided that $p(\cdot, x)\ : X \to [0, \infty]$ is lower semicontinuous.

Owing to the third condition of ω-distance in Definition 2.1, some properties of sequences with respect to ω-distance might be criteria on Cauchy sequences in the metric space.

LEMMA 2.2. [3] *Let X be a metric space with a metric d and p a ω-distance on X. Let $\{x_n\}$ be a sequence in X, $\{\alpha_n\}$ and $\{\beta_n\}$ sequences in $[0,\infty)$ converging to 0, and $x, y, z \in X$. Then the following hold :*

(a) *if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$,*

(b) *if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

DEFINITION 2.3. Let X be a nonempty set and $T : X \to X$ a selfmap. The point $x \in X$ is called a periodic point for T if there exists a positive integer k such that $T^k x = x$. If $k = 1$, then x is called a fixed point for T.

Let X be a metric space and $T : X \to X$ a self-map. The set $o(x, \infty) = \{x, Tx, T^2x, \cdots\}$ is called the orbit of x.

DEFINITION 2.4. A real valued function $G : X \to [0.\infty)$ is said to be T-orbitually lower semicontinuous (resp. weak lower semicontinuous) relative to $x \in X$ if $\{x_n\}$ is a sequence in $o(x, \infty)$ and $\lim_{n\to\infty} x_n = z$ implies $G(z) \leq \liminf_{n\to\infty} G(x_n)$ (resp. $G(z) \leq \limsup_{n\to\infty} G(x_n)$).

3. Main results

Using the concept of a ω-distance, Kada et al. proved a nonconvex minimization theorem which improves the result of Takahashi[7].

LEMMA 3.1. *Let X be a complete metric space, and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a ω-distance p on X such that for any $u \in X$ with $\liminf_{x\in X} f(x) < f(u)$, there exists an element $v \in X$ with $v \neq u$ and*

$$f(v) + p(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\liminf_{x\in X} f(x) = f(x_0)$

THEOREM 3.2. *Let X be a complete metric space, and let $\Phi : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Let T be a mapping from X into itself and $n : X \to \mathbb{N}$ a mapping. Assume that there exists a ω-distance p on X such that*

$$p(x, T^{n(x)}x) \leq \Phi(x) - \Phi(T^{n(x)}x)$$

for every $x \in X$. Then there exists a $x_0 \in X$ such that $T^{n(x_0)}x_0 = x_0$ and $p(x_0, x_0) = 0$.

PROOF. Since Φ is a proper function, there exists $u \in X$ such that $\Phi(u) < \infty$. Put

$$A = \{x \in X : \Phi(x) \leq \Phi(u)\}$$

Then A is a closed subspace of the complete space X because Φ is lower semicontinuous. Therefore A is also a complete space. If $x \in A$, then

$$\Phi(T^{n(x)}x) \leq \Phi(x) - p(x, T^{n(x)}x) \leq \Phi(x) \leq \Phi(u),$$

which implies $T^{n(x)}x \in A$. Assume that $T^{n(x)}x \neq x$ for every $x \in A$. Then, by Lemma 3.1, there exists $x_0 \in X$ such that $\liminf_{x \in A} \Phi(x) = \Phi(x_0)$. From the fact that $\Phi(T^{n(x_0)}x_0) + p(x_0, T^{n(x_0)}x_0) \leq \Phi(x_0)$ and $\liminf_{x \in A} \Phi(x) = \Phi(x_0)$, we have

$$\Phi(T^{n(x_0)}x_0) = \Phi(x_0) = \liminf_{x \in A} \Phi(x),$$

and $p(x_0, T^{n(x_0)}x_0) = 0$. Put $x_1 = T^{n(x_0)}x_0$. Applying the similar method to x_1 instead of x_0, we obtain

$$\Phi(T^{n(x_1)}x_1) = \Phi(x_1) = \Phi(T^{n(x_0)}x_0) = \liminf_{x \in A} \Phi(x),$$

and $p(T^{n(x_0)}x_0, T^{n(x_1)}x_1) = 0$. Since

$$p(x_0, T^{n(x_1)}x_1) \leq p(x_0, T^{n(x_0)}x_0) + p(T^{n(x_0)}x_0, T^{n(x_1)}x_1) = 0,$$

$p(x_0, T^{n(x_1)}x_1) = 0$ and thus $x_1 = T^{n(x_1)}x_1$ by Lemma 2.2 (a). This is a contradiction. Therefore. T has a periodic point x^* in A. Hence

$$p(x^*, x^*) = p(x^*, T^{n(x^*)}x^*) \leq \Phi(x^*) - \Phi(T^{n(x^*)}x^*) = \Phi(x^*) - \Phi(x^*) = 0,$$

which means $p(x^*, x^*) = 0$.

In Theorem 3.2, T need not have fixed points.

EXAMPLE 3. Let $X = [-\frac{5}{2}, -\frac{3}{2}] \cup [2,3]$ be a space with the usual metric d, $p = d$, $T : X \to X$ the function defined by $T(x) = -x + \frac{1}{2}$ for each $x \in X$ and $n \cdot X \to \mathbb{N}$ the constant function with its value 2. If $\Phi : X \to (-\infty, \infty]$ is a constant function, then it is a proper lower semicontinuous function, bounded from below. Since

$$p(x, T^2x) = p(x,x) = 0 \leq \Phi(x) - \Phi(T^2x) = 0,$$

for each $x \in X$. T has a periodic point but it has no fixed points.

In Theorem 3.2, if $n(x) = 1$ for each $x \in X$, we obtain the following result of Kada et al. as a corollary.

COROLLARY 3.3[3]. *Let X be a complete metric space, T a mapping from X into itself and $\Phi : X \to (-\infty, \infty]$ a proper lower semicontinuous function, bounded from below. Assume that there exists a ω-distance p on X such that*

$$p(x, Tx) \leq \Phi(x) - \Phi(Tx)$$

for every $x \in X$. Then there exists a $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$.

COROLLARY 3.4[6]. *Let T be a selfmap on a complete metric space X and $\Phi : X \to [0, \infty)$ a lower semicontinuous function. If for each $x \in X$,*

$$p(x, Tx) \leq \Phi(x) - \Phi(Tx),$$

then T has a fixed point.

The hypothesis of the lower semicontinuity of a function Φ in the previous discussion can be substituted the hypothesis of orbital continuity of a selfmap T as follows:

THEOREM 3.5. *Let $T : X \to X$ be a selfmap on a complete metric space (X, d), $n : X \to \mathbb{N}$ a mapping and $\Phi : X \to [0, \infty]$ a real valued function. Suppose that for some $x_0 \in X$, there exist a ω-distance p and*

a subsequence $S = \{x_n\}_{n=1}^{\infty}$ *in the orbit* $o(x, \infty)$ *such that* $T^{n(x_n)}x_n \in S$ *and*

$$p(x, T^{n(x)}x) \leq \Phi(x) - \Phi(T^{n(x)}x)$$

holds for each $x \in S$. *Then we have*

(a) $\lim_{n\to\infty} x_n = z$ *exists.*

(b) *If* $p(z, z) = 0$, *then* T *has a periodic point* z *if and only if* $G(x) = p(x, T^{n(x)}x)$ *is* T*-orbitally weak lower semicontinuous relative to* x_0.

(c) *If* $G(x) = p(x, T^{n(x)}x)$ *is* T*-orbitually weak lower semicontinuous relative to* x_0 *and for all* $y \in Cl[o(x_0, \infty)]$,

$$y \neq Ty \text{ implies } \Phi(T^m y) < \Phi(y)$$

for some positive integer $m = m(y)$, *then* T *has a fixed point.*

PROOF. Put $x_1 = T^{n(x_0)}x_0$ and define $x_{n+1} = T^{n(x_n)}x_n$ for each $n = 1, 2, \cdots$. Then

$$p(x_n, x_{n+1}) = p(x_n, T^{n(x_n)}x_n) \leq \Phi(x_n) - \Phi(x_{n+1}).$$

For $m \geq 0$,

$$\begin{aligned}\sum_{n=0}^{m} p(x_n, x_{n+1}) &\leq \sum_{n=0}^{m} \Phi(x_n) - \Phi(x_{n+1}) \\ &= \Phi(x_0) - \Phi(x_{m+1}) \\ &\leq \Phi(x_0).\end{aligned}$$

Since $\sum_{n=0}^{\infty} p(x_n, x_{n+1})$ is an infinite series of non-negative real numbers and the sequence of its partial sum is bounded above by $\Phi(x_0)$, it converges. We claim that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given. Then, there is $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$. From the fact that $\sum_{n=0}^{\infty} p(x_n, x_{n+1})$ converges in X, we can choose a positive integer n_0 such that $\sum_{n=n_0}^{\infty} p(x_n, x_{n+1}) < \delta$. Then for any n, $m \geq n_0$ $(n < m)$,

$$\begin{aligned}p(x_n, x_m) &\leq p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m) \\ &\leq \sum_{n=n_0}^{\infty} p(x_n, x_{n+1}) < \delta.\end{aligned}$$

Since $p(x_n, x_{n+1}) < \delta$ and $p(x_{n+1}, x_m) < \delta$, $d(x_n, x_m) < \varepsilon$, which shows that $\{x_n\}$ is a Cauchy sequence in a complete space. Therefore there exists $\lim_{n\to\infty} x_n = z$ for some $z \in X$. Thus we proved (a).

To show (b), assume that $p(z, z) = 0$. Suppose that $G(x) = p(x, T^{n(x)}x)$ is T-orbitally weak lower semicontinuous relative to x_0. Then,

$$\begin{aligned} G(z) &\leq \limsup_{n\to\infty} G(x_n) \\ &= \limsup_{n\to\infty} p(x_n, x_{n+1}). \end{aligned}$$

Since the infinite series $\sum_{n=0}^{\infty} p(x_n, x_{n+1})$ converges, the sequence $p(x_n, x_{n+1})$ converges to 0. Therefore $G(z) = p(z, T^{n(z)}z) = 0$. Since $p(z, z) = 0$, $z = T^{n(z)}z$, that is, z is a periodic point of T. Conversely, suppose that $z = T^{n(z)}z$ and $\{x_n\}$ is a sequence in the orbit $o(x, \infty)$ such that $\lim_{n\to\infty} x_n = z$. Then,

$$G(z) = p(z, T^{n(z)}z) = 0 \leq \limsup_{n\to\infty} G(x_n).$$

For (iii), assume that the conditions hold. Let z be a periodic point of T. Then the orbit $o(z, \infty)$ is a finite subset of X. Choose a point $y \in o(z, \infty)$ such that

$$\Phi(y) = \min\{\Phi(x) : \ x \in o(z, \infty)\}.$$

Assume that $y \neq Ty$. By the hypothesis, there exists $m \in \mathbb{N}$ such that $\Phi(T^m y) < \Phi(y)$. It contradicts the fact that $\Phi(y)$ is the minimum value on $o(z, \infty)$. Therefore $y = Ty$, that is, T has a fixed point.

Now we shall prove a common fixed point theorem for two mappings.

THEOREM 3.6. *Let S, $T : X \to X$ be selfmaps on a complete metric space (X, d), and $\Phi : X \to [0, \infty]$ a real valued function. Suppose that for some $x_0 \in X$, there exists a ω-distance p such that*

$$p(y, Ty) + p(Ty, STy) \leq \Phi(y) - \Phi(STy)$$

for each $y \in o_{ST}(x_0, \infty)$, *where*

$$o_{ST}(x_0, \infty) = \{x_0, Tx_0, STx_0, T(ST)x_0, \cdots, (ST)^n x_0, T(ST)^n x_0, \cdots\}.$$

Then we have

(*a*) $\lim_{n\to\infty}(ST)^n x_0 = \lim_{n\to\infty} T(ST)^n x_0 = z$ *exists, and*
(*b*) *if* $G_1(x) = p(x, Tx)$ *and* $G_2(x) = p(x, Sx)$ *are* (S,T)*-orbitally weak lower semicontinuous relative to* x_0 *and* $p(z,z) = 0$, *then* T *and* S *have a common fixed point.*

PROOF. Define $x_{2k} = (ST)^k x_0$ and $x_{2k+1} = Tx_{2k}$ for each $k = 0, 1, 2 \cdots$. Then,

$$\begin{aligned}\sum_{n=0}^{2k+1} p(x_n, x_{n+1}) &\le \sum_{n=0}^{k} p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}) \\ &\le \sum_{n=0}^{k} \Phi(x_{2n}) - \Phi(x_{2n+2}) \\ &= \Phi(x_0) - \Phi(x_{2n+2}) \le \Phi(x_0).\end{aligned}$$

Since the infinite series $\sum_{n=0}^{\infty} p(x_n, x_{n+1})$ of non-negative real numbers is bounded above, $\{x_n\}$ is a Cauchy sequence in a complete metric space X. Therefore there exist $\lim_{n\to\infty} x_n = z$ for some $z \in X$. Hence

$$\lim_{k\to\infty} x_{2k} = \lim_{n\to\infty} (ST)^n x_0 = z$$

and

$$\lim_{k\to\infty} x_{2k+1} = \lim_{n\to\infty} T(ST)^n x_0 = z.$$

Thus we have (*a*). For (b), assume that $G_1(x) = p(x, Tx)$ and $G_2(x) = p(x, Sx)$ are (S,T)-orbitally weak lower semicontinuous relative to x_0 and $p(z,z) = 0$. From (*b*) of Theorem 3.3, we have $Tz = z = Sz$.

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