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REMARKS ON FIXED POINT THEOREMS

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ABSTRACT. In this paper we show fixed point theorems related with the diameter of orbit on metric spaces. The results presented in this paper extend, improve and unify the results of Hegedüs [1], Kim, Kim, Leem and Ume [2], Kim and Leem [3], Ohta and Nikaido [4] and Tasković [5].

1. Introduction

Let f and g be mappings from a metric space (X,d) into itself, ω and N denote the sets of nonnegative integers and positive integers, respectively. For $x, y \in X$ and $A \subseteq X$, define

$$\begin{aligned} O_{f,g}(x) &= \{f^i g^j x : i,j \in \omega\}, \\ O_{f,g}(x,y) &= O_{f,g}(x) \cup O_{f,g}(y), \\ O_f(x) &= O_{f,f}(x), \\ O_f(x,y) &= O_f(x) \cup O_f(y), \\ \delta(x,A) &= \sup\{d(x,a) : a \in A\}, \\ \delta(A) &= \sup\{d(x,y) : x,y \in A\}. \end{aligned}$$

Recall that x is *regular* for f if $\delta(O_f(x)) < +\infty$ and the point x is *regular* for f and g if $\delta(O_{f,g}(x)) < +\infty$. The mapping f is called *closed*

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on X if whenever $\{x_n\}_{n=0}^{\infty}$ is a sequence in X and $x, y \in X$ satisfying $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} fx_n = y$, then $y = fx$. It is easy to see that each continuous mapping is closed. For each $t \in [0, +\infty)$, $[t]$ denotes the largest integer not exceeding t.

Recently, the existence of fixed point and common fixed point for the following mappings have been investigated by Hegedüs [1], Kim, Kim, Leem and Ume [2], Kim and Leem [3], Ohta and Nikaido [4] and Tasković [5] and others.

(1.1) (Hegedüs [1] and Tasković [5]) there exists $r \in [0, 1)$ such that for all $x, y \in X$,

$$d(fx, fy) \leq r\delta(O_f(x, y)).$$

(1.2) (Ohta and Nikaido [4]) there exist $k \in N$ and $r \in [0, 1)$ such that for all $x, y \in X$,

$$d(f^k x, f^k y) \leq r\delta(O_f(x, y)).$$

(1.3) (Kim and Leem [3]) there exist $k \in N$ and $r \in [0, 1)$ such that for all $x, y \in X$,

$$d(f^k x, g^k y) \leq r\delta(O_{f,g}(x, y)).$$

(1.4) (Kim and Leem [3]) there exist $k \in N$ and $r \in [0, 1)$ such that for all $x, y \in X$,

$$d((fg)^k x, (fg)^k y) \leq r\delta(O_{f,g}(x, y)).$$

(1.5) (Kim, Kim, Leem and Ume [2]) there exist $m, n \in N$ and $r \in [0, 1)$ such that for all $x, y \in X$,

$$d((fg)^m x, (fg)^n y) \leq r\delta(O_{f,g}(x, y))$$

The purpose of this paper is to establish fixed and common fixed point theorems for the following mappings.

(1.6) there exist $p, q, m, n \in \omega$ with $p + q, m + n \in N$ and $r \in [0, 1)$ such that for all $x, y \in X$,

$$d(f^p g^q x, f^m g^n y) \leq r\delta(O_{f,g}(x, y)).$$

(1.7) there exist $p \in \{1, 2\}$ and $r \in [0, 1)$ such that for all $x, y \in X$,

$$d(fx, f^p y) \leq r\delta(O_f(x, y)).$$

Our results extend, improve and unify the results of Hegedüs [1], Kim, Kim, Leem and Ume [2], Kim and Leem [3], Ohta and Nikaido [4], Tasković [5] and others.

2. Main results

THEOREM 2.1. *Let f and g be commuting mappings from a metric space (X,d) into itself such that fg is closed on X. Suppose that there exists a regular point $u \in X$ for f and g such that some subsequence of the sequence $\{(fg)^i u\}_{i\in N}$ converges to a regular point $w \in X$ for f and g for which the inequality* (1.6) *holds for all $x, y \in O_{f,g}(u,w)$. Then w is a common fixed point of f and g. Moreover*

$$d((fg)^i f^a g^b u, w) \leq r^{[\frac{i}{k}]}\delta(O_{f,g}(u))$$

for all $i \in N$ and $a, b \in \{0,1\}$, where $k = \max\{p,q,m,n\}$.

PROOF. For any $i, j, l, s, t \in \omega$, it follows form (1.6) that

$$\begin{aligned} &d(f^{i+k+j}g^{i+k+l}u, f^{i+k+s}g^{i+k+t}u) \\ &\leq r\delta(O_{f,g}(f^{i+k-p+j}g^{i+k-q+l}u, f^{i+k-m+s}g^{i+k-n+t}u)) \\ &\leq r\delta(O_{f,g}(f^{i+j}g^{i+l}u, f^{i+s}g^{i+t}u)) \\ &\leq r\delta(O_{f,g}(f^i g^i u, f^i g^i u)) \\ &= r\delta(O_{f,g}(f^i g^i u)), \end{aligned}$$

which means that

$$\delta(O_{f,g}(f^{i+k}g^{i+k}u)) \leq r\delta(O_{f,g}(f^i g^i u)) \tag{2.1}$$

for all $i \in \omega$. We assert that

$$d((fg)^i u, (fg)^{i+t}u) \leq r^{[\frac{i}{k}]}\delta(O_{f,g}(u)) \tag{2.2}$$

for all $i, t \in N$. In fact, we can write $i = ck + l$ uniquely for some $c, l \in \omega$ with $l \leq k-1$. From (2.1) we have

$$\begin{aligned} d((fg)^i u, (fg)^{i+t}u) &= d((fg)^{ck+l}u, (fg)^{ck+l+t}u) \\ &\leq \delta(O_{f,g}((fg)^{ck+l}u)) \\ &\leq r\delta(O_{f,g}((fg)^{(c-1)k+l}u) \\ &\leq r^2\delta(O_{f,g}((fg)^{(c-2)k+l}u)) \\ &\leq \cdots \\ &\leq r^c\delta(O_{f,g}((fg)^l u)) \\ &\leq r^c\delta(O_{f,g}(u)). \end{aligned}$$

That is, (2.2) holds. This implies that $\{(fg)^i u\}_{i\in N}$ is a Cauchy sequence. Note that there exists a subsequence of $\{(fg)^i u\}_{i\in N}$ converging to $w \in X$, that is, $w = \lim_{i\to\infty}(fg)^i u$. Since fg is closed on X,

$$w = \lim_{i\to\infty}(fg)^i u = \lim_{i\to\infty}(fg)^{i+1}u = \lim_{i\to\infty} fg(fg)^i u = fgw. \tag{2.3}$$

For any i, j, s, $t \in \omega$, by (1.6) and (2.3) we have

$$\begin{aligned}
& d(f^i g^j w, f^s g^t w) \\
&= d(f^{i+k} g^{j+k} w, f^{s+k} g^{j+k} w) \\
&\le r\delta(O_{f,g}(f^{i+k-p} g^{j+k-q} w, f^{s+k-m} g^{t+k-n} w)) \\
&\le r\delta(O_{f,g}(f^i g^j w, f^s g^t w)) \\
&\le r\delta(O_{f,g}(w)),
\end{aligned}$$

which implies that

$$\delta(O_{f,g}(w)) \le r\delta(O_{f,g}(w)).$$

That is, $\delta(O_{f,g}(w)) = 0$. Therefore $w = fw = gw$. It follows from (2.1) that

$$\begin{aligned}
d((fg)^i f^a g^b u, (fg)^{i+t} u) &\le \delta(O_{f,g}((fg)^i u)) \\
&\le r\delta(O_{f,g}((fg)^{i-k} u)) \\
&\le \cdots \\
&\le r^{[\frac{i}{k}]}\delta(O_{f,g}(u))
\end{aligned}$$

for all $i, t \in N$ and $a, b \in \{0, 1\}$. Letting t tend to infinity we have

$$d((fg)^i f^a g^b u, w) \le r^{[\frac{i}{k}]}\delta(O_{f,g}(u))$$

for all $i \in N$ and $a, b \in \{0, 1\}$. This completes the proof.

THEOREM 2.2. *Let f and g be commuting mappings from a bounded complete metric space (X,d) into itself such that fg is closed on X. Assume that* (1.6) *holds. Then f and g have a unique common fixed point $w \in X$ and*

$$d((fg)^i f^a g^b x, w) \leq r^{[\frac{i}{k}]}\delta(O_{f,g}(x)) \tag{2.4}$$

for all $x \in X$, $i \in N$ and $a, b \in \{0,1\}$, where $k = \max\{p,q,m,n\}$.

PROOF. Theorem 2.1 ensures that f and g have a common fixed point $w \in X$ and that (2.4) holds. The uniqueness of common fixed point follows from (1.6). This completes the proof.

REMARK 2.1. Theorem 2.1 extends, improves and unifies Theorems 3 and 4 of Kim and Leem [3].

REMARK 2.2. Theorem 2.2 includes Theorem 2.1 of Kim, Kim, Leem and Ume [2] and Theorem 3 of Ohta and Nikaido [4] as special cases.

THEOREM 2.3. *Let f be a mapping from a metric space (X,d) into itself. Suppose that there exists a regular point $u \in X$ for f such that some subsequence of the sequence $\{f^n u\}_{n\in N}$ converges to a regular point $w \in X$ for f for which the inequality* (1.7) *holds for all $x, y \in O_f(u,w)$. Then w is a fixed point of f and*

$$d(f^i u, w) \leq r^{[\frac{i}{p}]}\delta(O_f(u)) \tag{2.5}$$

for all $i \in N$.

PROOF. As in the proof of Theorem 2.1 we conclude that

$$\delta(O_f(f^{i+p}u)) \leq r\delta(O_f(f^i u)) \tag{2.6}$$

for all $i \in \omega$;

$$d(f^i u, f^{i+m}u) \leq r^{[\frac{i}{p}]}\delta(O_f(u)) \tag{2.7}$$

for all $m \in N$, and $w = \lim_{i\to\infty} f^i u$. Letting m tend infinity in (2.7), we immediately obtain that (2.5) holds. For each $\epsilon > 0$ there exists an integer $k > 2p$ such that $i > k - p$ implies $d(f^i u, w) < \epsilon$. For any m, $i \in N$ with $i > k$, (1.7) ensures that

$$\begin{aligned} d(w, f^m w) &\le d(w, f^i u) + d(f^m w, f^i u) \\ &< \epsilon + r\delta(O_f(f^{m-1}w, f^{i-p}u)) \\ &\le \epsilon + r\max\{2\epsilon, \delta(O_f(w)) + \epsilon\}. \end{aligned}$$

This implies that

$$\delta(w, O_f(w)) \le \epsilon + r\max\{2\epsilon, \delta(O_f(w)) + \epsilon\}.$$

Letting ϵ tend to zero, we have

$$\delta(w, O_f(w)) \le r\delta(O_f(w)). \tag{2.8}$$

We now distinguish two cases.

Case 1. $p = 1$. By (2.6) and (2.8) we have

$$\begin{aligned} \delta(O_f(w)) &= \max\{\delta(w, O_f(fw)), \delta(O_f(fw))\} \\ &\le \max\{\delta(w, O_f(fw)), r\delta(O_f(w))\} \\ &\le r\delta(O_f(w)), \end{aligned}$$

which implies that $\delta(O_f(w)) = 0$. That is, $w = fw$.

Case 2. $p = 2$. By (2.6), (1.7) and (2.8) we have

$$\begin{aligned} \delta(O_f(w)) &= \max\{\delta(w, O_f(fw)), \delta(fw, O_f(f^2w)), \delta(O_f(f^2w))\} \\ &\le \max\{r\delta(O_f(w)), \sup_{t\in\omega} r\delta(O_f(w, f^t w)), r\delta(O_f(w))\} \\ &= r\delta(O_f(w)), \end{aligned}$$

which implies that $\delta(O_f(w)) = 0$. That is, $w = fw$. This completes the proof.

From Theorem 2.3 we have the following theorem.

THEOREM 2.4. *Let f be a mapping from a bounded complete metric space (X, d) into itself satisfying* (1.7). *Then f has a unique fixed point $w \in X$ and*

$$d(f^i x, w) \leq r^{[\frac{i}{p}]}\delta(O_f(x))$$

for all $x \in X$ and $i \in N$.

REMARK 2.3. Theorems 2.3 and 2.4 extend Theorem 2 of Hegedüs [1] and the main result of Tasković [5].

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