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# GRAPHS AND NON-NORMAL OPERATOR (I)

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ABSTRACT. In this paper, we investigate the properties of non-normal (convexoid, hyponormal) adjacency operators for a graph under two operations, tensor product and Cartesian one.

## 1. Introduction

A directed graph $G = (V, E, \partial^+, \partial^-)$ is a system of sets $V, E$ and maps $\partial^\pm : E \to V$. An element $v \in V$ (resp. $e \in E$) is called a vertex (resp. arc). If $\partial^+(e) = u$ and $\partial^-(e) = v$ for some $e \in E$, then $u$ is a server of $v$ (or an initial vertex) and $v$ is a receiver of $u$ (or a terminal vertex). For each vertices $u, v$ in $V$, $d^+(u, v)$ (resp. $d^-(u, v)$) is the number of all common servers (resp. common receivers) of $u$ and $v$. The outdegree $d^+(v)$, the indegree $d^-(v)$, and the valency (or degree) $d(v)$ are defined by $d^+(v) = \#\{e \in E : \partial^+(e) = v\}$, $d^-(v) = \#\{e \in E : \partial^-(e) = v\}$ and $d(v) = d^+(v) + d^-(v)$, respectively.

A graph is called locally finite if every vertex has finite valency. A graph has bounded valency if there is a constant $M > 0$ such that $d(v) \leq M$ for any vertex $v \in V$. Now an adjacency matrix has been consider for finite graphs. In[8] Morhar defined an adjacency operator for infinite graph. In general, an adjacency operator is closed, but we restrict our attention to bounded adjacency operators.

Following [2], we recall adjacency operators of graphs. Let $H$ be the Hilbert space $\ell^2(V)$ with the canonical basis $\{e_v : e \in V\}$ defined by $e_v(u) = \delta_{v,u}$. Then a closed operator $A(G)$ is defined by

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$$Dom(A) = \{x = \sum_{v \in V} x_v e_v \in H : \sum_{u \in V} | \sum_{v \in D^-(u)} x_v|^2 < \infty\} \quad \text{and}$$

$$Ax = \sum_{u \in V} \sum_{v \in D^-(u)} x_v e_u,$$

where $D^-(u)$ is the set of all servers of $u$. We call $A(G)$ the adjacency operator of $G$, and $A(G)$ can be expressed as the operator on $\ell^2(V)$ whose matrix representation $(a_{ij})$ with respect to the basis $\{v_i\}$, where $v_i \in V$ and

$$a_{ij} = \begin{cases} 1, & \text{if there exists the arrows from } v_i \text{ to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

It is known by [2 . Th. 2] that $A(G)$ is bounded if and only if $G$ has bounded valency, i.e. the valency $d(v)$ is bounded , $\sup\{d(v) : v \in V\} < \infty\}$. In this case, the adjoint $A(G)^*$ of $A(G)$ is given by $A(G)^* x = \sum_{u \in V} \sum_{v \in D^+(u)} x_v e_u$ for $x = \sum_{v \in V} x_v e_v \in H$, where $D^+(u)$ is the set of all receivers of $u$. For a directed graph $G$, it makes sense to consider the directed graph $G^*$ whose arrows are all oppositely directed for $G$. In [5], the adjacency operator $A(G^*)$ of $G^*$ is the adjacency operator $A(G)^*$. Then $G^*$ is called the adjoint graph of $G$. that is, one of edges are exactly the converses for those of $G$. Note that $G^*$ does not denote what is called the line graph of $G$. In [3], the adjacency operator $A(G)$ on $\ell^2(V)$ can be defined by the dyadic representation: $A(G)) = \sum_{(w,v) \in E} e_v \otimes e_w$, where a dyad $x \otimes y$ means the rank 1 operator defined by $(x \otimes y)z = (z, y)x$.

Note that above summation converges in the strong operator topology. Next we see the tensor product of graphs. Let $G = (V, E)$ and $H = (W, F)$ be directed graph. Then the adjacency operator $A(G)$ and $A(H)$ act on Hilbert space $\ell^2(V)$ and $\ell^2(W)$ respectively. So the tensor product $A(G) \otimes A(H)$ acting on $\ell^2(V) \otimes \ell^2(W)$ is defined by $((A(G) \otimes A(H))x \otimes y, x \otimes y) = (A(G)x \otimes A(H)y, x \otimes y) = (A(G)x, x)(A(H)y, y)$ for $x \in \ell^2(V)$ and $y \in \ell^2(W)$. Then the tensor product $G \otimes H$ of $G$ and $H$ is defined as a graph $G \otimes H$ with vertices $V \times W$ such that $((v,w),(x,y)) \in E(G \otimes H)$ if and only if $((A(G) \otimes A(H))e_v \otimes e_w, e_v \otimes e_w) = 1$. Thus the tensor product $G \otimes H$ of $G$ and $H$ is a graph whose adjacency operator is $A(G) \otimes A(H)$. In the finite case, the tensor product of graphs is obtained by making use of the Kronecker product of the adjacency matrices.

In [4], the adjacency operator of the Cartesian product $G \oplus H$ for simple graphs $G$ and $H$ is given by $A(G \oplus H) = A(G) \otimes I + I \otimes A(H)$. For a directed graph $G$, the spectrum $\sigma(G)$, the approximate point spectrum $\sigma_{ap}(G)$, the normal approximate spectrum $\prod_n(G)$, the numerical range $W(G)$, the spectral radius $\gamma(G)$, the numerical radius $\omega(G)$ and the norm $\|G\|$ of a graph $G$ are defined as $\sigma(A(G))$, $\sigma_{ap}(A(G))$, $\prod_n(A(G))$, $\gamma(A(G))$, $\omega(A(G))$, and $\|A(G)\|$, respectively

Throughout this paper, a graph stands for a locally finite directed graph without multiple arces

In section 2, we shall show that $\prod_n(G) \cdot \prod_n(H) \subset \prod_n(G \otimes H)$, $\|G \oplus H\| \leq \|G\| + \|H\|$, and $\overline{Co}(W(T) \ \ W(H)) \subset \overline{W(G \otimes H)}$, where $\overline{Co}X$ denotes the closed convex hull of $X$, moreover we shall give examples of proper inclusions in the above relations.

In section 3, we shall investigate the properties of non-normal(convexoid and hyponormal ) adjacency operators of simple graphs $G$ and $H$ under two operations, tensor product and Cartesian product.

## 2. Spectras and numerical ranges of graphs

B.Mohar and M.Omladič[9] showed that

$$\sigma(G \otimes H) = \sigma(G) \ \ \sigma(H) \tag{1}$$

for any locally finite graphs $G$ and $H$. J.I. Fujii [1] showed that $\gamma(G \otimes H) = \gamma(G) \cdot \gamma(G)$. However the equality (1) does not hold for other spectras. We shall show here.

THEOREM 2.1 *We have*

$$\prod_n(G) \cdot \prod_n(H) \subset \prod_n(G \otimes H) \tag{2}$$

*for graphs $G$ and $H$.*

PROOF. Since

$$A(G) \otimes A(H) - \alpha\beta = (A(G) - \alpha) \otimes \beta + A(G) \otimes (A(H) - \beta).$$

we have

$$\begin{aligned}
&\|(A(G) \otimes A(H) - \alpha\beta)x_n \otimes y_n\| \\
&= \|\{(A(G) - \alpha) \otimes \beta + A(G) \otimes (A(H) - \beta)\}x_n \otimes y_n\| \\
&\leq \|(A(G) - \alpha)x_n \otimes \beta y_n\| + \|A(G)x_n \otimes (A(H) - \beta)y_n\| \\
&= \|(A(G) - \alpha)x_n\|\|\beta y_n\| + \|A(G)x_n\|\|(A(H) - \beta)y_n\| \\
&\leq \|(A(G) - \alpha)x_n\|\|\beta\| + \|A(G)\|\|(A(H) - \beta)y_n\| \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\|(A(G) \otimes A(H) - \alpha\beta)^* x_n \otimes y_n\| \\
&\leq \|(A(G) - \alpha)^* x_n\|\|\beta y_n\| + \|A(G)^* x_n\|\|(A(H) - \beta)^* y_n\| \\
&= \|(A(G) - \alpha)^* x_n\||\beta| + \|A(G)^*\|\|(A(H) - \beta)^* y_n\| \to 0
\end{aligned}$$

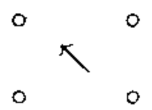
as $n \to \infty$, whenever $\alpha \in \prod_n(G)$ (resp. $\beta \in \prod_n(H)$) and $\{x_n\}$ (resp. $\{y_n\}$ ) is a sequence of unit vectors satisfying $\|(A(G) - \alpha)x_n\| \to 0$ (resp. $\|(A(G) - \beta)y_n\| \to 0$) and $\|(A(G) - \alpha)^* x_n\| \to 0$ (resp. $\|(A(G) - \beta)^* y_n\| \to 0$). Hence $\alpha\beta \in \prod_n(G \otimes H)$.

We shall show that there are adjacency operators $A(G)$ and $A(H)$ for which the equality in (2) does not hold as follows :

EXAMPLE 2.2. Let $G$ be the graph

$$G : \circ \leftarrow \circ\ , \quad A(G) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and let $G \otimes G$ be

$G \otimes G :$ 

$$A(G) \otimes A(G) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We have observed that $A(G)$ have empty normal approximate spectrum $\prod_n(G)$ of a graph $G$ by a theorem due to Kasahara and Takai[7]. But $A(G) \otimes A(G)$ has a normal approximate proper value , that is. $0 \in \prod_n(G \otimes G)$.

By using same manner as a proof in Theorem 2.1, we can easily show the following.

COROLLARY 2.3. *We have* $\sigma_{ap}(G) \cdot \sigma_{ap}(H) \subset \sigma_{ap}(G \otimes H)$.

THEOREM 2.4. *Let* $A(G)$ *and* $A(H)$ *be any adjacency operators acting on Hilbert space* $\ell^2(V)$ *and* $\ell^2(W)$, *respectively. Then*

(i) $Co(W(G) \cdot W(H)) \subset W(G \otimes H)$ *and* $\omega(G) \cdot \omega(H) \leq \omega(G \otimes H)$,
(ii) $Co(W(G) \cup W(H)) = W(G \oplus H)$ *and* $\omega(G) + \omega(H) = \omega(G \oplus H)$, *and*
(iii) $\|G \otimes H\| = \|G\|\|H\|$ *and* $\|G \oplus H\| \leq \|G\| + \|H\|$

PROOF. (i) Let $\lambda \in W(G)$ and $\mu \in W(H)$. Then there exist unit vectors $x \in \ell^2(V)$ and $y \in \ell^2(W)$ such that $\lambda = (A(G)x, x)$ and $\mu = (A(H)y, y)$ Thus we have

$$\lambda\mu = (A(G)x, x)(A(H)y, y) = ((A(G) \otimes A(H))x \otimes y, x \otimes y) \in W(G \otimes H)$$

Since $W(G \otimes H)$ is convex, $Co(W(G) \cdot W(H)) \subset W(G \otimes H)$. It follows that $\omega(G) \cdot \omega(H) \leq \omega(G \otimes H)$.

(ii) It is clear from (i) that there is a scalar $t$ in $[0, 1]$ such that $t\lambda + (1-t)\mu \in Co(W(G) \cup W(H))$. Hence we have

$$\begin{aligned} t\lambda + (1-t)\mu &= t(A(G)x, x) + (1-t)(A(H)y, y) \\ &= (A(G)\sqrt{t}x, \sqrt{t}x) + (A(H)\sqrt{1-t}y, \sqrt{1-t}y) \\ &= ((A(G) \oplus A(H))(\sqrt{t}x, \sqrt{1-t}y), (\sqrt{t}x, \sqrt{1-t}y)) \end{aligned}$$

Since $\|x\| = \|y\| = 1$,

$$\|(\sqrt{t}x, \sqrt{1-t}y)\|^2 = \|\sqrt{t}x\|^2 + \|\sqrt{1-t}y\|^2 = 1$$

Hence we conclude that $t\lambda + (1-t)\mu \in W(G \oplus H)$

Conversely, let $z \in W(G \oplus H)$. Then we have

$$\begin{aligned} z &= (A(G \oplus H)(x,y),(x,y)) \\ &= ((A(G) \otimes I + I \otimes A(H))x \otimes y, x \otimes y) \\ &= (A(G)x,x)(y,y) + (x,x)(A(H)y,y) \\ &= (A(G)x,x) + (A(H)y,y) \in Co(W(G) \cup W(H)). \end{aligned}$$

Therefore we have $Co(W(G) \cup W(H)) = W(G \oplus H)$, and hence it follows that $\omega(G) + \omega(H) = \omega(G \oplus H)$. In (iii) the equality is well-known and the inequality is clear.

In Theorem 2.4 $\overline{W(G \otimes H)}$ (resp. $\|G \oplus H\|$) is not always equal to $\overline{Co}(W(G) \cdot W(H))$ (resp. $\|G\| + \|H\|$). Here we give an elementary example :

EXAMPLE 2.5. Let $G$ and $H$ be the following graphs :

$$G : \circ \leftarrow \circ\ ,\ A(G) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$H : \circ \rightarrow \circ\ ,\ A(H) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we have

$$G \otimes H : \begin{array}{ccc} \circ & & \circ \\ & \nwarrow & \\ \circ & & \circ \end{array}\ ,\ A(G) \otimes A(H) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G \oplus H : \begin{array}{ccc} \circ & \rightarrow & \circ \\ \uparrow & & \uparrow \\ \circ & \rightarrow & \circ \end{array}\ ,\ A(G) \oplus A(H) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

It is easy to show that $\overline{W(G)} = \overline{W(H)} = \{\lambda : |\lambda| \leq \frac{1}{2}\}$, from which we have $\overline{Co}(W(G) \cdot W(H)) \subset \{\lambda : |\lambda| \leq \frac{1}{4}\}$. Let $x_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $x_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$ and $x = (x_1, x_2)$. Then it follows that $((A(G) \otimes A(H))x, x) = \frac{1}{2}$ Thus $\frac{1}{2}$ is not in the set $\{\lambda : \||\lambda| \leq \frac{1}{4}\}$. Moreover, it is clear that $\omega(G) \cdot \omega(H) = \frac{1}{4} < \frac{1}{2} \leq \omega(G \otimes H)$. Also we have $\|G \oplus H\| = \sqrt{2} < 2 = |G\| + \|H\|$.

## 3. Non-normal adjacency operators

We list definitions on adjacency operators for graphs discussed in the below

(1) $A(G)$ is normal if $A(G)^*A(G) = A(G)A(G)^*$.
(2) $A(G)$ is hyponormal if $A(G)^*A(G) \geq A(G)A(G)^*$. and
(3) $A(G)$ is convexoid if $\overline{W(G)} = Co\sigma(G)$.

Also we state some definitions for the graph $G$. A source of a directed graph $G$ is a vertex $v$ whose $d^-(v) = 0$. A source $v$ is called non-trivial if $d^+(v) \neq 0$. A sink of $G$ is a vertex $v$ whose $d^+(v) = 0$. A sink $v$ is called non-trivial if $d^-(v) \neq 0$. We recall that a graph $G$ is normally symmetric if $d^-(u, v) = d^+(u, v)$ for any $u, v \in V$. In general, it follows from Theorem 2.1 and Theorem 2.4(i) that

$$Co(\prod_n(G) \cdot \prod_n(H)) \subset \overline{W(G \otimes H)}. \tag{3}$$

By the equality in (3) and the following lemma. we shall investigate the properties of non-normal (Convexoid and hyponormal) adjacency operators under two operators, tensor product and Cartesian one.

LEMMA 3.1. *The adjacency operator $A(G)$ of a graph $G$ is convexoid if and only if the closed numerical range $\overline{W(G)}$ is spanned by the normal approximate spectrum $\prod_n(G)$ of $A(G)$ in the sense that $\overline{W(G)} = Co\prod_n(G)$.*

PROOF. Suppose that $A(G)$ is convexoid. Then $\overline{W(G)} = Co\sigma(G)$, and so an extrem point of $\overline{W(G)}$ belongs to $\sigma(G)$ and so $\sigma_{ap}(G)$. Thus we have $\overline{W(G)} = Co\ \ ext(\overline{W(G)}) = Co(\sigma(G) \cap \partial\overline{W(G)})$, where $extX$ denotes the set of all extreme points of $X$ and $\partial X$ denotes the boundary of $X$. On the other hand, we have $\sigma(G) \cap \partial\overline{W(G)} \subset \prod_n(G)$ by Hildebrant's theorem [6]. Then it follows $\overline{W(G)} \subset Co(\sigma(G) \cap \partial\overline{W(G)}) \subset Co\prod_n(G) \subset \overline{W(G)}$. Thus $\overline{W(G)} = Co\prod_n(G)$.

Conversely, if $A(G)$ satisfies the equality $\overline{W(G)} = Co\prod_n(G)$, then $\overline{W(G)} = Co\prod_n(G) \subset Co\sigma(G) \subset \overline{W(G)}$, so that the equality $\overline{W(G)} = Co\sigma(G)$ holds. Therefore $A(G)$ is convexoid.

THEOREM 3.2. *Let $A(G)$ and $A(H)$ be convexoid adjacency operators for graphs $G$ and $H$, respectively. Then the equality $\overline{W(G\otimes H)} = Co(\prod_n(G)\cdot\prod_n(H))$ holds if and only if $A(G)\otimes A(H)$ is convexoid.*

PROOF. Since $A(G)$ and $A(H)$ are convexoid operators, it follows from Lemma 3.1 that $\overline{W(G)} = Co\prod_n(G)$ and $\overline{W(G)} = Co\prod_n(H)$. Assume that the equality in (3) holds, then we have

$$\begin{aligned}\overline{W(G\otimes H)} &= Co(\prod_n(G)\cdot\prod_n(H)) = Co(Co\prod_n(G)\cdot Co\prod_n(H)) \\ &= Co(\overline{W(G)}\cdot\overline{W(H)}) = Co(Co\sigma(G)\cdot Co\sigma(H)) = Co(\sigma(G\otimes H)) \\ &= Co(\prod_n(G\otimes H)).\end{aligned}$$

Thus $A(G)\otimes A(H)$ is a convexoid.

Conversely, if $A(G)\otimes A(H)$ is a convexoid adjacency operator, then it follows from Lemma 3.1 that $\overline{W(G\otimes H)} = Co(\prod_n(G\otimes H))$. Hence we have that

$$\begin{aligned}Co\prod_n(G\otimes H) &\subseteq Co\sigma(G\otimes H) = Co(\sigma(G)\cdot\sigma(H)) \\ &= Co(\overline{W(G)}\cdot\overline{W(H)}) \subset \overline{W(G\otimes H)}\end{aligned}$$

by (i) in Theorem 2.4, and

$$Co(\overline{W(G)}\cdot\overline{W(H)}) = Co(Co\prod_n(G)\cdot Co\prod_n(H)) = Co(\prod_n(G)\cdot\prod_n(H)).$$

Therefore the equality $\overline{W(G \otimes H)} = Co(\prod_n \prod_n(H))$ holds

In connection with Theorem 3.2. we can easily see the following :

COROLLARY 3.3 *Let $A(G)$ and $A(H)$ be convexoid adjacency operators for graphs $G$ and $H$, respectively. Then the equality $\overline{W(G \otimes H)} = \overline{Co}(W(G) \cdot W(H))$ is true if and only if $A(G) \otimes A(H)$ is convexoid.*

The tensor product of two adjacency operators is not necessarily convexoid even if two adjacency operators are convexoid as follows.

EXAMPLE 3.4. Let $A(G)$ be the adjacency operator acting on $\ell^2(V)$ defined by

$$A(G) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } G . \circ \to \circ.$$

Let $A(H)$ be the adjacency operator acting on $\ell^2(W)$ such that

$$\overline{W(H)} = Co\prod_n(H) = \{\lambda \cdot |\lambda| \leq \frac{1}{2}\}$$

Setting $A(J) = A(G) \oplus A(H)$ and $A(K) = A(J)^*$. we have

$$\overline{W(J)} = \overline{W(G \oplus H)} = \overline{Co}(W(G) \cup W(H)) = \{\lambda . |\lambda| \leq \frac{1}{2}\}$$

and

$$\overline{W(K)} = \overline{W(G^* \oplus H^*)} = \overline{Co}(W(G^*) \cup W(H^*)) = \{\lambda : |\lambda| \leq \frac{1}{2}\}.$$

Since it is clear that

$$Co\prod_n(J) = Co\prod_n(G \oplus H) = Co(\prod_n(G) \cup \prod_n(H))$$
$$= \{\lambda : |\lambda| \leq \frac{1}{2}\} = \overline{W(J)}$$

and $\overline{W(K)} = Co\prod_n(K)$. it follows from Lemma 3.1 that $A(J)$ and $A(K)$ are convexoid operators. But we have

$$Co\prod_n(J \otimes K) \subsetneq \overline{W(J \otimes K)}$$

THEOREM 3.5. *Let $A(G)$ and $A(H)$ be convexoid adjacency operators for graphs $G$ and $H$, respectively. Then $A(G) \oplus A(H)$ is convexoid.*

PROOF. If $A(G)$ and $A(H)$ are convexoid, then it follows from Lemma 3.1 that $\overline{W(G)} = Co\prod_n(G)$ and $\overline{W(H)} = Co\prod_n(H)$. Thus we have

$$\begin{aligned}\overline{W(G \oplus H)} &= Co\overline{W(G)} \cup \overline{W(H)}) \\ &= Co(Co\prod_n(G) \cup Co\prod_n(H)) = Co\prod_n(G \oplus H),\end{aligned}$$

and hence $A(G) \oplus A(H)$ is convexoid.

Both adjacency operators $A(G)$ and $A(H)$ are not always convexoid even if the Cartesian product of $A(G)$ and $A(H)$ is convexoid. Now we shall give an example as follows:

EXAMPLE 3.6. Let $G$ and $H$ be the following graphs:

$$G : \circ \leftarrow \circ\ , \ A(G) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$H : \begin{array}{ccc} & \circ & \\ \nearrow & & \searrow \\ \circ & \leftarrow & \circ \end{array}, \ A(H) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$G \oplus H : \begin{array}{ccccc} \circ & \rightarrow & \circ & \rightarrow & \circ \\ \uparrow & & & & \downarrow \\ \circ & \leftarrow & \circ & \leftarrow & \circ, \end{array} \ A(G \oplus H) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $A(H)$ is a normal adjacency operator and so $A(H)$ is convexoid. Thus we easily see that $\overline{W(H)}$ is the interior and the boundary of the equilateral triangle whose vertices are $\sigma(H) = \{1, \omega, \omega^2\}$ where $\omega = \frac{-1+\sqrt{3}i}{2}$.

It follows that $\overline{W(H)} = Co\prod_n(H)$ and $\overline{W(G)} = \{\lambda . |\lambda| \leq \frac{1}{2}\} \subset Co\prod_n(H)$. Thus we have

$$\begin{aligned}\overline{W(G \oplus H)} &= \overline{Co}(W(G) \cup W(H)) = Co\prod_n(H) \\ &= Co(\prod_n(G) \cup \prod_n(H)) = Co\prod_n(G \oplus H).\end{aligned}$$

Therefore $A(G) \oplus A((H)$ is a convexoid operator but $A(G)$ is not convexoid.

Fujii, Sasaka and Watatani [2] showed the following :

**Lemma 3.7.** *Let $A(G)$ be a hyponormal adjacency operator for a graph $G$. Then there does not exist a non-trivial sink of $G$.*

**Theorem 3.8** *Let $A(G)$ and $A(H)$ be hyponormal adjacency operators for a graphs $G$ and $H$, respectively. Then*

(i) *$A(G) \otimes A(H)$ is hyponormal,*
(ii) *There does not exits a non-trivial sink of $G \otimes H$, and*
(iii) *The equality in (3) holds.*

**Proof.** (i) If $A(G)$ and $A(H)$ are hyponormal, then we have

$$\begin{aligned}&(((A(G) \otimes A(H))^*(A(G) \otimes A(H)) - (A(G) \otimes A(H))(A(G) \otimes A(H))^*) \\ &e_v \otimes e_w, e_v \otimes e_w) = ((A(G)^*A(G) \otimes A(H)^*A(H))e_v \otimes e_w, e_v \otimes e_w) \\ &\qquad - ((A(G)A(G)^* \otimes A(H)A(H)^*)e_v \otimes e_w, e_v \otimes e_w) \\ &\qquad = (A(G)^*A(G)e_v, e_v)(A(H)^*A(H)e_w, e_w) \\ &\qquad - (A(G)A(G)^*e_v, e_v)(A(H)A(H)^*e_w, e_w) \\ &\qquad = \|A(G)e_v\|^2\|A(H)e_w\|^2 - \|A(G)^*e_v\|^2\|A(H)^*e_w\|^2 \\ &\qquad \geq 0\end{aligned}$$

Thus $A(G) \otimes A(H)$ is hyponormal.

(ii) It follows from Lemma 3.7 that there are not non-trivial sinks $v$ and $w$ of $G$ and $H$, respectively.

Assume that there exists a non-trivial sink $v \otimes w$ of $G \otimes H$. Then we have

$$\begin{aligned}
&(((A(G) \otimes A(H))^*(A(G) \otimes A(H)) - (A(G) \otimes A(H))(A(G) \otimes A(H))^*)) \\
&e_v \otimes e_w, e_v \otimes e_w) = \|A(G)e_v\|^2\|A(H)e_w\|^2 - \|A(G)^*e_v\|^2\|A(H)^*e_w\|^2 \\
&\qquad = d^+(v)^2d^+(w)^2 - d^-(v)^2d^-(w)^2 \\
&\qquad = -d^-(v)^2d^-(w)^2 \leq 0,
\end{aligned}$$

by the definition of non-trivial sink. Thus $A(G) \otimes A(H)$ is not hyponormal.

(iii) It follows from (i) that $A(G) \otimes A(H)$ is convexoid. Thus we have $\overline{W(G \otimes H)} = Co\prod_n(G \otimes H) = Co(\prod_n(G) \cdot \prod_n(H))$.

COROLLARY 3.9. *Let $A(G)$ and $A(H)$ be hyponormal adjacency operators for graphs $G$ and $H$, respectively. Then*

(i) *$A(G) \oplus A(H)$ is hyponormal,*

(ii) *There does not exist a non-trivial sink of $G \oplus H$, and*

(iii) *The equality $\overline{W(G \oplus H)} = Co\prod_n(G \oplus H)$ holds*

PROOF (i) If $A(G)$ and $A(H)$ are hyponormal, then we have

$$\begin{aligned}
&(((A(G) \oplus A(H))^*(A(G) \oplus A(H)) - (A(G) \oplus A(H))(A(G) \oplus A(H))^*)) \\
&e_v \oplus e_w, e_v \oplus e_w) = (A(G)^*A(G)e_v, e_v) + (A(H)^*A(H)e_w, e_w) \\
&\qquad - (A(G)A(G)^*e_v, e_v) - (A(H)A(H)^*e_w, e_w) \\
&\qquad = \|A(G)e_v\|^2 + \|A(H)e_w\|^2 - \|A(G)^*e_v\|^2 - \|A(H)^*e_w\|^2 \\
&\qquad \geq 0.
\end{aligned}$$

Thus $A(G) \oplus A(H)$ is hyponormal.

(ii) Assume that exists a non-trivial sink $v \oplus w$ of $G \oplus H$. Then we have

$$\begin{aligned}
&(((A(G) \oplus A(H))^*(A(G) \oplus A(H)) - (A(G) \oplus A(H))(A(G) \oplus A(H))^*)) \\
&e_v \oplus e_w, e_v \oplus e_w) = \|A(G)e_v\|^2 + \|A(H)e_w\|^2 - (\|A(G)^*e_v\|^2 + \|A(H)^*e_w\| \\
&\qquad = d^+(v)^2 + d^+(w)^2 - (d^-(v)^2 + d^-(w)^2) \\
&\qquad = -(d^-(v)^2 + d^-(w)^2) < 0.
\end{aligned}$$

Thus $A(G) \oplus A(H)$ is not hyponormal.

(iii) Since $A(G) \oplus A(H)$ is hyponormal, $A(G) \oplus A(H)$ is convexoid. It follows from Lemma 3.1 that the equality $\overline{W(G \oplus H)} = Co \prod_n (G \oplus H)$ holds.

From Theorem 3.8 we can easily see the following:

COROLLARY 3.10 *Let $A(G)$ and $A(G)^*$ be hyponormal adjacency operators. Then $A(G)$ is normally symmetric and so $A(G)$ is normal adjacency operator.*

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