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ON THE RANDERS CHANGES OF FINSLER SPACES WITH THE KROPINA TYPE OF DOUGLAS TYPE

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1. Introduction

A Finsler space F^n with the vanishing Douglas tensor D is said to be *of Douglas type* or called a *Douglas space* ([3], [13]). It is known that if a Finsler space F^n is projective to a Berwald space, then F^n is of Douglas type ([2]). Recently, S. Bácsó and M. Matsumoto([3]) have introduced the new notion of Douglas space as a generalization of a Berwald space from the viewpoint of geodesic equations.

A Finsler metric $L(x,y)$ is called an (α,β)*-metric*, when L is a positively homogeneous function $L(\alpha,\beta)$ of degree one in two variables : $\alpha^2 = a_{ij}(x)y^iy^j$ and 1-form $\beta = b_i(x)y^i$. The theories of Finsler spaces with (α,β)-metric have contributed to the development of Finsler geometry ([11]), and Berwald spaces with an (α,β)-metric have been treated by some authors ([1], [7], [10]). Since a Berwald space is a kind of Douglas spaces, the noteworthy point of the present paper is to observe that, comparing with the condition of Berwald space, to what extent the condition of Douglas space relaxed. The (α,β)-metric $L(\alpha,\beta)$ satisfying $L = (c_1\alpha^2 + c_2\beta^2)/\beta$, where c_i's are constants, is called a *Kropina type.* Some properties of the Finsler metric L satisfying the Kropina type have been investigated by ([2], [9]).

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On the other hand, the Randers metric $L = \alpha + \beta$ is considered as the modification of a Riemannian metric α by 1-form β. We can consider generally the change of Finsler metric $L \to \overline{L} = L + \rho$, where ρ is a 1-form. This change is called the *Randers change* by ρ.

In the present paper, first we are devoted to studying the general Randers change of the Finsler space F^n which is of Douglas type. Next, we deal with the condition for a Finsler space F^n with the Kropina type to be of Douglas type. Next, we investigate the condition that the Finsler space $\overline{F}^n$ with the Kropina type obtained from a special Randers change by β is also of Douglas type. Finally, in order to compare with the Douglas space, we find the condition for F^n to be a Berwald space.

Throughout the present paper the terminology and notation are referred to Matsumoto's monograph ([14]).

2. Preliminaries

The geodesics of a Finsler space $F^n = (M^n, L)$ are given by the system of differential equations

$$\frac{d^2x^i}{dt^2}y^j - \frac{d^2x^j}{dt^2}y^i + 2\{G^i(x,y)y^j - G^j(x,y)y^i\} = 0, \quad y^i = \frac{dx^i}{dt}$$

in a parameter t. The functions $G^i(x,y)$ are given by

$$2G^i(x,y) = g^{ij}(y^r\dot{\partial}_j\partial_r F - \partial_j F) = \left\{ {i \atop j\,k} \right\} y^j y^k,$$

where $\partial_i = \partial/\partial x^i$, $\dot{\partial}_i = \partial/\partial y^i$, $F = L^2/2$, $g^{ij}(x,y)$ are the inverse of Finsler metric $g_{ij}(x,y)$ and $\left\{ {i \atop j\,k} \right\}$ are Christoffel symbols constructed from $g_{ij}(x,y)$ with respect to x^i.

A Finsler space F^n is said to be of *Douglas type* or called a *Douglas space* ([3]) if

$$D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i$$

are homogeneous polynomials in (y^i) of degree three. It is shown that F^n is of Douglas type, if and only if the *Douglas tensor*

$$D_i{}^h{}_{jk} = G_i{}^h{}_{jk} - \frac{1}{n+1}(G_{ijk}y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j)$$

vanishes everywhere, where $G_i{}^h{}_{jk} = \dot\partial_k G_i{}^h{}_j$ is the hv-curvature tensor of the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i_j)$, $G_{ij} = G_i{}^r{}_{jr}$ and $G_{ijk} = \dot\partial_k G_{ij}$ ([2]).

On the other hand, F^n is said to have an (α,β)-metric, if L is a positively homogeneous function of (α,β) of degree one in α and β, where $\alpha^2 = a_{ij}(x)y^iy^j$ and $\beta = b_i(x)y^i$. The space $R^n = (M^n,\alpha)$ is called the associated Riemannian space with F^n ([2], [6], [11]). In R^n we have the Christoffel symbols $\gamma_j{}^i{}_k(x)$ and the covariant differentiation with respect to $\gamma_j{}^i{}_k(x)$. We shall use the symbols as follows:

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}),$$
$$s^i{}_j = a^{ir}s_{rj}, \quad s_j = b_r s^r{}_j.$$

It is noted that $s_{ij} = \dfrac{1}{2}(\partial_j b_i - \partial_i b_j)$.

There are two kinds of Finsler spaces with an (α,β)-metric which are specially interesting and important in the geometrical point of view as well as in applications to physics ;

Randers spaces with $L = \alpha + \beta$ and the Kropina spaces with $L = \alpha^2/\beta$.

Those spaces of Douglas type have been considered in the previous paper ([3]):

LEMMA 2.1. *A Randers space is of Douglas type if and only if* $s_{ij} = 0$. *Moreover* $2G^i = \gamma_0{}^i{}_0 + r_{00}y^i/L$.

It has been shown ([7], [10]) that a Randers space is a Berwald space, if and only if $b_{i;j} = 0$. Therefore the condition in Lemma 2.1 is certainly more relaxed than that of Berwald space.

Now we consider the functions $G^i(x,y)$ of F^n with an (α,β)-metric. According to ([8], [10]), they are written in the forms

$$(2.1)\qquad \begin{aligned} 2G^i &= \gamma_0{}^i{}_0 + 2B^i, \\ B^i &= \frac{E}{\alpha}y^i + \frac{\alpha L_\beta}{L_\alpha}s^i_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}C^*\left(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i\right), \end{aligned}$$

where

$$E = \frac{\beta L_\beta}{L}C^*, \quad C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \quad b^i = a^{ij}b_j,$$
$$\gamma^2 = b^2\alpha^2 - \beta^2, \quad b^2 = a^{ij}b_ib_j.$$

Since $\gamma_0{}^i{}_0 = \gamma_j{}^i{}_k(x)y^jy^k$ are homogeneous polynomials in (y^i) of degree two, we have :

PROPOSITION 2.2. *A Finsler space F^n with (α,β)-metric is a Douglas space if and only if $B^{ij} = B^iy^j - B^jy^i$ are homogeneous polynomials in (y^i) of degree three.*

Thus the equation (2.1) gives

$$(2.2)\qquad B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0y^j - s^j_0y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^iy^j - b^jy^i).$$

Here, we shall state the following lemma for the later frequent use ([8]):

LEMMA 2.3. *If $\alpha^2 \equiv 0 \pmod{\beta}$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

Throughout the paper, we shall say "*homogeneous polynomial(s) in (y^i) of degree r*" as $hp(r)$ for brevities. Thus $\gamma_0{}^i{}_0$ are $hp(2)$ and, if the space is of Douglas type, then D^{ij} and B^{ij} are $hp(3)$.

On the other hand, we consider the properties of Randers change by ρ. Let $L_i = \dot\partial_iL$, $L_{ij} = \dot\partial_i\dot\partial_iL$, $L_{ijk} = \dot\partial_k\dot\partial_j\dot\partial_iL$. Then we have

$$L_i = l_i,\ LL_{ij} = h_{ij},\ L^2L_{ijk} = h_{ij}l_k + h_{jk}l_i + h_{ki}l_j.$$

And we put

$$2E_{ij} = \rho_{i|j} + \rho_{j|i}, \quad 2F_{ij} = \rho_{i|j} - \rho_{j|i}, \tag{2.3}$$

where $(_|)$ denotes the h-covariant derivative with respect to the Cartan connection $C\Gamma = (F_k{}^i{}_j, G^i{}_j, C_k{}^i{}_j)$.

The following Lemma 2.4 ([13]) is used later.

LEMMA 2:4. *A system of linear equations*

$$L_{ir}X^r = Y_i, \quad (l_r + \rho_r)X^r = Y, \quad (Y_i y^i = 0),$$

in X^i *has the unique solution*

$$X^i = LY^i + \frac{1}{\tau}(Y - LY^r\rho_r)l^i,$$

where $Y^i = g^{ir}Y_r$ *and* $\tau = \overline{L}/L$.

3. Randers change of Douglas type

In general, we are devoted to investigating the condition that a Finsler space $\overline{F}_n$, obtained from Randers change by ρ with respect to a Finsler space F_n, is of Douglas type. For a Randers change : $L(x,y) \longrightarrow \overline{L}(x,y) = L(x,y) + \rho(x,y), \quad \rho(x,y) = \rho(x)_i y^i$,

we may put

$$\overline{G}^i = G^i + D^i. \tag{3.1}$$

So $\overline{G}^i{}_j = G^i{}_j + D^i{}_j$ and $\overline{G}_j{}^i{}_k = G_j{}^i{}_k + D_j{}^i{}_k$, where $D^i{}_j = \dot{\partial}_j D^i$ and $D_j{}^i{}_k = \dot{\partial}_k D^i{}_j$. The tensors D^i, $D^i{}_j$ and $D_j{}^i{}_k$ are positively homogeneous in y^i of degree two, one and zero respectively. In the following the explicit form of D^i is necessary. To find this, we deal with equation $L_{ij|k} = 0$, where $L_{ij|k}$ is the h-covariant derivative of $L_{ij} = h_{ij}/L$ in $C\Gamma$. Note that

$$\partial_k L_{ij} = L_{ijr}G^r{}_k + L_{rj}F_i{}^r{}_k + L_{ir}F_j{}^r{}_k.$$

Since $\overline{L}_{ij} = L_{ij}$ and $\overline{L}_{ijk} = L_{ijk}$ hold,

$$\overline{L}_{ijk} = L_{ijr}(G^r{}_k + D^r{}_k) + L_{rj}(F_i{}^r{}_k - D_i{}^r{}_k) + L_{ir}(F_j{}^r{}_k + D_j{}^r{}_k),$$

which imply

$$L_{ijr}D^r{}_k + L_{rj}D_i{}^r{}_k + L_{ir}D_j{}^r{}_k = 0.$$

Thus transvection of this equation by y^k yields

$$2L_{ijr}D^r + L_{rj}D^r{}_i + L_{ir}D^r{}_j = 0. \tag{3.2}$$

Next, we deal with $L_{i|j} = 0$, that is,

$$\partial_j L_i = L_{ir}G^r{}_j + L_r F_i{}^r{}_j,$$

$$\partial_j \overline{L}_i = L_{ir}(G^r{}_j + D^r{}_j) + (L_r + \rho_r)(F_i{}^r{}_j + {}^cD_i{}^r{}_j),$$

where ${}^cD_i{}^r{}_j = \overline{F}_i{}^r{}_k - F_i{}^r{}_k$. Substitution of the equations above in $\partial_j\overline{L}_i = \partial_j L_i + \partial_j \rho_i$ leads to

$$\partial_j \rho_i - \rho_r F_i{}^r{}_j = L_{ir}D^r{}_j + (l_r + \rho_r)\ {}^cD_i{}^r{}_j.$$

Then we have

$$2E_{ij} = L_{ir}D^r{}_j + L_{jr}D^r{}_i + 2(l_r + \rho_r)\ {}^cD_i{}^r{}_j, \tag{3.3}$$

$$2F_{ij} = L_{ir}D^r{}_j - L_{jr}D^r{}_i. \tag{3.4}$$

Therefore (3.2) and (3.4) give

$$L_{ir}D^r{}_j = F_{ij} - L_{ijr}D^r \tag{3.5}$$

and transvection of (3.3) by y^i gives

$$(l_r + \rho_r)D^r{}_j = E_{ij}y^i - L_{jr}D^r. \tag{3.6}$$

Furthermore transvection of (3.5) and (3.6) by y^j leads to

$$\text{(a)}\quad L_{ir}D^r = F_{ij}y^j, \qquad \text{(b)}\quad (l_r + \rho_r)D^r = \frac{1}{2}E_{ij}y^i y^j. \tag{3.7}$$

The equations (3.7)(a) and (3.7)(b) constitute a system of linear equations respectively. Applying Lemma 2.5 to (3.7), we have

$$D^i = LF^i{}_0 + \frac{1}{\overline{L}}\left(\frac{1}{2}E_{00} - LF_0\right)y^i, \tag{3.8}$$

where $F^i{}_j = g^{ir}F_{rj}$ and $F_j = \rho_r F^r{}_j$.

Thus we have the following :

PROPOSITION 3.1. ([13]) *The tensor D^i of* (3.1) *arising from a Randers change is given by* (3.8).

From (3.1) and (3.8) we have

$$\overline{G}^i y^j - \overline{G}^j y^i = G^i y^j - G^j y^i + L(F^i{}_0 y^j - F^j{}_0 y^i). \tag{3.9}$$

Suppose F^n is a Douglas space, that is, $G^i y^j - G^j y^i$ be *hp* (3). Therefore the Randers change $\overline{F}^n$ of F^n by ρ is also a Douglas space if and only if $L(F^i{}_0 y^j - F^j{}_0 y^i)$ is *hp* (3). Thus we have the following :

THEOREM 3.2. *Let F^n be a Douglas space and $\overline{F}^n$ a Finsler space which is obtained by Randers change by ρ. Then $\overline{F}^n$ is a Douglas space if and only if $L(F^i{}_0 y^j - F^j{}_0 y^i)$ is hp* (3).

From (3.9), $\overline{G}^i y^j - \overline{G}^j y^i = G^i y^j - G^j y^i$ if and only if $F^i{}_0 y^j$ is symmetric in i and j. Thus we have the following :

THEOREM 3.3. *Let F^n be a Finsler space satisfying that $F^i{}_0 y^j$ are symmetric in i, j, and $\overline{F}^n$ be the Randers change of F^n. If F^n is a Douglas space, then $\overline{F}^n$ is also a Douglas space, and vice versa.*

The Randers changes are called *projective Randers changes* if all the geodesic curves are preserved under the Randers changes. According to Hashiguchi-Ichijyō ([5]), a Randers change is projective if and only if $F_{ij} = 0$, that is, ρ_i is a locally gradient vector field. In this case, (3.8) is reduced to $D^i = E_{00} y^i / 2\overline{L}$. Therefore we have $D^i y^j - D^j y^i = 0$. Thus $\overline{G}^i y^j - \overline{G}^j y^i = G^i y^j - G^j y^i$. Consequently, we have the following

THEOREM 3.4. *Let $F^n(M^n, L) \longrightarrow \overline{F}^n(M^n, L+\rho)$ be a projective Randers change. If F^n is a Douglas space, then $\overline{F}^n$ is also a Douglas space, and vice versa.*

4. Douglas space with a Kropina type

We consider a condition for a Finsler space F^n with Kropina type satisfying

$$L = \frac{c_1\alpha^2 + c_2\beta^2}{\beta} \tag{4.1}$$

to be of Douglas type. As it has been remarked in [4], $b^2 \neq 0$ may be supposed for F^2, and hence Lemma 2.4 shows $\alpha^2 \not\equiv 0 \ (mod\,\beta)$. Then from (4.1) we obtain

$$\beta L_\alpha = 2c_1\alpha, \quad \beta^2 L_\beta = c_2\beta^2 - c_1\alpha^2, \quad \beta L_{\alpha\alpha} = 2c_1. \tag{4.2}$$

Thus (2.2) gives

$$\begin{aligned} B^{ij} = &\left\{\frac{c_2\beta}{2c_1}(s^i{}_0 y^j - s^j{}_0 y^i) + \frac{c_1 r_{00} - c_2\beta s_0}{2c_1 b^2}(b^i y^j - b^j y^i)\right\} \\ &+ \left\{-\frac{\alpha^2}{2\beta}(s^i{}_0 y^j - s^j{}_0 y^i) + \frac{\alpha^2 s_0}{2b^2\beta}(b^i y^j - b^j y^i)\right\}. \end{aligned} \tag{4.3}$$

Since the term

$$(c_2\beta/2c_1)(s^i{}_0 y^j - s^j{}_0 y^i) + \{(c_1 r_{00} - c_2\beta s_0)/2c_1 b^2\}(b^i y^j - b^j y^i)$$

is $hp(3)$, this term may be neglected in our discussion and we deal only with

$$B'^{ij} = \frac{\alpha^2}{2\beta}\left\{\frac{s_0}{b^2}(b^i y^j - b^j y^i) - (s^i{}_0 y^j - s^j{}_0 y^i)\right\}. \tag{4.4}$$

It follows from $\alpha^2 \not\equiv 0 \ (mod\,\beta)$ that (4.4) leads to

$$\frac{s_0}{b^2}(b^i y^j - b^j y^i) - s^i{}_0 y^j + s^j{}_0 y^i = \beta u^{ij},$$

where $u^{ij} = u^{ij}_k(x) y^k$. This can be written in the form

$$\frac{1}{b^2}\{b^i(s_h\delta^j_k + s_k\delta^j_h) - i/j\} - (s^i{}_h\delta^j_k + s^i{}_k\delta^j_h - i/j) = b_h u^{ij}_k + b_k u^{ij}_h, \tag{4.5}$$

where i/j denotes the interchange of indices i, j of the previous terms. Transvecting (4.5) by a^{hk}, we have

$$\frac{1}{b^2}(b^i s^j - b^j s^i) - 2s^{ij} = b^r u_r^{ij}. \tag{4.6}$$

Next, transvection of (4.5) by b^h leads to

$$s^i \delta_k^j + b^i s^j{}_k - i/j = b^2 u_k^{ij} + b_k b^r u_r^{ij}. \tag{4.7}$$

Further, contracting (4.5) by $j = h$, we get

$$n\left(\frac{1}{b^2} b^i s_k - s^i{}_k\right) = b_r u_k^{ir} + b_k u_r^{ir}. \tag{4.8}$$

Substituting $b^r u_r^{ij}$ of (4.6) in (4.7), we get

$$b^2 u_k^{ij} = 2s^{ij} b_k + \left(b^i s^j{}_k + s^i \delta_k^j + \frac{1}{b^2} s^i b^j b_k - i/j\right),$$

which implies

$$b^2 u_r^{ir} = (n-1)s^i, \quad b^2 b_r u_k^{ir} = b^i s_k - b^2 s^i{}_k.$$

Consequently (4.8) leads to

$$s_{ik} = \frac{1}{b^2}(b_i s_k - b_k s_i). \tag{4.9}$$

Then (4.4) gives

$$B'^{ij} = \frac{\alpha^2}{2b^2}(s^i y^j - s^j y^i),$$

which is $hp(3)$.

Therefore we get (4.9) as a necessary and sufficient condition for F^n to be of Douglas type.

In particular, we shall deal with the two-dimensional Kropina type F^2 to be of Douglas type. On account of [8], the skew-symmetric tensor

s_{ij} can be written as $s_{ij} = s(u_i v_j - u_j v_i)$ in the Berwald frame (u, v) of the associated Riemannian space R^2. Since $b_i = (\beta/\alpha)u_i + Bv_i$, we get $s^i{}_0 = -s\alpha v^i$ and $s_i = (s\beta/\alpha)v_i - esBu_i$, where e is the signature of R^2. Thus

$$b_i s_k - b_k s_i = \left(\frac{\beta^2}{\alpha^2} + eB^2\right) s(u_i v_k - u_k v_i).$$

Since B satisfies $eB^2 + (\beta/\alpha)^2 = b^2$, above equation just coincides with (4.9).

Therefore we have the following :

THEOREM 4.1. *A Kropina type F^n is of Douglas type if and only if the equation in (4.9) is satisfied.*

5. Douglas space with Kropina type by a special Randers change

In this section, we consider a condition for which a Finsler space $\overline{F}^n$, obtained from a special Randers change by β of the Finsler metric L satisfying (4.1), is of Douglas type, where the modification 1-form ρ is coincided with β of (4.1). Let $\overline{F}^n = (M^n, \overline{L})$ be a Finsler space which is obtained by Randers change of L satisfying

$$\overline{L} = \frac{c_1\alpha^2 + c_2\beta^2}{\beta} + \beta. \tag{5.1}$$

Then we obtain

$$\begin{gathered} \beta\overline{L}_\alpha = 2c_1\alpha, \quad \beta^2\overline{L}_\beta = (c_2+1)\beta^2 - c_1\alpha^2, \quad \beta L_{\alpha\alpha} = 2c_1, \\ 2c_1 b^2\alpha C^* = c_1 r_{00}\beta - s_0\{(c_2+1)\beta^2 - c_1\alpha^2\}. \end{gathered} \tag{5.2}$$

Substituting (5.2) in (2.2), we have

$$\begin{aligned} \overline{B}^{ij} = & \left\{\frac{c_1 r_{00} - (c_2+1)\beta s_0}{2c_1 b^2}(b^i y^j - b^j y^i)\right. \\ & \left. + \frac{(c_2+1)\beta}{2c_1}(s^i{}_0 y^j - s^j{}_0 y^i)\right\} \\ & + \frac{\alpha^2}{2\beta}\left\{\frac{s_0}{b^2}(b^i y^j - b^j y^i) - (s^i{}_0 y^j - s^j{}_0 y^i)\right\}. \end{aligned} \tag{5.3}$$

Because the term

$$\left\{\frac{c_1 r_{00} - (c_2+1)\beta s_0}{2c_1 b^2}(b^i y^j - b^j y^i) + \frac{(c_2+1)\beta}{2c_1}(s^i{}_0 y^j - s^j{}_0 y^i)\right\}$$

is $hp(3)$, this term may be neglected in our discussion and we treat only

$$\overline{W}^{ij} = \frac{\alpha^2}{2\beta}\left\{\frac{s_0}{b^2}(b^i y^j - b^j y^i) - (s^i{}_0 y^j - s^j{}_0 y^i)\right\}. \tag{5.4}$$

For $n > 2$, $\alpha^2 \not\equiv 0\ (mod\,\beta)$ ([3]). Therefore there exists $hp(1)$ $v^{ij} = v^{ij}_k(x)y^k$ such that

$$s_0(b^i y^j - b^j y^i) - b^2(s^i{}_0 y^j - s^j{}_0 y^i) = b^2\beta v^{ij}. \tag{5.5}$$

This equation is written as follows:

$$\begin{aligned}&\frac{1}{b^2}\{b^i(s_h\delta^j_k + s_k\delta^j_h) - b^j(s_h\delta^i_k + s_k\delta^i_h)\}\\ &\qquad -(s^i{}_h\delta^j{}_k + s^i{}_k\delta^j{}_h) + (s^j{}_h\delta^i{}_k + s^j{}_k\delta^i{}_h) = b_h v^{ij}_k + b_k v^{ij}_h.\end{aligned} \tag{5.6}$$

Transvection of (5.6) by a^{hk} leads to

$$(b^i s^j - b^j s^i) - 2b^2 s^{ij} = b^2 b^r v^{ij}_r. \tag{5.7}$$

Next, transvecting (5.6) by b^h, we have

$$(s^i\delta^j_k + b^i s^j{}_k) - (s^j\delta^i_k + b^j s^i{}_k) = b^2 v^{ij}_k + b_k b^r v^{ij}_r. \tag{5.8}$$

Contraction of (5.6) with j and h leads to

$$n\Big(b^i s_k - b^2 s^i{}_k\Big) = b^2(b_r v^{ir}_k - b_k v^{ir}_r). \tag{5.9}$$

Substituting $b^r v^{ij}_r$ of (5.7) in (5.8), we have

$$b^2 v^{ij}_k = 2s^{ij}b_k + \left\{b^i s^j{}_k - b^j s^i{}_k + s^i\delta^j_k - s^j\delta^i_k + \frac{1}{b^2}(s^i b^j b_k - s^j b^i b_k)\right\},$$

which implies

$$v^{ir}_{r} = \frac{(n-1)}{b^2}s^i, \qquad b_r v^{ir}_{k} = \frac{b^i}{b^2}s_k - s^i{}_k.$$

Consequently (5.9) leads to

$$b^2 s_{ij} = b_i s_j - b_j s_i. \tag{5.10}$$

Then (5.4) gives

$$\overline{W}^{ij} = \frac{\alpha^2}{2b^2}(s^i y^j - s^j y^i),$$

which is $hp(3)$. Therefore (5.10) is a necessary and sufficient condition for $\overline{F}^n$ to be of Douglas type.

Thus we have the following :

THEOREM 5.1. *Let F^n be a Finsler space ($n > 2$) with an (α, β)-metric L satisfying (3.1) and $b^2 \neq 0$. Suppose F^n is a Douglas space and $\overline{F}^n$ is a Finsler space which is obtained from a special Randers change of $F^n (n > 2)$ by β. Then $\overline{F}^n (n > 2)$ is also a Douglas space if and only if the equation (5.10) holds.*

6. Berwald space with a Kropina type

In this section, in order to compare with the Douglas space with respect to the Finsler space F^n with the Kropina type, we deal with a condition for which the Finsler space F^n satisfying (4.1) is a Berwald space.

Since $B\Gamma$ ([15]) is L-metrical, the equation $\partial_i L - G^k{}_i \dot{\partial}_k L = 0$ is rewritten as follows:

$$L_\alpha B_j{}^k{}_i y^j y_k = \alpha L_\beta (b_{i\,j} - B_j{}^k{}_i b_k) y^j, \tag{6.1}$$

where $y_k = a_{ki} y^i$. In the following, the raising and lowering of indices are done by means of the Riemannian $a_{ij}(x)$. Substituting (4.2) in (6.1), we obtain

$$2c_1 \beta B_j{}^k{}_i y^j y_k + (c_2 \beta^2 - c_1 \alpha^2)(B_j{}^k{}_i b_k - b_{j;i}) y^j = 0. \tag{6.2}$$

Now, we assume that the Finlser space F^n with an (α, β)-metric given by (4.1) is a Berwald space, that is, $G_j{}^i{}_k$ is a function of the position alone, or we have $B_j{}^k{}_i = B_j{}^k{}_i(x)$. Then the left-hand side of the above equation is a polynomial of order three in (y^i) and, this guarantees the existence of function $p_i(x)$ satisfying

$$B_j{}^k{}_i y^j y_k = p_i(x)(c_2\beta^2 - c_1\alpha^2), \tag{6.3.1}$$

$$(b_{j,i} - B_j{}^k{}_i b_k)y^j = 2c_1 p_i(x)\beta. \tag{6.3.2}$$

Using $y_k = a_{kl}y^l$ and differentiating (6.3.1) by y^m and y^n, we get

$$B_j{}^k{}_i a_{kl}(\delta^j_m\delta^l_n + \delta^j_n\delta^l_m) = p_i(x)(c_2 b_j b_k - c_1 a_{jk})(\delta^j_m\delta^k_n + \delta^j_n\delta^k_m). \tag{6.4}$$

Thus (6.4) is rewritten in the form

$$B_m{}^k{}_i a_{kn} + B_n{}^k{}_i a_{km} = 2p_i(x)(c_2 b_m b_n - c_1 a_{mn}). \tag{6.5}$$

By Christoffel processes with respect to i, m, n to (6.5) we obtain

$$\begin{aligned} B_m{}^k{}_i a_{kn} =& p_i(x)(c_2 b_m b_n - c_1 a_{mn}) + p_m(x)(c_2 b_n b_i - c_1 a_{ni}) \\ & - p_n(x)(c_2 b_i b_m - c_1 a_{im}). \end{aligned} \tag{6.6}$$

Transvecting (6.6) by a^{nl}, we have

$$\begin{aligned} B_j{}^k{}_i =& p_i(x)(c_2 b_j b^k - c_1\delta^k_j) + p_j(x)(c_2 b^k b_i - c_1\delta^k_i) \\ & - p^k(x)(c_2 b_i b_j - c_1 a_{ij}). \end{aligned} \tag{6.7}$$

Differentiating (6.3.2) by y^l, we get

$$b_{j,i} = B_j{}^k{}_i b_k + 2c_1 p_i(x) b_j. \tag{6.8}$$

Substituting (6.7) in (6.8), we obtain

$$\begin{aligned} b_{j,i} =& (c_2 b^2 + c_1)p_i(x)b_j + (c_2 b^2 - c_1)p_j(x)b_i \\ & - p_b(c_2 b_i b_j - c_1 a_{ij}), \end{aligned} \tag{6.9}$$

where $p_b = b_k p^k(x)$.

Conversely, if there exists a vector $p_i(x)$ satisfying (6.9), then we have $L = 0$ with respect to $G_j{}^i{}_k = \gamma_j{}^i{}_k + B_j{}^i{}_k$, where $B_j{}^i{}_k$ is given by (6.7). Hence, by the well-known Hashiguchi-Ichijyō's theorem ([6]), the Finsler space is a Berwald space.

Thus we have the following :

THEOREM 6.1. *Let F^n be the Finsler space with an (α,β)-metric given by (4.1) and the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$ given by $G^i{}_j = \gamma_0{}^i{}_j + B^i{}_j$, $G_j{}^i{}_k = \gamma_j{}^i{}_k + B_j{}^i{}_k$. Then F^n is a Berwald space if and only if there exists a covariant vector $p_i(x)$ satisfying (6.9), and the Berwald connection is written as $(\gamma_j{}^i{}_k + B_j{}^i{}_k, \gamma_0{}^i{}_j + B_0{}^i{}_j, 0)$, where $B_j{}^i{}_k$ are given by (6.7).*

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