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HOM AND EXT FUNCTORS OF GENERALIZED INVERSE POLYNOMIAL MODULES

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ABSTRACT. Northcott and McKerrow proved that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective left $R[x]$-module. Park generalize Northcott and McKerrow's result so that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-S}]$ is an injective left $R[x^S]$-module, where S is a submonoid of N (N is the set of all natural numbers). In this paper we show

$$\mathrm{Hom}_{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong \mathrm{Hom}_R(M, N)[[x^S]]$$

and using the above result and this isomorphism, finally we show that

$$Ext^i_{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong Ext^i_R(M, N)[[x^S]]$$

1. Introduction

Northcott [3] considered the module $K[x^{-1}]$ of inverse polynomial over the polynomial ring $K[x]$ (with K a field), and Northcott and McKerrow [1] proved that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective left $R[x]$-module. In [6] Park generalize Northcott and McKerrow's result so that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-S}]$

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is an injective left $R[x^S]$-module, where S is a submonoid of N (N is the set of all natural numbers). In this paper we prove

$$\mathrm{Hom}_{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong \mathrm{Hom}_R(M, N)[[x^S]]$$

and by using this isomorphism and the above result we show

$$Ext^i_{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong Ext^i_R(M, N)[[x^S]].$$

Inverse polynomial modules were developed in [4] [5] and recently in [2].

DEFINITION 1.1. Let R be a ring and M be a left R-module. then $M[x^{-1}]$ is a left $R[x]$-module such that

$$x(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = m_1 + m_2x^{-1} + \cdots + m_nx^{-n+1}$$

and

$$r(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \cdots + rm_nx^{-n}$$

where $r \in R$.

Similarly, we also can define $M[[x^{-1}]], M[x, x^{-1}], M[x, x^{-1}]]$, and also $M[[x, x^{-1}]$ as left $R[x]$-modules where, for example, $M[[x, x^{-1}]$ is the set of Laurent series in x with coefficients in M, i.e., the set of all formal sums $\sum_{k \geq n_0} m_kx^k$ with n_0 any element of Z (Z is the set of all integers).

DEFINITION 1.2. Let R be a ring and M be a left R-module. and $S = \{0, k_1, k_2, \cdots\}$ be a submonoid of N (N is the set of all natural numbers), then $M[x^{-S}]$ is a left $R[x^S]$-module such that

$$\begin{aligned} &x^{k_i}(m_0 + m_1x^{-k_1} + m_2x^{-k_2} \cdots + m_nx^{-k_n}) \\ &= m_1x^{-k_1+k_i} + m_2x^{-k_2+k_i} + \cdots + m_nx^{-k_n+k_i} \end{aligned}$$

where

$$x^{-k_j+k_i} = \begin{cases} x^{-k_j+k_i}, & \text{if } \; k_j - k_i \in S \\ 0, & \text{if } \; k_j - k_i \notin S. \end{cases}$$

For example, if $S = \{0, 2, 3, 4, 5, \cdots\}$, then $m_0 + m_2x^{-2} + m_3x^{-3} + \cdots + m_ix^{-i} \in M[x^{-S}]$ and if $S = \{0, 1, 2, 3, 4, \cdots\}$, then $M[x^{-S}] = M[x^{-1}]$.

Similarly, we can define $M[[x^{-S}]]$ as a left $R[x^S]$-module.

THEOREM 1.3. *[6] Let R be a commutative noetherian ring and S be a submonoid, and E be an injective left R-module. Then $E[x^{-S}]$ is an injective left $R[x^S]$-module.*

PROOF. Let $S = \{0, k_1, k_2, \cdots\}$ be a submonoid. Then

$$\mathrm{Hom}_R(R[x^S],\ E) \cong E[[x^{-S}]]$$

is an injective left $R[x^S]$-module. Define $\phi : E[[x^{-S}]] \to E[[x^{-S}]]$ by

$$\phi(f) = x^{k_1} f$$

for $f \in E[[x^{-S}]]$, then ϕ is not locally nilpotent on $E[[x^{-S}]]$. So $E[[x^{-S}]]$ is not an essential extension of $Ker(\phi)$. Let $\bar{E}$ be an injective envelope of $Ker(\phi)$, then

$$Ker(\phi) \subset \bar{E} \subset E[[x^{-S}]].$$

Then the restriction $\phi|_{\bar{E}} : \bar{E} \to \bar{E}$, is locally nilpotent on $\bar{E}$. So $\bar{E} \subset E[x^{-S}]$. But $E[x^{-S}]$ is an essential extention of $Ker(\phi)$, so that $E[x^{-S}]$ is an essential extention of $\bar{E}$. Therefore, $\bar{E} = E[x^{-S}]$. Hence, $E[x^{-S}]$ is an injective left $R[x^S]$-module.

2. Hom functor of generalized inverse polynomial modules

LEMMA 2.1. *Let M and N be left R-modules and let $S = \{0, k_1, k_2, k_3, \cdots\}$ be a submonoid. If $\phi : M[x^{-S}] \to N[x^{-S}]$ is a $R[x^S]$-linear map, then $\phi(M + Mx^{-k_1} + Mx^{-k_2} + \cdots + Mx^{-k_i}) \subset N + Nx^{-k_1} + Nx^{-k_2} + \cdots + Nx^{-k_i}$.*

PROOF. Let $m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i}$ be an element of $M + Mx^{-k_1} + Mx^{-k_2} + \cdots + Mx^{-k_i}$ and

$$\begin{aligned}\phi(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i}) \\ = n_0 + n_{k_1}x^{-k_1} + n_{k_2}x^{-k_2} + \cdots + n_{k_i}x^{-k_i} \\ + n_{k_{i+1}}x^{-k_{i+1}} + \cdots + n_{k_j}x^{-k_j},\end{aligned}$$

and without loss of generality, we assume that $j \geq i$. Then

$$\begin{aligned}\phi(x^{k_i}(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i}))\\ = \phi(m_{k_i}) \in N.\end{aligned}$$

On the other hand,

$$\begin{aligned}x^{k_i}&(\phi(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i}))\\ &= x^{k_i}(n_0 + n_{k_1}x^{-k_1} + n_{k_2}x^{-k_2} + \cdots + n_{k_i}x^{-k_i}\\ &\quad + n_{k_{i+1}}x^{-k_{i+1}} + \cdots + n_{k_j}x^{-k_j})\\ &= n_{k_i} + \alpha,\end{aligned}$$

where $\alpha = n_{k_{i+1}}x^{-k_{i+1}+k_i} + \cdots + n_{k_j}x^{-k_j+k_i}$. Since $\phi(m_{k_i}) = n_{k_i} + \alpha$ and n_{k_i} is an element of N, all coefficients of α have to be zero. Therefore,

$$\phi(M + Mx^{-k_1} + \cdots + Mx^{-k_i}) \subset N + Nx^{-k_1} + \cdots + Nx^{-k_i}.$$

THEOREM 2.2. *Let M and N be left R-modules, and let $S = \{0, k_1, k_2, \cdots$ be a submonoid. Then*

$$Hom_{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong Hom_R(M, N)[[x^S]].$$

PROOF. Let $\sigma : M[x^{-S}] \to N[x^{-S}]$ be a $R[x^S]$-linear map. and

$$M[x^{-S}] = M \oplus Mx^{-k_1} \oplus Mx^{-k_2} \oplus \cdots,$$

$$N[x^{-S}] = N \oplus Nx^{-k_1} \oplus Nx^{-k_2} \oplus \cdots.$$

Then

$$\sigma(M + Mx^{-k_1} + \cdots + Mx^{-k_i}) \subset N + Nx^{-k_1} + \cdots + Nx^{-k_i},$$

by Lemma 2.1.

Define $\phi_{k_i k_j} = \pi_j \sigma \iota_i : Mx^{-k_i} \to Nx^{-k_j}$, where $\iota_i : Mx^{-k_i} \to M[x^{-S}]$ the i^{th} inclusion and $\pi_j : N[x^{-S}] \to Nx^{-k_j}$ the j^{th} projection map.

If we let

$$_{N+Nx^{-k_1}+Nx^{-k_2}}|\,\sigma\,|_{M+Mx^{-k_1}+Mx^{-k_2}} = \begin{pmatrix} \phi_{k_0k_0} & \phi_{k_0k_1} & \phi_{k_0k_2} \\ \phi_{k_1k_0} & \phi_{k_1k_1} & \phi_{k_1k_2} \\ \phi_{k_2k_0} & \phi_{k_2k_1} & \phi_{k_2k_2} \end{pmatrix},$$

then $\phi_{k_0k_1} = \phi_{k_0k_2} = \phi_{k_1k_2} = 0$, and since

$$\sigma\{x^{k_2}(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2})\} = \sigma(m_{k_2}) = \phi_{k_0k_0}(m_{k_2})$$

and

$$\begin{aligned} x^{k_2}&\{\sigma(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2})\} \\ &= x^{k_2}(\phi_{k_0k_0}(m_0) + \phi_{k_1k_0}(m_{k_1}) + \phi_{k_2k_0}(m_{k_2}) \\ &\quad + \phi_{k_1k_1}(m_{k_1})x^{-k_1} + \phi_{k_2k_1}(m_{k_2})x^{-k_1} + \phi_{k_2k_2}(m_{k_2})x^{-k_2}) \\ &= \phi_{k_2k_2}(m_{k_2}), \end{aligned}$$

so we have $\phi_{k_0k_0} = \phi_{k_1k_1} = \phi_{k_2k_2}$. And

$$\begin{aligned} \sigma\{x^{k_1}(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2})\} &= \sigma(m_{k_1} + m_{k_2}x^{-k_2+k_1}) \\ &= \phi_{k_0k_0}(m_{k_1}) + \phi_{k_1k_0}(m_{k_2}) + \phi_{k_1k_1}(m_{k_2})x^{-k_2+k_1}. \end{aligned}$$

$$\begin{aligned} x^{k_1}&\{\sigma(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2})\} \\ &= x^{k_1}(\phi_{k_0k_0}(m_0) + \phi_{k_1k_0}(m_{k_1}) + \phi_{k_2k_0}(m_{k_2}) \\ &\quad + \phi_{k_1k_1}(m_{k_1})x^{-k_1} + \phi_{k_2k_1}(m_{k_2})x^{-k_1} + \phi_{k_2k_2}(m_{k_2})x^{-k_2}) \\ &= \phi_{k_1k_1}(m_{k_1}) + \phi_{k_2k_1}(m_{k_2}) + \phi_{k_2k_2}(m_{k_2})x^{-k_2+k_1}. \end{aligned}$$

We have $\phi_{k_2k_1} = \phi_{k_1k_0}$. So let

$$f_{k_0} = \phi_{k_0k_0} = \phi_{k_1k_1} = \phi_{k_2k_2}$$

$$f_{k_1} = \phi_{k_1 k_0} = \phi_{k_2 k_1}$$

$$f_{k_2} = \phi_{k_2 k_0}.$$

If we let

$$_{N+Nx^{-k_1}+\cdots+Nx^{-k_t}} \mid \sigma \mid_{M+Mx^{-k_1}+\cdots+Mx^{-k_t}}$$

$$= \begin{pmatrix} \phi_{k_0k_0} & \phi_{k_0k_1} & \cdots & \phi_{k_0k_t} \\ \phi_{k_1k_0} & \phi_{k_1k_1} & \cdots & \phi_{k_1k_t} \\ \phi_{k_2k_0} & \phi_{k_2k_1} & \cdots & \phi_{k_2k_t} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k_tk_0} & \phi_{k_tk_1} & \cdots & \phi_{k_tk_t} \end{pmatrix}, \quad .$$

then

$$\phi_{k_0k_1} = \phi_{k_0k_2} = \cdots = \phi_{k_0k_t} = \phi_{k_1k_2} = \cdots = \phi_{k_1k_t} = \cdots = \phi_{k_{t-1}k_t} = 0,$$

and since

$$\begin{aligned} &\sigma\{x^{k_t}(m_0 + m_{k_1}x^{-k_1} + \cdots + m_{k_t}x^{-k_t})\} \\ &\qquad = \sigma(m_{k_t}) = \phi_{k_0k_0}(m_{k_t}), \end{aligned}$$

$$\begin{aligned} &x^{k_t}\{\sigma(m_0 + m_{k_1}x^{-k_1} + \cdots + m_{k_t}x^{-k_t})\} \\ &\quad = x^{k_t}\{(\phi_{k_0k_0}(m_0) + \phi_{k_1k_0}(m_{k_1}) + \cdots + \phi_{k_tk_0}(m_{k_t})) \\ &\qquad + (\phi_{k_1k_1}(m_{k_1}) + \phi_{k_2k_1}(m_{k_2}) + \cdots + \phi_{k_tk_1}(m_{k_t}))x^{-k_1} \\ &\qquad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\qquad + (\phi_{k_{t-1}k_{t-1}}(m_{k_{t-1}}) + \phi_{k_tk_{t-1}}(m_{k_t}))x^{-k_{t-1}} \\ &\qquad + \phi_{k_tk_t}(m_{k_t})x^{-k_t}\} \\ &\quad = \phi_{k_tk_t}(m_{k_t}), \end{aligned}$$

we have $\phi_{k_0k_0} = \phi_{k_1k_1} = \cdots = \phi_{k_ik_i}$. And

$$
\begin{aligned}
&\sigma\{x^{k_1}(m_0 + m_{k_1}x^{-k_1} + \cdots + m_{k_i}x^{-k_i})\}\\
&= \sigma(m_{k_1} + m_{k_2}x^{-k_2+k_1} + \cdots + m_{k_i}x^{-k_i+k_1})\\
&= (\phi_{k_0k_0}(m_{k_1}) + \phi_{k_1k_0}(m_{k_2}) + \cdots + \phi_{(k_i-k_1)(k_0)}(m_{k_i}))\\
&\quad + (\phi_{k_1k_1}(m_{k_2}) + \phi_{k_2k_1}(m_{k_3}) + \cdots + \phi_{(k_i-k_1)(k_1)}(m_{k_i}))x^{-k_2+k_1}\\
&\quad + (\phi_{k_2k_2}(m_{k_3}) + \phi_{k_3k_2}(m_{k_4}) + \cdots + \phi_{(k_i-k_1)(k_2)}(m_{k_i}))x^{-k_3+k_1}\\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad + (\phi_{(k_{i-1}-k_1)(k_{i-1}-k_1)}(m_{k_{i-1}}) + \phi_{(k_i-k_1)(k_{i-1}-k_1)}(m_{k_i}))x^{-k_{i-1}+k_1}\\
&\quad + \phi_{(k_i-k_1)(k_i-k_1)}(m_{k_i})x^{-k_i+k_1},
\end{aligned}
$$

$$
\begin{aligned}
&x^{k_1}\{\sigma(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i})\}\\
&\quad = x^{k_1}\{(\phi_{k_0k_0}(m_{k_0}) + \phi_{k_1k_0}(m_{k_1}) + \phi_{k_2k_0}(m_{k_2}) + \cdots + \phi_{k_ik_0}(m_{k_i}))\\
&\qquad\quad + (\phi_{k_1k_1}(m_{k_1}) + \phi_{k_2k_1}(m_{k_2}) + \cdots + \phi_{k_ik_1}(m_{k_i}))x^{-k_1}\\
&\qquad\quad + (\phi_{k_2k_2}(m_{k_2}) + \phi_{k_3k_2}(m_{k_3}) + \cdots + \phi_{k_ik_2}(m_{k_i}))x^{-k_2}\\
&\qquad\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad\quad + (\phi_{k_{i-1}k_{i-1}}(m_{k_{i-1}}) + \phi_{k_ik_{i-1}}(m_{k_i}))x^{-k_{i-1}}\\
&\qquad\quad + \phi_{k_ik_i}(m_{k_i})x^{-k_i}\}\\
&\qquad = (\phi_{k_1k_1}(m_{k_1}) + \phi_{k_2k_1}(m_{k_2}) + \cdots + \phi_{k_ik_1}(m_{k_i}))\\
&\qquad\quad + (\phi_{k_2k_2}(m_{k_2}) + \phi_{k_3k_2}(m_{k_3}) + \cdots + \phi_{k_ik_2}(m_{k_i}))x^{-k_2+k_1}\\
&\qquad\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad\quad + (\phi_{k_{i-1}k_{i-1}}(m_{k_{i-1}}) + \phi_{k_ik_{i-1}}(m_{k_i}))x^{-k_{i-1}+k_1}\\
&\qquad\quad + \phi_{k_ik_i}(m_{k_i})x^{-k_i+k_1}.
\end{aligned}
$$

Then

$$\phi_{k_0k_0} = \phi_{k_1k_1} = \cdots = \phi_{k_ik_i},$$

$$\phi_{k_1k_0} = \phi_{k_2k_1} = \cdots = \phi_{k_ik_{i-1}},$$

$$\phi_{k_2k_0} = \phi_{k_3k_1} = \cdots = \phi_{k_ik_{i-2}},$$
$$\phi_{k_3k_0} = \phi_{k_4k_1} = \cdots = \phi_{k_ik_{i-3}},$$
$$\vdots$$
$$\phi_{k_{i-1}k_0} = \phi_{k_ik_1}.$$

So let

$$f_{k_0} = \phi_{k_0k_0} = \phi_{k_1k_1} = \cdots = \phi_{k_ik_i},$$
$$f_{k_1} = \phi_{k_1k_0} = \phi_{k_2k_1} = \cdots = \phi_{k_ik_{i-1}},$$
$$f_{k_2} = \phi_{k_2k_0} = \phi_{k_3k_1} = \cdots = \phi_{k_ik_{i-2}},$$
$$f_{k_3} = \phi_{k_3k_0} = \phi_{k_4k_1} = \cdots = \phi_{k_ik_{i-3}},$$
$$\vdots$$
$$f_{k_{i-1}} = \phi_{k_{i-1}k_0} = \phi_{k_ik_1}.$$

By the same process we can choose $f_{k_3}, f_{k_4}, \cdots$. Then

$$f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots \in \mathrm{Hom}_R(M, N)[[x^S]].$$

Conversely given

$$f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots \in \mathrm{Hom}_R(M, N)[[x^S]],$$

define $\sigma \in \mathrm{Hom}_{R[x^S]}(M[x^{-S}], N[x^{-S}])$ by

$$\begin{aligned}
&\sigma(m_0 + m_{k_1}x^{-k_1} + \cdots + m_{k_i}x^{-k_i}) \\
&= (f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots)(m_0 + m_{k_1}x^{-k_1} + \cdots + m_{k_i}x^{-k_i}) \\
&= f_{k_0}(m_0) + f_{k_0}(m_{k_1})x^{-k_1} + f_{k_0}(m_{k_2})x^{-k_2} \\
&\quad + \cdots + f_{k_0}(m_{k_i})x^{-k_i} \\
&\quad + f_{k_1}(m_{k_1}) + f_{k_1}(m_{k_2})x^{-k_2+k_1} + f_{k_1}(m_{k_3})x^{-k_3+k_1} \\
&\quad + \cdots + f_{k_1}(m_{k_i})x^{-k_i+k_1} \\
&\quad + f_{k_2}(m_{k_2}) + f_{k_2}(m_{k_3})x^{-k_3+k_2} + f_{k_2}(m_{k_4})x^{-k_4+k_2} \\
&\quad + \cdots + f_{k_2}(m_{k_i})x^{-k_i+k_2} \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + f_{k_{i-1}}(m_{k_{i-1}}) + f_{k_{i-1}}(m_{k_i})x^{-k_i+k_{i-1}} + f_{k_i}(m_{k_i})
\end{aligned}$$

$$\begin{cases} f_{k_\alpha}(m_{k_\beta})x^{-k_\beta+k_\alpha} = 0, & \text{if } -k_\beta + k_\alpha \notin -S \\ f_{k_\alpha}(m_{k_\beta})x^{-k_\beta+k_\alpha} = f_{k_\alpha}(m_{k_\beta})x^{-k_\beta+k_\alpha}, & \text{if } -k_\beta + k_\alpha \in -S \end{cases}$$

$$= (f_{k_0}(m_0) + f_{k_1}(m_{k_1}) + \cdots + f_{k_t}(m_{k_t})) + (f_{k_0}(m_{k_1}) + S.O.C.)x^{-k_1} + (f_{k_0}(m_{k_2}) + S.O.C.)x^{-k_2} + \cdots + (f_{k_0}(m_{k_t}))x^{-k_t}.$$

Now σ is a $R[x^S]$-linear map. And, easily, σ is a group homomorphism. Hence .

$$\mathrm{Hom}_{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong \mathrm{Hom}_R(M, N)[[x^S]].$$

3. Ext functor of generalized inverse polynomial module

THEOREM 3.1. *Let*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence of R-modules. Then there exists a short exact sequence of $R[x^{-S}]$-modules

$$0 \longrightarrow L[x^{-S}] \xrightarrow{\bar{f}} M[x^{-S}] \xrightarrow{\bar{g}} N[x^{-S}] \longrightarrow 0.$$

PROOF. Define $\bar{f} : L[x^{-S}] \to M[x^{-S}]$ by

$$\bar{f}(a_{k_0}+a_{k_1}x^{-k_1}+\cdots+a_{k_t}x^{-k_t}) = f(a_{k_0})+f(a_{k_1})x^{-k_1}+\cdots+f(a_{k_t})x^{-k_t}.$$

Then $\bar{f}$ is well-defined injective $R[x^S]$-linear map.
Define $\bar{g} : M[x^{-S}] \to N[x^{-S}]$ by

$$\bar{g}(a_{k_0}+a_{k_1}x^{-k_1}+\cdots+a_{k_t}x^{k_t}) = g(a_{k_0})+g(a_{k_1})x^{-k_1}+\cdots+g(a_{k_t})x^{-k_t}$$

Then $\bar{g}$ is a well-defined surjective $R[x^S]$-linear map. Now we show

$$\mathrm{Im}\bar{f} = \ker\bar{g}.$$

Let $a_{k_0} + a_{k_1}x^{-k_1} + \cdots + a_{k_i}x^{-k_i} \in L[x^{-S}]$.
Then

$$\begin{aligned}
&(\bar{g}\circ\bar{f})(a_{k_0} + a_{k_1}x^{-k_1} + \cdots + a_{k_i}x^{-k_i})\\
&= \bar{g}(\bar{f}(a_{k_0} + a_{k_1}x^{-k_1} + \cdots + a_{k_i}x^{-k_i}))\\
&= \bar{g}(f(a_{k_0}) + f(a_{k_1})x^{-k_1} + \cdots + f(a_{k_i})x^{-k_i})\\
&= g(f(a_{k_0})) + g(f(a_{k_1}))x^{-k_1} + \cdots + g(f(a_{k_i}))x^{-k_i}\\
&= (g\circ f)(a_{k_0}) + (g\circ f)(a_{k_1})x^{-k_1} + \cdots + (g\circ f)(a_{k_i})x^{-k_i}\\
&= 0.
\end{aligned}$$

Let $a_{k_0} + a_{k_1}x^{-k_1} + \cdots + a_{k_i}x^{-k_i} \in \ker\bar{g}$.
Then

$$\begin{aligned}
&\bar{g}(a_{k_0} + a_{k_1}x^{-k_1} + \cdots + a_{k_i}x^{-k_i})\\
&\quad = g(a_{k_0}) + g(a_{k_1})x^{-k_1} + \cdots + g(a_{k_i})x^{-k_i}.
\end{aligned}$$

Thus

$$g(a_{k_j}) = 0, \text{ for } j = \{0,1,2,\cdots,i\}$$

$$a_{k_j} \in \ker g = \mathrm{Im} f.$$

Let $f(b_{k_0}) = a_{k_0},\ f(b_{k_1}) = a_{k_1},\ f(b_{k_2}) = a_{k_2},\ \cdots,\ f(b_{k_i}) = a_{k_i}$

$$b_{k_j} \in L, \text{ for } j = \{0,1,2,\cdots,i\}.$$

Then

$$\begin{aligned}
&b_{k_0} + b_{k_1}x^{-k_1} + \cdots + b_{k_i}x^{-k_i} \in L[x^{-s}].\\
&\bar{f}(b_{k_0} + b_{k_1}x^{-k_1} + \cdots + b_{k_i}x^{-k_i})\\
&\quad = f(b_{k_0}) + f(b_{k_1})x^{-k_1} + \cdots + f(b_{k_i})x^{-k_i}\\
&\quad = a_{k_0} + a_{k_1}x^{-k_1} + \cdots + a_{k_i}x^{-k_i}.
\end{aligned}$$

Therefore, $\mathrm{Im}\bar{f} = \ker\bar{g}$. Hence,

$$0 \longrightarrow L[x^{-S}] \xrightarrow{\bar{f}} M[x^{-S}] \xrightarrow{\bar{g}} N[x^{-S}] \longrightarrow 0$$

is a short exact sequence of $R[x^S]$-modules.

THEOREM 3.2. *Let R be a ring. and M, N be left R-modules. Then*

$$Ext^i_{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong Ext^i_R(M, N)[[x^S]]$$

PROOF. Let $0 \longrightarrow N \xrightarrow{\varepsilon} E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} E^2 \xrightarrow{d_2} \cdots$ be an injective resolution of a left R-module N.
Then by Theorem 1.3 $E^i[x^{-S}]$ is an injective $R[x^S]$-module for each i. and

$$0 \longrightarrow N[x^{-S}] \longrightarrow E^0[x^{-S}] \longrightarrow E^1[x^{-S}] \longrightarrow E^2[x^{-S}] \longrightarrow \cdots$$

is an injective resolution of a left $R[x^S]$-module $N[x^{-S}]$.
Consider the deleted injective resolution

$$0 \longrightarrow 0 \longrightarrow E^0[x^{-S}] \xrightarrow{d^0[x^{-S}]} E^1[x^{-S}] \xrightarrow{d^1[x^{-S}]} E^2[x^{-S}] \longrightarrow \cdots,$$

then we have the complex

$$\mathrm{Hom}(M[x^{-S}], E^0[x^{-S}]) \xrightarrow{\mathrm{Hom}(M[x^{-S}], d^0[x^{-S}])} \mathrm{Hom}(M[x^{-S}], E^1[x^{-S}])$$
$$\xrightarrow{\mathrm{Hom}(M[x^{-S}], d^1[x^{-S}])} \mathrm{Hom}(M[x^{-S}], E^2[x^{-S}]) \to \cdots$$

By the Theorem 2.2, we also have an isomorphism

$$f_i \; : \; \mathrm{Hom}(M[x^{-S}], E^i[x^{-S}]) \cong \mathrm{Hom}(M, E^i)[[x^S]]$$

for each i.
Thus there is a complex

$$0 \to \mathrm{Hom}(M, E^0)[[x^S]] \xrightarrow{\mathrm{Hom}(M, d^0)[[x^S]]} \mathrm{Hom}(M, E^1)[[x^S]]$$
$$\xrightarrow{\mathrm{Hom}(M, d^1)[[x^S]]} \mathrm{Hom}(M, E^2)[[x^S]] \to \cdots$$

Now consider the following diagram :

$$\begin{array}{ccc} \mathrm{Hom}(M[x^{-S}], E^i[x^{-S}]) & \xrightarrow{\mathrm{Hom}(M[x^{-S}], d^i[x^{-S}])} & \mathrm{Hom}(M[x^{-S}], E^{i+1}[x^{-S}]) \\ f_i \downarrow & & \downarrow f_{i+1} \\ \mathrm{Hom}(M, E^i)[[x^S]] & \xrightarrow{\mathrm{Hom}(M,d^i)[[x^S]]} & \mathrm{Hom}(M, E^{i+1})[[x^S]] \end{array}$$

Let $\phi \in \mathrm{Hom}(M[x^{-S}], E^i[x^{-S}])$, then $f(\phi) = \phi_{00} + \phi_{10}x^{k_1} + \phi_{20}x^{k_2} + \cdots$, where $\phi(m_i x^{-k_i}) = \phi_{k_i 0}(m_i) + \phi_{k_i-10}(m_i)x^{k_1} + \cdots + \phi_{00}(m_i)x^{k_i}$ for any k_i so, $\mathrm{Hom}(M, d^i)[[x^S]](\phi_{00} + \phi_{10}x^{k_1} + \phi_{20}x^{k_2} + \cdots) = d^i \circ \phi_{00} + d^i \circ \phi_{10}x^{k_1} + d^i \circ \phi_{20}x^{k_2} + \cdots$ and $\mathrm{Hom}(M[x^{-S}], d^i[x^{-S}])(\phi) = d^i[x^{-S}] \circ \phi$. Now

$$\begin{aligned} & d^i[x^{-S}] \circ \phi(m_i x^{-k_i}) \\ &= d^i[x^{-S}](\phi_{k_i 0}(m_i) + \phi_{k_i-10}(m_i)x^{-k_1} + \cdots + \phi_{00}(m_i)x^{-k_i}) \\ &= d^i(\phi_{k_i 0}(m_i)) + d^i(\phi_{k_i-10}(m_i)x^{-k_1}) + \cdots + d^i(\phi_{00}(m_i)x^{-k_i}) \\ &= d^i \circ \phi_{k_i 0}(m_i) + d^i \circ \phi_{k_i-10}(m_i)x^{-k_1} + \cdots + d^i \circ \phi_{00}(m_i)x^{-k_i} \end{aligned}$$

for any k_i.
So, $f_{i+1}(d^i[x^{-k_i}] \circ \phi) = d^i \circ \phi_{00} + d^i \circ \phi_{10}x^{k_1} + d^i \circ \phi_{20}x^{k_2} + \cdots$.
Therefore, $\mathrm{Hom}(M, d^i)[[x^S]](\phi_{00} + \phi_{10}x^{k_1} + \phi_{20}x^{k_2} + \cdots) = f_{i+1}(d^i[x^{-s}] \circ \phi)$.
Hence, the diagram of the above is commutative, and

$$\begin{aligned} Ext^i_{R[x^S]}(M[x^{-S}], N[x^{-S}]) &= \frac{\ker(\mathrm{Hom}(M[x^{-S}], d^i[x^{-S}]))}{\mathrm{Im}(\mathrm{Hom}(M[x^{-S}], d^{i-1}[x^{-S}]))} \\ &\cong \frac{\ker(\mathrm{Hom}(M, d^i)[[x^S]])}{\mathrm{Im}(\mathrm{Hom}(M, d^{i-1})[[x^{-S}]])} \\ &= \frac{\ker(\mathrm{Hom}(M, d^i))[[x^S]]}{\mathrm{Im}(\mathrm{Hom}(M, d^{i-1}))[[x^{-S}]]} \\ &= \frac{\ker(\mathrm{Hom}(M, d^i))}{\mathrm{Im}(\mathrm{Hom}(M, d^{i-1}))}[[x^S]] \\ &= Ext^i_R(M, N)[[x^S]]. \end{aligned}$$

Hence,

$$Ext^i_{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong Ext^i_R(M, N)[[x^S]].$$

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