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SOME RESULTS ON A DIFFERENTIAL POLYNOMIAL RING OVER A REDUCED RING

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ABSTRACT. In this paper, a differential polynomial ring $R[x, \delta]$ of ring R with a derivation δ are investigated as follows. For a reduced ring R, a ring R is Baer(resp. quasi-Baer, p.q.-Baer, p.p.-ring) if and only if the differential polynomial ring $R[x, \delta]$ is Baer(resp. quasi-Baer, p.q.-Baer, p.p.-ring).

Throughout this paper, all rings are associative with unity. A ring R is called *(quasi-) Baer* if the right annihilator of every (right ideal) nonempty subset of R is generated by an idempotent. In [3], a ring R is called a *right(resp. left) principally quasi-Baer* (or simply *right(resp. left) p.q.-Baer*) if the right(resp. left) annihilator of a principal right(resp. left) ideal is generated by an idempotent. A ring R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer. Another generalization of Baer ring is the p.p.-ring. A ring R is called a *right(resp. left) p.p.-* ring if the right(resp. left) annihilator of an element of R is generated by an idempotent. Also, a ring R is called a *p.p.*-ring if it is both right and left p.p. In [3], the following fact was proved;

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PROPOSITION 1. *The following are equivalent;*

(1) *R is a right p.q.-Baer ring.*

(2) *The right annihilator of any finitely generated right ideal is generated (as a right ideal) by an idempotent.*

(3) *The right annihilator of every principal right ideal is generated (as a right ideal) by an idempotent.*

(4) *The right annihilator of every finitely generated ideal is generated (as a right ideal) by an idempotent.*

Note that this statement is true if "right" is replaced by "left" throughout.

PROOF. See [3].

In [3], they also have shown the following results;

THEOREM A. *R is a right p.q.-Baer ring if and only if the polynomial ring $R[x]$ is a right p.q.- Baer ring.*

THEOREM B. *For a ring R, the following are equivalent;*

(1) *R is a quasi-Baer ring;*

(2) *the polynomial ring $R[x]$ over R is a quasi-Baer ring;*

(3) *the formal power series ring $R[[x]]$ over R is a quasi-Baer ring.*

Now we try to apply those results for Ore extension and so we recall the following : Let α be an endomorphism of a ring R. An $\alpha-$*derivation* of R is an additive map $\delta : R \longmapsto R$ such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. The Ore extension $R[x; \alpha, \delta]$ is the ring of polynomials in x over R with usual addition and new multiplication defined by $xa = \alpha(a)x + \delta(a)$ for each $a \in R$. If $\delta = 0$, we write $R[x; \alpha]$ for $R[x; \alpha, \delta]$ and is called an *Ore extension of endomorphism type* (also called a skew polynomial ring). While if $\alpha = 1$, we write $R[x; \delta]$ for $R[x; 1, \delta]$ and is called an *Ore extension of derivation type* (also called a differential polynomial ring). Also, we recall that R is called a *reduced* ring if it has no nonzero nilpotent elements and R is called an *abelian* ring if every idempotent of R is central. We can easily observe that every reduced ring is abelian and in a reduced ring R the left and right annihilaters coincide for any subset U of R, where the left (right) annihilator of U is denoted by $l_R(U) = \{a \in R \mid aU = 0\}(r_R(U) = \{a \in R \mid Ua = 0\})$.

Of course, for a reduced ring R, it is clear that the following are equivalent :

(1) R is a right p.p.-ring,
(2) R is a left p.p.-ring;
(3) R is a right p.q.-Baer ring;
(4) R is a left p q.-Baer ring.

For a quasi-Baer(or p.q.-Baer) ring, we have the following questions.

QUESTION.

(1) *If R is a quasi-Baer(or left p.q.-Baer) ring, then is the Ore extension $R[x; \alpha, \delta]$ quasi-Baer(or left p.q.-Baer)? Here α is an endomorphism of R and δ is an α-derivation of R.*
(2) *Is the converse of (1) true?*

In the following example, we give answers negatively to this question

EXAMPLE 2.

(1) [2, Example 11] The ring $R = Z_2[x]/(x^2)$ is not a quasi-Baer, where Z_2 is the field of two elements and (x^2) is the ideal of the ring $Z_2[x]$ generated by x^2. In fact, $l_R(R(x + (x^2))$ is not generated by an idempotent of R.
But since $R[y; \delta] \simeq Mat_2(Z_2[y^2])$, where the derivation δ is defined by $\delta(x+(x^2)) = 1+(x^2)$, $R[y; \delta]$ is a quasi-Baer because $Z_2[y^2]$ is a quasi-Baer and so $Mat_2(Z_2[y^2])$ is also a quasi-Baer.
(2) Let R be the ring of upper triangular matrices over Z_2. For our notation, e_{ij} denotes the matrix with 1 in (i, j)-position and 0 for otherwise. Define an endomorphism α of R by $\alpha(ae_{11} + be_{12} + ce_{13} + de_{22} + ee_{23} + fe_{33}) = de_{22}$ Let δ_β be an inner derivation of R (i.e. $\delta_\beta(r) = \beta r - r\beta$ for all $r \in R$), where $\beta = e_{11}$ Then $S = R[x; \alpha, \delta_\beta]$ is an Ore extension of R.
Consider the principal left ideal Se_{13} generated by e_{13}. Since $xe_{13} = \alpha(e_{13})x + \delta_\beta(e_{13}) = e_{13}$. we get $x^n e_{13} = e_{13}$ for all $n = 1, 2, \ldots$ Also, since $(ae_{11} + be_{12} + ce_{13} + de_{22} + ee_{23} + fe_{33})e_{13} = ae_{13}$. we have $Se_{13} = \{ae_{13} \mid a \in Z_2\}$.

Note that all possible idempotents in S that generate $l_S(Se_{13})$ are $e_{22}+e_{33}$, $e_{12}+e_{22}+e_{33}$, $e_{13}+e_{22}+e_{33}$ and $e_{12}+e_{13}+e_{22}+e_{33}$. Directional calculation yields that $l_S(Se_{13})$ is not generated by an idempotent. So S is not left p.q.-Baer.

By Example 2, the answer to the question is "no" and so we can ask "under what conditions, the answer to the question is yes?". In (1), Armendariz proved that if R is a reduced ring, then R is a p.p.-ring if and only if the polynomial ring $R[x]$ is a p.p.-ring. We will generalize this results by showing that if R is a reduced ring. then R is a p.p.(resp. Baer)-ring if and only if the differential polynomial ring $R[x;\delta]$ is a p.p.(resp. Baer)-ring. Based on these facts, we have the following :

LEMMA 3. *Let R be a reduced ring with a derivation δ. Then for all a, b, c, and $d \in R$,*

(1) *$ab = 0$ if and only if $ba = 0$;*
(2) *If $ab = 0$ and $cb + ad = 0$, then $cb = ad = 0$;*
(3) *If $ab = 0$, then $\delta^k(a)\delta^{n-k}(b) = 0$ for all positive integers n and k $(0 \le k \le n)$;*
(4) *$\delta(e) = 0$ for all itempotent $e \in R$.*

PROOF. (1) is clear.
(2) If $ab = 0$ and $cb+ad = 0$, then $0 = (cb+ad)a = c(ba)+(ad)a = ada$ and so $ad = 0$. Hence $cb = 0$.
(3) If $ab = 0$, then $0 = \delta(ab) = a\delta(b)+\delta(a)b$. By (2), $a\delta(b) = \delta(a)b = 0$ If $a\delta(b) = 0$, then $\delta(a)\delta(b) = a\delta^2(b) = 0$ by (2). Similarly, if $\delta(a)b = 0$, then we also have $\delta^2(a)b = 0$. Continuing in this way, we get the results by induction on n.
(4) Let e be an idempotent of R. Then $\delta(e) = \delta(e^2) = e\delta(e) + \delta(e)e$ and then $e\delta(e) = e\delta(e) + e\delta(e)e$, and so $e\delta(e)e = 0$, which implies that $e\delta(e) = \delta(e)e = 0$. Hence we get $\delta(e) = 0$.

LEMMA 4. *Let R be a ring with a derivation δ. Then for each $a \in R$ and any positive integer n, we have $x^n b = \sum_{i=0}^{n} \binom{n}{i} \delta^i(b) x^{n-i}$ in the ring $R[x;\delta]$, where $\delta^0(b) = b$.*

PROOF. The proof is straightforward by an induction on n.

LEMMA 5. *Let R be a reduced ring with a derivation δ and let f and $g \in R[x;\delta]$ with $f = \sum_{i=0}^{n} a_i x^i$, $g = \sum_{i=0}^{m} b_i x^i$. Then $fg = 0$ if and only if $a_i b_j = 0$ for all i and j $(0 \le i \le n, 0 \le j \le m)$.*

PROOF. Suppose that $fg = 0$. We can assume that $m = n$ without loss of generality. Let $f_1 = \sum_{i=0}^{m-1} a_i x^i$ and $g_1 = \sum_{i=0}^{m-1} b_i x^i$. Then $0 = fg = (f_1 + a_m x^m)(g_1 + b_m x^m) = f_1 g_1 + f_1(b_m x^m) + (a_m x^m) g_1 + (a_m x^m)(b_m x^m)$. Let $f_1 g_1 = \sum_{i=0}^{2m-2} c_i x^i$ and $fg = \sum_{k=0}^{2m} d_k x^k$. Then by Lemma 4 we get the following equations

(i) For $0 \le k \le m-1$,

$$d_k = c_k + \sum_{i=0}^{k} \binom{m}{m-k+i} a_m \delta^{m-k+i}(b_i)$$

(ii) For $m \le k \le 2m$,

$$\begin{aligned} d_k = & c_k + \sum_{i=k-m}^{m-1} \binom{i}{i-(k-m)} a_i \delta^{i-(k-m)}(b_m) \\ & + \sum_{i=k-m}^{m} \binom{m}{i-(k-m)} a_m \delta^{i-(k-m)}(b_i). \end{aligned}$$

Observe that $c_{2m-1} = c_{2m} = 0$, $d_{2m-1} = a_{m-1} b_m + m a_m \delta(b_m) + a_m b_{m-1}$ and $d_{2m} = a_m b_m$. Since $fg = 0$, $d_k = 0$ for all $k (0 \le k \le 2m)$. We will proceed the proof by induction on m. First, if $m = 0$, then it is clear. Assume that $a_i b_j = 0$ for all i and j $(0 \le i, j \le m-1)$. By Lemma 4, $c_k = 0$, for all $k (0 \le k \le 2m-2)$. It is enough to show that $a_i b_m = a_m b_j = 0$ for all i and $j (0 \le i, j \le m)$. Since $d_{2m} = a_m b_m$, we have $0 = d_{2m-1} = a_{m-1} b_m + m a_m \delta(b_m) + a_m b_{m-1} = a_{m-1} b_m + a_m b_{m-1}$ by Lemma 3-(3), and also $a_{m-1} b_m = a_m b_{m-1} = 0$ by Lemma 3-(2). By continuing in this way and induction, we have the result.

Conversely, if $a_i b_j = 0$ for all i and j $(0 \le i, j \le m)$, then by Lemma 1 and equations (i) and (ii), $d_k = 0$ for all k $(0 \le k \le 2m)$. Hence $fg = 0$.

COROLLARY 6. *Let R be a ring with a derivation δ. Then R is a reduced ring if and only if $R[x;\delta]$ is a reduced ring.*

PROOF. Suppose that a ring R is reduced. If $f^2 = 0$ for any $f = a_0 + a_1x + \cdots + a_nx^n \in R[x;\delta]$, then by Lemma 5 $a_ia_j = 0$ for all i, j $(0 \leq i, j \leq n)$. In particular, $a_i^2 = 0$ for all i. Since R is reduced, $a_i = 0$ for all i. Hence $f = 0$ and so $R[x;\delta]$ is a reduced ring. The converse is clear.

COROLLARY 7. *If R is a reduced ring with a derivation δ and $f \in R[x;\delta]$ is an idempotent, then $f \in R$, that is, every idempotent of $R[x;\delta]$ is an idempotent of R.*

PROOF. Let $f = a_0 + a_1x + \cdots + a_nx^n \in R[x;\delta]$ be an idempotent. Then $0 = f - f^2 = f(1-f)$. By Lemma 5, $a_0(1-a_0) = 0$ and $a_i^2 = 0$ for each i $(1 \leq i \leq n)$, and we get $a_0 = a_0^2$ and so $a_i = 0$ for each i $(1 \leq i \leq n)$. Hence $f = a_0 \in R$.

COROLLARY 8. *Let R be a reduced ring with a derivation δ and let $T \subseteq R[X;\delta]$. If $S_f = \{a_0, a_1, \ldots, a_n\}$, where $f = a_0 + a_1x + \cdots + a_nx^n \in T$, then $r_{R[x;\delta]}(T) = r_R(S_T)[x;\delta]$, where $S_T = \cup_{f\in T} S_f$.*

PROOF. If $g = b_0 + b_1x + \cdots + b_mx^m \in r_{R[x;\delta]}(T)$, then $Tg = 0$, i.e., $fg = 0$ for all $f \in T$. By Lemma 5, $a_ib_j = 0$ for all i and j $(0 \leq i \leq m$, $0 \leq j \leq n)$, which implies that $b_j \in r_R(S_T)$, and so $g \in r_R(S_T)[x;\delta]$. Hence $r_{R[x;\delta]}(T) \subseteq r_R(S_T)[x;\delta]$. The other inclusion is obvious.

THEOREM 9. *Let R be a reduced ring with a derivation δ. Then $R[x;\delta]$ is a p.p.-ring if and only if R is a p.p.-ring.*

PROOF. ($\Longrightarrow$) If $R[x;\delta]$ is a p.p.-ring and $a \in R$, then $r_R(a) = R \cap r_{R[x;\delta]}(a) = R \cap eR[x;\delta]$ for some idempotent $e \in R[x;\delta]$. By Corollary 7, $e \in R$, and so $r_R(a) = eR$. Hence R is a p.p.-ring.
($\Longleftarrow$) Assume that R is a p.p. ring. Note that for any finite subset T of R, $r_R(T) = eR$ for some idempotent $e \in R$. If $f \in R[x;\delta]$, then by Corollary 8, $r_{R[x;\delta]}(S_f) = r_R(S_f)[x;\delta] = eR[x;\delta]$ for some idempotent $e \in R$ because S_f is a finite subset of R and e is central. Hence $R[x;\delta]$ is a p.p. ring.

Since a p.p.-ring is equivalent to a p.q.-Baer ring for a reduced ring, we have the following;

COROLLARY 10. *Let R be a reduced ring with a derivation δ. Then $R[x;\delta]$ is a left(right) p.q.-Baer ring if and only if R is a left(right) p.q.-Baer ring.*

Similarly we can also have the following Theorem;

THEOREM 11. *Let R be a reduced ring with a derivation δ. Then $R[x;\delta]$ is a Baer ring if and only if R is a Baer ring.*

PROOF. ($\Longrightarrow$) If $R[x;\delta]$ is Baer, then for any subset T of R, $r_{R[x;\delta]}(T) = fR[x;\delta]$ for some idempotent $f \in R[x;\delta]$. By Corollary 7, $f \in R$, and then $r_R(T) = R \cap r_{R[x;\delta]}(T) = R \cap fR[x;\delta] = fR$. Hence R is a Baer ring
($\Longleftarrow$) Suppose that R is Baer and let S be an arbitrary nonempty subset of $R[x;\delta]$ Let $T = \cup_{f\in S} S_f$ Since R is Baer, $r_R(T) = eR$ for some idempotent $e \in R$. By Corollary 8, $r_{R[x;\delta]}(S) = r_R(T)[x;\delta] = eR[x;\delta]$. Thus $R[x;\delta]$ is Baer.

Theorems 9 and 11 extend Armendariz's results[1, Theorem A and B]. Also, for a reduced ring R, the following are equivalent clearly:

(1) R is a Baer ring.
(2) R is a quasi-Baer ring.

Hence we have the following:

COROLLARY 12

Let R be a reduced ring with a derivation δ. Then $R[x;\delta]$ is a quasi-Baer ring if and only if R is a quasi-Baer ring.

PROOF. ($\Longrightarrow$) Assume that $R[x;\delta]$ is a quasi-Baer ring. Since R is a reduced ring, $R[x;\delta]$ is a reduced ring by Corollary 6 and a Baer ring by above note According to Theorem 10, R is a Baer ring and hence is a quasi-Baer ring.
($\Longleftarrow$) If R is a quasi-Baer and reduced ring, then $R[x;\delta]$ is a reduced ring by Corollary 6 and a Baer ring by above note Hence $R[x;\delta]$ is a quasi-Baer ring

Finally we may raise the following question;

QUESTION. *For an abelian ring, are the results in this paper true?*

REFERENCES

[1] E. P. Armendariz, *A note on extensions of Baer and p.p. rings*, Australian Math. Soc. **18** (1974), 470 – 473.

[2] E. P. Armendariz, H. K. Koo and J. K. Park, *Isomorphic Ore extensions*, Comm. in Algebra. **15(12)** (1987), 2633 – 2652

[3] G. F. Birkenmeier, J. Y. Kim and J. K. Park, *On extensions of quasi-Baer and principally quasi-Baer rings, preprint.*

[4] K. R. Goodearl, and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, Cambridge. 1989.

[5] J. C. McConnell, and J. C. Robson, *Noncommutative Noetherian rings*, John Wiley & Sons Ltd. 1987.

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