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SPACE-LIKE COMPLEX SUBMANIFOLDS OF AN INDEFINITE COMPLEX SPACE FORM

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ABSTRACT. The purpose of this paper is to study the Chern-type problem of the complete space-like submanifolds of an indefinite complex space form.

1. Introduction

The theory of indefinite complex submanifolds of an indefinite complex space form is one of the most interesting topics in differential geometry and it is investigated by many geometers from the various different points of view. See [2], [3], [7] and [8] for examples

Let M be an n-dimensional space-like complex submanifold of an $(n+p)$-dimensional indefinite Kähler manifold M' of index $2p$ We denote by $H'(P', Q')$ the holomorphic bisectional curvature of M' for any holomorphic planes P' and Q'. In particular, the holomorphic bisectional curvature $H'(P', Q')$ for any two space-like holomorphic planes P' and Q' is said to be *space-like* and that for any space-like holomorphic plane P' and any time-like holomorphic plane Q' is said *time-like*. We call it simply a *space-like* or *time-like holomorphic bisectional curvature*. Then the authors in [5] proved the following

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THEOREM A. *Let M be an $n(\geqq 2)$-dimensional complete space-like complex hypersurface of an $(n+1)$-dimensional indefinite Kähler manifold M' of index 2. If the ambient space is locally symmetric and if it has non-negative space-like holomorphic bisectional curvatures and non-positive time-like holomorphic bisectional curvatures, then M is totally geodesic.*

The Chern-type problem in the space-like Kähler geometry is as follows;

PROBLEM. *Let M be an n-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphic sectional curvature c of index $2p(>0)$. Then does there exist a constant h in such a way that if it satisfies $h_2 > h$, then M is totally geodesic?, where h_2 is the squared norm of the second fundamental form α of M.*

In [4], the authors recently treated this problem from the different point of view, and they obtained partial solutions under the additional conditions. The authors in [7] generalized also recently Theorem A in the case where M is a space-like complex submanifold. This is, they proved

THEOREM B. *Let M be an n-dimensional complete space-like complex submanifold of an $(n+2)$-dimensional indefinite locally symmetric Kähler manifold M' of index 4. Assume that the normal connection of M is proper. If M' has non-negative space-like holomorphic bisectional curvatures and non-positive time-like holomorphic bisectional curvatures, then M is totally geodesic.*

In this paper, we investigate the case where M is a space-like complex submanifold of an indefinite locally symmetric Kähler manifold. In particular, we research the Chern-type problem of complete space-like complex submanifolds of an indefinite complex space form.

2. Space-like complex submanifolds

This section is concerned with space-like complex submanifolds of an indefinite Kähler manifold First of all, the basic formulas for the theory of space-like complex submanifolds are prepared. Let M' be an $(n+p)$-dimensional connected indefinite Kähler manifold of index $2p$ with the indefinite Kähler structure (g', J'). Let M be an n-dimensional connected space-like complex submanifold of M' and let g be the induced Kähler metric tensor on M from g'. We can choose a local field $\{U_A\} = \{U_1, \ldots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that restricted to M, $U_1, \ldots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated :

$$A, B, \cdots = 1, \ldots, n, n+1, \ldots, n+p;$$
$$i, j, \cdots = 1, \ldots, n, \qquad x, y, \cdots = n+1, \ldots, n+p$$

With respect to the frame field $\{U_A\}$, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame field. Then the indefinite Kähler metric tensor g' of M' is given by $g' = 2\sum_A \epsilon_A \omega_A \otimes \bar\omega_A$ where $\{\epsilon_A\} = \{\epsilon_i, \epsilon_x\}$, $\epsilon_i = 1$ and $\epsilon_x = -1$. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$
\begin{aligned}
&d\omega_A + \sum_B \epsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar\omega_{AB} = 0,\\
(2.1) \qquad &d\omega_{AB} + \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB},\\
&\Omega'_{AB} = \sum_{C,D} \epsilon_C \epsilon_D R'_{\bar A B C \bar D} \omega_C \wedge \bar\omega_D,
\end{aligned}
$$

where Ω'_{AB} (resp. $R'_{\bar A B C \bar D}$) denotes the curvature form with respect to the frame field $\{U_A\}$ (resp. the components of the indefinite Riemannian curvature tensor R') of M'. Restricting these forms to the submanifold M, we have

$$(2.2) \qquad \omega_x = 0,$$

and the induced Kähler metric tensor g of M is given by $g = 2\sum_j \epsilon_j \omega_j \otimes \bar\omega_j$. Then $\{U_j\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_j\}$ is a local dual frame field due to $\{U_j\}$, which consists of complex valued 1-forms of type (1.0) on M. Moreover, $\omega_1, \ldots, \omega_n, \bar{\omega}_1, \ldots, \bar{\omega}_n$ are linearly independent, and ω_j are the canonical forms on M. It follows from (2.2) and Cartan's lemma that the exterior derivatives of (2.2) give rise to

$$\omega_{xi} = \sum_j \epsilon_j h^x_{ij} \omega_j, \quad h^x_{ij} = h^x_{ji}. \tag{2.3}$$

The quadratic form $\alpha = \sum_{x,i,j} \epsilon_x \epsilon_i \epsilon_j h^x_{ij} \omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle NM on M in M' is called the *second fundamental form* of the submanifold M. From the structure equations for M', the structure equations for M are similarly given by

$$\begin{aligned} &d\omega_i + \sum_j \epsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \\ &d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \quad \Omega_{ij} = \sum_{k,m} \epsilon_k \epsilon_m R_{\bar{i}jk\bar{m}} \omega_k \wedge \bar{\omega}_m. \end{aligned} \tag{2.4}$$

Moreover the following relationships are obtained ;

$$d\omega_{xy} + \sum_z \epsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \quad \Omega_{xy} = \sum_{k,m} \epsilon_k \epsilon_m R_{\bar{x}yk\bar{m}} \omega_k \wedge \bar{\omega}_m, \tag{2.5}$$

where Ω_{xy} is called the *normal curvature form* of M. For the Riemannian curvature tensors R and R' of M and M', respectively. it follows from (2.1), (2.3) and (2.4) that we have the Gauss equation

$$R_{\bar{i}jk\bar{m}} = R'_{\bar{i}jk\bar{m}} - \sum_x \epsilon_x h^x_{jk} \bar{h}^x_{im}.$$

And by means of (2.1), (2.3) and (2.5), we have

$$R_{\bar{x}yk\bar{m}} = R'_{\bar{x}yk\bar{m}} + \sum_j \epsilon_j h^x_{kj} \bar{h}^y_{jm}.$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{i\bar{j}} = \sum_k \epsilon_k R'_{k\bar{k}i\bar{j}} - h_{i\bar{j}}{}^2,$$
$$r = 2(\sum_{j,k} \epsilon_j \epsilon_k R'_{j\bar{j}k\bar{k}} - h_2),$$

where $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{x,k} \epsilon_x \epsilon_k h^x_{ik} \bar{h}^x_{kj}$ and $h_2 = \sum_j \epsilon_j h_{j\bar{j}}{}^2$.

In particular, let the ambient space M' be an $(n+p)$-dimensional indefinite complex space form $M_p^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2p$. Then, we get

$$R_{\bar{i}jk\bar{m}} = \frac{c}{2}\epsilon_j \epsilon_k (\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm}) - \sum_x \epsilon_x h^x_{jk} \bar{h}^x_{im},$$

(2.6) $$S_{i\bar{j}} = \frac{c}{2}(n+1)\epsilon_i \delta_{ij} - h_{i\bar{j}}{}^2, \quad r = cn(n+1) - h_2.$$

Next, we calculate the Laplacian of the squared norm $h_2 = |\alpha|_2$ of the second fundamental form α on M. The matrix $A = (A_y{}^x)$ of order p defined by $A_y{}^x = \sum_{i,j} h^x_{ij} \bar{h}^y_{ij}$ is a Hermitian one. Since M is space-like and the normal space is time-like, it is a positive semi-definite Hermitian matrix of order p. Hence its eigenvalues λ_x are all non-negative real valued functions on M and it can easily be proved that

(2.7) $$\sum_x \lambda_x = TrA = -h_2 , \quad h_2{}^2 \geqq TrA^2 = \sum_x \lambda_x{}^2 \geqq \frac{1}{p} h_2{}^2.$$

Since the Laplacian of the squared norm h_2 of the second fundamental form on M is by definition given as

$$\Delta h_2 = -\sum_{i,j,k} \left\{ \left(\sum_x h^x_{ij} \bar{h}^x_{ij} \right)_{kk} + \left(\sum_x h^x_{ij} \bar{h}^x_{ij} \right)_{\bar{k}k} \right\},$$

we have

(2.8)

$$\begin{aligned}\Delta h_2 =&2|\nabla\alpha|_2 + 2\sum_{x,i,j,k} R'_{\bar{x}ij\bar{k}.k}\bar{h}^x_{ij} + 2\sum_{x,i,j,k} R'_{\bar{i}xk\bar{j}\,\bar{k}}h^x_{ij}\\ &-8\sum_{x,y,i,j,k} R'_{\bar{x}yj\bar{k}}h^y_{ki}\bar{h}^x_{ij} - 2\sum_{x,y,k} A_x{}^y R'_{\bar{x}yk\bar{k}}\\ &-4\sum_{x,i,j,k,l} R'_{\bar{k}ij\bar{l}}h^x_{kl}\bar{h}^x_{ij} + 4\sum_{i,j,k} R'_{\bar{i}jk\bar{k}}h_{ij}{}^2 - 4h_4 - 2TrA^2.\end{aligned}$$

where $h_4 = \sum_{i,j} h_{i\bar{j}}{}^2 h_{j\bar{i}}{}^2$ and the squared norm $|\nabla\alpha|_2$ of the covariant derivative $\nabla\alpha$ of the second fundamental form α on M is defined by $|\nabla\alpha|_2 = -\sum_{x,i,j,k}(h^x_{ijk}\bar{h}^x_{ijk} + h^x_{ij\bar{k}}\bar{h}^x_{ij\bar{k}})$ (for details, see [7]).

Lastly, we introduce here a more extended property than the generalized maximum principle due to Omori [9] and Yau [11].

THEOREM 2.1. *[7] Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below and let F be any polynomial of one variable f with*

$$F(f) = c_0 f^n + c_1 f^{n-1} + \cdots + c_k f^{n-k} + c_{k+1},$$

where $c_0, \cdots, c_{k+1}$ are constants such that $c_0 > c_{k+1}$ and $n > 1$, $n-k > 0$. If a C^2-function f satisfies $\Delta f \geqq F(f)$, then we have $F(sup\ f) \leqq 0$.

3. Some results

In this section, let M' be an $(n+p)$-dimensional indefinite Kähler manifold of index $2p$ with the indefinite Kähler structure (g', J') and let M be an $n(\geqq 3)$-dimensional space-like subcomplex manifold of M'. Assume that M' is locally symmetric. the normal connection of M is proper and it satisfies the following the conditions ;

(∗1) The space-like totally real bisectional curvature is bounded from below by a_1.

(∗2) The time-like holomorphic bisectional curvature is bounded from above by a_2.

Then M' is said to satisfy the *condition* $(*)$ if it satisfies the above conditions $(*1)$ and $(*2)$. For a local field $\{E_A, E_A^*\}$ of orthonormal frames on a neighborhood of the manifold M', we have

$$H'(P'_j, P'_k) = H'(E_j, E_k) = \epsilon_j \epsilon_k R'_{\bar{j}jk\bar{k}} \geqq a_1 \ (j \neq k),$$
$$H'(P'_x, P'_k) = H'(E_x, E_k) = \epsilon_x \epsilon_k R'_{\bar{x}xk\bar{k}} \leqq a_2,$$

where $H'(P'_A, P'_B)$ is the holomorphic bisectional curvature for the holomorphic plane $P'_A = [E_A, J'E_A]$.

REMARK 3.1. Let M' be an $(n+p)$-dimensional indefinite complex space form $M_p^{n+p}(c)$ of index $2p$ and of constant holomorphic sectional curvature c. Then M' is locally symmetric and it satisfies the condition $(*)$ and we may consider $a_1 = a_2 = \frac{c}{2}$ if c is non-negative and $a_1 = c$, $a_2 = \frac{c}{2}$ if c is non-positive

Since the ambient space M' is locally symmetric and the squared norm $|\nabla\alpha|_2$ of the covariant derivative $\nabla\alpha$ of the second fundamental form α is non-positive, the equation (2.8) is estimated as follows (for details, see [6]):

$$\begin{aligned}\Delta h_2 \leqq &- 8(a_2 h_2 - h_4 + \frac{1}{p} {h_2}^2) - 2n a_2 h_2 \\ &+ \frac{2}{n-2}\{2(n-1)(n+4)a_1 - r'_0\}h_2 \\ &+ \frac{2}{n-2}\{4(n-1)(n+1)a_1 - r'_0\}h_2 - 4h_4 - 2TrA^2,\end{aligned}$$

where $r'_0 = \sum_{j,k} R'_{\bar{j}jk\bar{k}}$. Accordingly, since $r'_0 \geqq 2n(n+1)a_1$ and $h_2^2 \geqq h_4$, we obtain by (2.7)

$$(3.1) \qquad \Delta h_2 \leqq A_0 {h_2}^2 + A_1 h_2,$$

where the coefficients A_0 and A_1 are constants given by

$$A_0 = \frac{2}{p}(2p-5), \quad A_1 = 2\{2(n+3)a_1 - (n+4)a_2\}.$$

Now, since the totally real bisectional curvature of M is bounded from below by a constant, by the assumption that the scalar curvature of M is bounded from above, Lemma 3.1 implies that the Ricci curvature of M is bounded from below. Remark that the dimensional condition $n \geqq 3$ is here used. Let f be the non-negative function defined by $-h_2$. Then, by (3.1) we have

$$(3.2) \quad \Delta f \geqq c_0 f^2 + c_1 f + c_2 = F(f), \quad c_0 = -A_0 \ , \ c_1 = A_1 \ , \ c_2 = 0,$$

where F is the polynomial of the variable f with the constant coefficients.

The first assertion of the following theorem is originally proved by Aiyama, Kwon and Nakagawa [1]. Here we prove the the theorem in a different way.

THEOREM 3.1. *Let M' be an $(n+p)$-dimensional indefinite complex space form $M_p^{n+p}(c)$ of index $2p$ and let M be an $n(\geqq 3)$-dimensional complete space-like complex submanifold of M'. The following assertions hold ;*

(1) If $c \geqq 0$, then M is totally geodesic.

(2) In the case $c < 0$, there exists a negative constant h so that if $h_2 \leqq h$, then M is totally geodesic.

PROOF. Since the ambient space M' is of constant holomorphic sectional curvature c, it is locally symmetric and it satisfies the condition $(*)$ such that $a_1 = a_2 = \frac{c}{2}$ Furthermore, from $h_4 \geqq \frac{1}{n}{h_2}^2$ and $TrA^2 \geqq \frac{1}{p}{h_2}^2$ we have directly the following inequality by (2.8).

(3.3)

$$\Delta f \geqq c_0 f^2 + c_1 f + c_2 = F(f), \quad c_0 = \frac{2}{np}(n+2p), \ c_1 = c(n+2), \ c_2 = 0,$$

regardless of (3.2). On the other hand, from the norm of

$$\sum_x h^x_{im} \bar{h}^x_{jk} + \frac{2}{n(n+1)}(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm})h_2$$

we get $TrA^2 \geqq \frac{2}{n(n+1)}h_2{}^2$, where the equality holds if and only if M is a complex space form of constant holomorphic sectional curvature $\frac{c}{2}$
By (2 8) we have
(3.4)
$$\Delta f \geqq c_0 f^2 + c_1 f + c_2 = F(f), \quad c_0 = \frac{4}{n(n+1)}(n+2),\ c_1 = c(n+2),\ c_2 = 0.$$

Since $S_{j\bar{j}} \geqq \frac{c}{2}(n+1)$ by the first equation of (2.6). the Ricci curvature of M is bounded from below.

In the first assertion, the coefficients satisfy $c_0 > 0 = c_2$ by (3.3) or (3.4) and $deg\ F = 2$, which implies that we can apply Theorem 2.1 to the function f and hence we get

$$F(sup\ f) = sup\ f(c_0 sup\ f + c_1) \leqq 0.$$

Accordingly, we have $sup\ f \leqq 0$ because the the function f is non-negative and $c_1 \geqq 0$. Hence f vanishes identically on M, which means that M is totally geodesic.

In the second assertion, we have by Theorem 2 1

$$sup\ f = 0 \text{ or } sup\ f \leqq -\frac{c_1}{c_0} \tag{3.5}$$

because $c_0 > 0$ and $c_1 < 0$. Hence for a negative constant h such that $h < \frac{c_1}{c_0}$ we suppose that $h_2 \leqq h$. Then we get $sup\ f \geqq -h > -\frac{c_1}{c_0}$ which means that $sup\ f = 0$ by (3.5). Hence f vanishes identically on M.

It completes the proof.

REMARK 3.2. For the complex coordinate system (z_A, z_{2n+1}) in C_s^{2n+1}, let $M = M(b_j)$ be the complex hypersurface in given by the equation

$$z_{2n+1} = \sum_j (z_j + b_j z_{j^*})^2, \qquad j^* = j - n$$

for any complex number b_j such that $|b_j| = 1$. Then it has been shown in [3] and [10] that M is a family of complete indefinite complex hypersurfaces of index $2s$, which are Ricci flat and not flat Thus we see $c_1 = 0$, but it is not totally geodesic This means that in Theorem 3.1 the condition that M is space-like is essential.

REMARK 3.3. A complex quadratic Q_n is a complete space-like complex hypersurface in a complex hyperbolic space $CH_1^{n+1}(c)$ and it is Einstein. The scalar curvature r satisfies $r = cn^2$ and $h_2 = \frac{c}{2}n$. Then we see $c_0 = \frac{2}{n}(n+2)$ and $c_1 = c(n+2)$ in (3.3). Thus the estimation of Theorem 3.1(2) shows that according to (3.3) for a negative constant $h < \frac{c_1}{c_0} = \frac{c}{2}n$, if $h_2 \leqq h$, then M is totally geodesic.

REMARK 3.4. A complex hyperbolic space $CH^n(\frac{c}{2})$ is a complete space-like complex submanifold in $CH_n^{n+p}(c)$ and it is not totally geodesic. The scalar curvature r satisfies $r = \frac{c}{2}n(n+1)$ and $h_2 = \frac{c}{4}n(n+1)$. Then we see $c_0 = \frac{4}{n(n+1)}(n+2)$ and $c_1 = c(n+2)$ in (3.4). Hence the estimation of Theorem 3.1(2) shows that according to (3.4) for a negative constant $h < \frac{c_1}{c_0} = \frac{c}{4}n(n+1)$, if $h_2 \leqq h$, then M is totally geodesic.

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