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A STUDY ON BAER AND P.P. NEAR-RINGS

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ABSTRACT. Baer rings were introduced by Kaplansky [3] to abstract various properties of rings of operators on a Hilbert spaces. On the other hand, p.p. rings were introduced by A. Hattori [2] to study the torsion theory. In this paper we introduce Baer near-rings and p.p. near-rings and study some properties and give some examples.

1. Introduction

In this paper we introduce Baer near-rings and p.p. near-rings and study some properties and give some examples. Let G be an additively written (but not necessarily abelian) group with zero 0 and $M_0(G) = \{f : G \to G \mid f(0) = 0\}$, the near-ring of all zero respecting mappings on G. We show that $M_0(G)$ is a Baer near-ring. As a corollary we show that every zero-symmetric near-ring can be embedded into a Baer near-ring. Let R be a commutative ring with identity. It is well known that R is a Baer (resp. p.p.) ring if and only if the polynomial ring $R[x]$ is a Baer (resp. p.p.) ring (See e.g., Armendariz [1]). Corresponding to this result, we will prove that the zero-symmetric part of $R[x]$ is a Baer (resp. p.p.) near-ring if and only if R is a Baer (resp. p.p.) ring. We also study the structure of the near-ring $R \oplus M$, where R is an associative ring with identity and M is a unital left R-module. Then $R \oplus M$ is a p.p. near-ring if and only if R is a p.p. ring.

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2. Baer near-rings and p.p. near-rings

A (right) near-ring is a set N with two binary operations $+$ and such that $(N,+)$ is a not necessarily abelian group with identity 0, $(N,\cdot)$ is a semigroup and $(x+y)z = xz + yz$ for all $x, y, z \in N$. Some basic definitions and concepts in near-ring theory can be found in Meldrum [4] and Pilz [5].

For a subset S of a near-ring N, the set $\{n \in N \mid nS = 0\}$ is called the annihilator of S in N and is denoted by $\mathrm{Ann}_N(S) = \mathrm{Ann}(S)$.

A near-ring is called a *Baer* near-ring if, for any subset S of N, $\mathrm{Ann}(S) = \mathrm{Ann}(e)$ for some idempotent $e \in N$. The following propositions is obvious.

PROPOSITION 1. *Let N_i $(i \in I)$ be a family of near-rings. Then the direct product $\prod_{i \in I} N_i$ is a Bear near-ring if and only if N_i is a Bear near-ring for each $i \in I$.*

EXAMPLE 1. (1) Every integral near-ring with identity is a Baer near-ring

(2) Every constant near-ring is a Baer near-ring.

(3) A direct product of integral near-rings with identity is a Baer near-ring.

Let G be an additively written (but not necessarily abelian) group with zero 0 and $M_0(G) = \{f : G \to G \mid f(0) = 0\}$, the near-ring of all zero respecting mappings on G.

THEOREM 2. *The near-ring $M_0(G)$ is Baer.*

PROOF. Let S be a subset of $M_0(G)$ and let $H = \{s(g) \mid s \in S,\ g \in G\}$. Let e be a mapping on G such that $e(x) = x$ for each $x \in H$. and $e(y) = 0$ for $y \in G - H$. Then e is an idempotent of $M_0(G)$ and $\mathrm{Ann}(S) = \mathrm{Ann}(e)$. This implies that $M_0(G)$ is a Baer near-ring.

COROLLARY 3. *Every zero-symmetric near-ring can be embedded into a Baer near-ring.*

PROOF. By [5, 1. 102], every zero-symmetric near-ring can be embedded into a zero-symmetric near-ring with identity. Let N be a zero-symmetric near-ring with identity. By Theorem 2, $M_0(N)$ is a

Baer near-ring. For $r \in N$, the mapping $f_r : t \in N \rightarrow rt \in N$ is an element of $M_0(N)$. Since N contains an identity it follows that the mapping $f : N \rightarrow M_0(N)$, defined by $f(r) = f_r$, is a near-ring monomorphism.

An associative ring R called a (left) $p.p.$ ring if every principal left ideal of R is projective. This is equivalent to the condition that, for any $a \in R$, $\mathrm{Ann}(a) = \mathrm{Ann}(e)$ for some idempotent $e \in R$. Similarly we can define for near-rings. N is called a p.p. near ring if for any $a \in N$, $\mathrm{Ann}(a) = \mathrm{Ann}(e)$ for some idempotent $e \in N$.

EXAMPLE 2. Recall that a near-ring N is called regular if, for any $x \in N$, there exists $y \in N$ such that $xyx = x$. Then xy is an idempotent and $\mathrm{Ann}(x) = \mathrm{Ann}(xy)$. Hence every regular near-ring is a p.p. near-ring.

Let R be a commutative ring with identity and let $R[x]$ denote the set of all polynomials in one indeterminate over R. Under usual addition $+$, and substitution $\circ$ of polynomials, $(R[x], +, \circ)$ becomes a near-ring. Following Pilz [5], $R_0[x]$ denote the zero symmetric part of $R[x]$, that is $R_0[x] = \{\sum_{i=1}^{n} a_i x^i \mid n \geq 1,\ a_i \in R\}$.

A ring(or near-ring) without non-zero nilpotent element is called *reduced*.

THEOREM 4. *Let R be a commutative ring with identity. The the following conditions are equivalent:*

1) $R_0[x]$ is a p.p. near-ring.

2) R is a p.p. ring.

PROOF. 1) $\Rightarrow$ 2) First we claim that R is reduced. Suppose that $a \in R$ with $a^2 = 0$. By hypothesis, there exists an idempotent $f \in R_0[x]$ such that $\mathrm{Ann}(ax) = \mathrm{Ann}(f)$. Let $f = a_1x + a_2x^2 + \cdots + a_nx^n$ with $a_i \in R$. Since f is an idempotent, we have $a_1^2 = a_1$. Since $ax \in \mathrm{Ann}(ax)$, $ax \circ f = af = 0$. In particular, $aa_1 = 0$. Since $x - f \in \mathrm{Ann}(f)$, $0 = (x - f) \circ ax = ax - f(ax)$. Hence $ax = a_1ax = 0$, that is $a = 0$. This proves that R is reduced.

Since R is reduced, the set of idempotents of $R_0[x]$ is just $\{ex \mid e^2 = e \in R\}$. Now let r be an arbitrary element of R. By hypothesis there exists an idempotent $e \in R$ such that $\text{Ann}(rx) = \text{Ann}(ex)$. Clearly this implies that $\{s \in R \mid sr = 0\} = R(1-e)$. Hence R is a p.p. ring.

2) $\Rightarrow$ 1). Let $f = a_1x + \cdots + a_nx^n \in R_0[x]$ and $g = b_1x + \cdots + b_mx^m \in R_0[x]$. First we claim that $f \circ g = 0$ if and only if $a_ib_j = 0$ for all i, j. It suffices to prove the 'only if' part. Let P be an arbitrary prime ideal of R and let $\bar{f}$ and $\bar{g}$ denote the image of f and g in $(R/P)[x]$, respectively. Since R/P is an integral domain and since $\bar{f} \circ \bar{g} = 0$, we can easily see that either $\bar{f} = 0$ or $\bar{g} = 0$ holds. Hence $a_ib_j \in P$ for all i, j. Since a prime ideal P is arbitrary, this implies that $a_ib_j \in \text{Rad}(R)$, where $\text{Rad}(R)$ denote the prime radical of R. Since R is a p.p. ring. R is reduced and hence $\text{Rad}(R) = 0$. This proves our claim. Therefore $a_1, \ldots, a_n \in \text{Ann}_R(b_1, \ldots, b_m)$ Since R is a p p ring, there exist idempotents $e_i \in R$ such that $\text{Ann}(b_i) = \text{Ann}(e_i)$ for all i. If $n = 2$. then $f = e_1 + e_2 - e_1e_2$ is an idempotent and $\text{Ann}_R(b_1, b_2) = \text{Ann}(f)$. Using induction on n, we can find an idempotent e of R such that $\text{Ann}_R(b_1, \ldots, b_m) = \text{Ann}(e)$. Then ex is an idempotent of $R_0[x]$ and $\text{Ann}(g) = \text{Ann}(ex)$. Therefore $R_0[x]$ is a p.p. near-ring.

THEOREM 5. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) $R_0[x]$ is a Baer near-ring.

2) R is a Baer ring.

PROOF. 1) $\Rightarrow$ 2). Let T be a subset of R and consider the suset $S = \{tx \mid t \in T\}$ of $R_0[x]$. As we have shown in the proof of 1) $\Rightarrow$ 2) in Theorem 4, the set of idempotents of $R_0[x]$ is just $\{ex \mid e^2 = e \in R\}$. Since $R_0[x]$ is Baer, $\text{Ann}(S) = \text{Ann}(ex)$ for some idempotent $e \in R$ Now we can easily see that $\text{Ann}_R(T) = \text{Ann}_R(e)$. Hence R is a Baer ring.

2) $\Rightarrow$ 1). Let S be a subset of $R_0[x]$ and consider the set T of all coefficients of $g(x) \in S$. Let $f = a_1x + \cdots + a_nx^n \in \text{Ann}(S)$. As in the proof of 2) $\Rightarrow$ 1) in Theorem 4, $a_i \in \text{Ann}_R(T)$ for all i. Since R is a Baer ring, there exists an idempotent e such that $\text{Ann}_R(T) = \text{Ann}_R(e)$. Now we can easily see that $\text{Ann}(S) = \text{Ann}(ex)$. This proves that $R_0[x]$ is a Baer near-ring.

Let R be an associative ring with identity and let M be a unital left

R-module. If we define a multiplication on the additive group $R \oplus M$ by $(a,b)\circ(c,d) = (ac, ad+b)$ for any (a,b), $(c,d) \in R\oplus M$, then $R\oplus M$ becomes a near-ring with identity $(1,0)$.

THEOREM 6. *Let R be an associative ring with identity and let M be a unital left R-module. Then the following conditions are equivalent:*
1) $R \oplus M$ is a p.p. near-ring.
2) R is a p.p. ring.

PROOF. 2) $\Rightarrow$ 1). We can easily see that, for $(c,d) \in R \oplus M$, $\mathrm{Ann}(c,d) = \{(a, -ad) \mid a \in \mathrm{Ann}(a)\}$. Since R is a left p.p ring, there is an idempotent $e \in R$ such that $\mathrm{Ann}_R(c) = \mathrm{Ann}(e)$ Then $(e,(1-e)d)$ is an idempotent of $R \oplus M$ and $\mathrm{Ann}(c,d) = \mathrm{Ann}(e,(1-e)d)$. Thus $R \oplus M$ is a p.p. near-ring. 1) $\Rightarrow$ 2). We first note that the set of idempotents of $R\oplus M$ is equal to $\{(e,(1-e)x) \mid e = e^2 \in R,\ x \in M\}$. Hence, for any $c \in R$, there exists idempotent $e \in R$ and $x \in M$ such that $\mathrm{Ann}(c,0) = \mathrm{Ann}(e,(1-e)x)$. By the way, $\mathrm{Ann}(c,0) = \{(a,0) \mid a \in \mathrm{Ann}(c)\}$. On the other hand, $(1-e, -(1-e)x) \in \mathrm{Ann}\ (e,(1-e)x)$. Hence $(1-e)x = 0$, and so $\mathrm{Ann}\ (c,0) = \mathrm{Ann}\ (e,0)$ This implies $\mathrm{Ann}\ (c) = \mathrm{Ann}(e)$ Therefore R is a p.p ring

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REFERENCES

[1] E P. Armendariz, *A note on extensions of Baer and p p rings*, Australian Math. Soc **18** (1974), 470 – 473.

[2] A Hattori *A foundation of the torsion theory over general rings*, Nagoya Math J. **17** (1960), 147 – 158.

[3] I. Kaplansky, *Rings of Operators*, Math Lecture Notes Series, Benjamin, New York, 1965

[4] J. D P Meldrum, *Near-Rings and Their Links with Groups*, Pitman Advanced Publishing Program Boston-London-Melbourne 1985.

[5] G. Pilz, *Near-Rings*, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1983

[6] D. Ramakotaiah and V. Sambasivarao, *Reduced near-ring*, in Near-rings and Near-fields, G. Betsch (editor), North-Holland (1987), 233-243

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