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# METRIC COMPLETENESS AND ORDER COMPLETENESS

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Abstract. We give nine necessary and sufficient conditions for a metric space to be complete.

## 1. Introduction

In 1980, Turinici [2] introduced the notion of order complete metric spaces, pointed out that it is a nontrivial extension of the classical notion of complete metric spaces and gave a characterization of order completeness. In 1993, Conserva and Rizzo [1] characterized a class of order complete metric spaces as those ones in which every map of a suitable family has at least one fixed point.

The purpose of this note is to investigate the connections between metric completeness and order completeness. We obtain several characterizations of metric completeness by using fixed point and stationary point theorems on a class of ordered metric spaces.

Throughout this note, $(X, d)$ is a metric space and $\leq$ is an order (i.e., a reflexive, anti-symmetric and transitive relation) on $X$. $2^X$ and $N$ denote the power set of $X$ and the set of all positive integers, respectively. For $A \subset X$, $\delta(A)$ denotes the diameter of $A$. An order $\leq$ on $X$ is called closed if $X(x)$ is closed for all $x$ in $X$, where

$$X(x) = \{y : x \leq y \text{ and } y \in X\}.$$

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$(X, d)$ is called $\leq$ −complete if every nondecreasing Cauchy sequence in $X$ converges. A sequence $\{x_n\}_{n \in N}$ in $X$ is said to be asymptotic if

$$d(x_n, x_{n+1}) \to 0$$

as $n \to \infty$. $(X, d)$ is said to be $\leq$ −asymptotic if every nondecreasing sequence in $X$ is an asymptotic sequence. Define

$$CO(X) = \{\leq : \leq \text{ is a closed order on } X \text{ and } (X, d) \text{ is } \leq -\text{asymptotic}\}.$$

## 2. Main results

We begin with the following lemma which plays a crucial role in the proof of main theorem in this paper.

LEMMA. *Let $(X, d)$ be a metric space and let $X = X_0 \supset X_1 \supset \cdots \supset X_n \supset \cdots$ be a sequence of nonempty closed subsets of $X$ such that $\delta(X_n) \to 0$ as $n \to \infty$, but $\cap_{n \in N} X_n = \emptyset$. Define a mapping $i : X \to N \cup \{0\}$ by $i(x) = n$ if $x \in X_n$ and $x \notin X_{n+1}$. Define a relation $\leq$ on $X$ by $x \leq y$ in $X$ if and only if*

$$y \in \{x\} \cup X_{i(x)+1}.$$

*Then the relation $\leq$ is an order on $X$.*

PROOF. Actually, reflexivity is clear. For antisymmetry, suppose that $x \leq y$, $y \leq x$ and $x \neq y$. Then $y \in X_{i(x)+1}$ and $x \in X_{i(y)+1}$, which implies that $i(y) \geq i(x) + 1$ and $i(x) \geq i(y) + 1$. Therefore $i(y) \geq i(x) + 1 \geq i(y) + 2$. This is a contradiction. For transitivity, suppose that $x \leq y$ and $y \leq z$. If either $x = y$ or $y = z$, then $x \leq z$ ; if $x \neq y$ and $y \neq z$, then $i(z) \geq i(y) + 1 \geq i(x) + 2 > i(x) + 1$ ; i.e., $z \in X_{i(z)} \subset X_{i(x)+1}$ and hence $x \leq z$. This completes the proof.

We are now in a position to prove the result of this paper.

THEOREM. *For any metric space, $(X,d)$ the following statements are equivalent:*

(1) *$(X,d)$ is complete;*
(2) *If $\leq$ is an order on $X$, then $(X,d)$ is $\leq -$complete;*
(3) *If $\leq$ is in $CO(X)$, then $(X,d)$ is $\leq -$ complete;*
(4) *If $\leq$ is in $CO(X)$, then there exists a maximal element in $X$;*
(5) *If $\leq$ is in $CO(X)$ and $f : X \to 2^X$ is a multimap such that for each $x \in X \setminus f(x)$ there exists $y \in X \setminus \{x\}$ with $x \leq y$, then $f$ has a fixed point;*
(6) *If $\leq$ is in $CO(X)$ and $f$ is a selfmap on $X$ such that for each $x \in X \setminus \{f(x)\}$ there exists $y \in X \setminus \{x\}$ with $x \leq y$, then $f$ has a fixed point;*
(7) *If $\leq$ is in $CO(X)$ and $f : X \to 2^X \setminus \{\emptyset\}$ is a multimap satisfying $x \leq y$ for each $x \in X$ and each $y \in f(x) \setminus \{x\}$, then $f$ has a stationary point $w$ in $X$; i.e., $fw = \{w\}$;*
(8) *If $\leq$ is in $CO(X)$ and $f : X \to 2^X \setminus \{\emptyset\}$ is a multimap satisfying $x \leq y$ for each $x \in X$ and each $y \in fx$, then $f$ has a stationary point $w$;*
(9) *If $\leq$ is in $CO(x)$ and $\mathcal{F}$ is a family of selfmaps on $X$ satisfying $x \leq fx$ for all $x$ in $X$ and all $f$ in $\mathcal{F}$, then $\mathcal{F}$ has a common fixed point;*
(10) *If $\leq$ is in $CO(X)$ and $f$ is a selfmap on $X$ satisfying $x \leq f(x)$ for all $x$ in $X$, then $f$ has a fixed point.*

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (7) $\Rightarrow$ (8) are clear. (3) $\Rightarrow$ (4) is assured by Theorem 3.1 of Turinici [2].

(4) $\Rightarrow$ (5). It follows from (4) that there exists a maximal element $w$ in $X$. Suppose that $w \notin fw$, then there is $y \in X - \{w\}$ with $w \leq y$. This is a contradiction. Hence $w \in fw$.

(5) $\Rightarrow$ (6). Let $g(x) = \{f(x)\}$ for all $x$ in $X$. By (5), $g$ has a fixed point $w$ in $X$, Therefore, $w = f(w)$.

(6) $\Rightarrow$ (7). Suppose that $f$ has no stationary point. Then $f(x) \setminus \{x\} \neq \emptyset$ for all $x$ in $X$. Choose a choice function $g$ on $\{f(x) \setminus \{x\} : x \in X\}$. Then $g$ is a selfmap on $X$ and $g$ has no fixed point. Note that for each $x \in X \setminus \{g(x)\}$ there exists $g(x) \in f(x) \setminus \{x\} \subset X \setminus \{x\}$ with $x \leq g(x)$. In view of (6), $g$ has a fixed point. This is a contradiction.

(8) $\Rightarrow$ (9). Define a multimap $g : X \to 2^X \setminus \{\emptyset\}$ by $g(x) = \{f(x) : f \in \mathcal{F}\}$ for all $x$ in $X$. Since $x \leq f(x)$ for all $x$ in $X$ and all $f$ in $\mathcal{F}$, by (8), $g$ has a stationary point $w$ in $X$, which is a common fixed point of $\mathcal{F}$.

(9) $\Rightarrow$ (10). Take $\mathcal{F} = \{f\}$.

(10) $\Rightarrow$ (1). Suppose that $(X, d)$ is not complete. Then there exists a sequence $X = F_0 \supset F_1 \supset \cdots \supset F_n \supset \cdots$ of nonempty closed subsets of $X$ such that $\delta(F_n) \to 0$ as $n \to \infty$, but $\cap_{n\in N} F_n = \emptyset$. Define a mapping $i : X \to N \cup \{0\}$ by $i(x) = n$ if $x \in F_n$ and $x \notin F_{n+1}$. Define a relation $\leq$ on $X$ by $x \leq y$ in $X$ if and only if $y \in \{x\} \cup F_{i(x)+1}$. Then the the relation $\leq$ is an order on $X$ from Lemma. We next prove that the order $\leq$ is in $CO(X)$. Note that for all $x$ in $X$, $X(x) = \{x\} \cup F_{i(x)+1}$ is closed. Therefore $\leq$ is closed. Assume that $\{x_n\}_{n\in N} \subset X$ is a nondecreasing sequence. To prove it is asymptotic, we consider two cases :

Case 1. There exists $k \in N$ such that $x_n = x_k$, for all $n \geq k$. Then $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ ;

Case 2. For every $k \in N$, there exists $n > k$ such that $x_n \neq x_k$. It is easy to see that there exists a sequence $\{n_k\}_{k\in N} \subset N$ such that $n_k < n_{k+1}$ and $x_{n_k} \neq x_{n_{k+1}}$ for all $k$ in $N$. Since $x_{n_k} \leq x_{n_{k+1}}$ and $x_{n_k} \neq x_{n_{k+1}}$, it follows that $x_{n_{k+1}} \in F_{i(x_{n_k})+1}$. This means that $i(x_{n_{k+1}}) \geq i(x_{n_k}) + 1$. Consequently $i(x_{n_k}) \to \infty$ as $k \to \infty$. For each $m > n_1$, we can easily find $k(m)$ in $N$ such that $n_{k(m)} \leq m < n_{k(m)+1}$. Note that $i(x_{n_{k(m)}}) \to \infty$ as $m \to \infty$. It follows that

$$d(x_m, x_{m+1}) \leq \delta(F_{i(x_m)}) \leq \delta(F_{i(x_{n_{k(m)}})}) \to 0$$

as $m \to \infty$. Note that $F_{i(x)+1} \neq \emptyset$ for each $x$ in $X$. Choose a choice function $f$ on $\{F_{i(x)+1} : x \in X\}$. It is easy to see that $x \leq fx$ for all $x$ in $X$. It follows from (10) that $f$ has a fixed point $w$ in $X$ ; i.e., $w = fw \in F_{i(w)+1}$. However, by the definition of $i(x)$ we have $x \notin F_{i(x)+1}$ for all $x$ in $X$. This is a contradiction. The proof is completed.

## References

[1] V. Conserva and S. Rizzo, *Fixed points and completeness*, Math. Japonica **38** (1993), 901-903

[2] M. Turinici, *Maximal elements in a class of order complete metric spaces*, Math. Japonica **25** (1980), 511-517

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