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PRIMITIVE POLYNOMIAL RINGS

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ABSTRACT. We show that the intersection of two standard torus knots of type (λ_1, λ_2) and (β_1, β_2) induces an automorphism of the cyclic group $\mathbb{Z}_d$, where d is the intersection number of the two torus knots and give an elementary proof of the fact that all non-trivial torus knots are strongly invertiable knots. We also show that the intersection of two standard knots on the 3-torus $S^1 \times S^1 \times S^1$ induces an isomorphism of cyclic groups

Throughout this paper all rings are associative with identity. Given a ring R, $R[x]$ denotes the polynomial ring over R with x its indeterminate. In this note we study the primitivity of polynomial rings, concerning the contraposition of the condition in [8] that is both a Morita invariant property and a generalization of the following two conditions:

(1) the quasi-duo condition, which was initiated by Yu in [9] and is related to the Bass' conjecture in [2],

(2) the pm condition that was studied by Birkenmeier-Kim-Park in [3].

A ring R is called *maximally right bounded* if every maximal right ideal of R contains a maximal ideal of R. Consider a condition: (*)

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there exists a maximal right ideal that does not contain a maximal ideal. Clearly a ring R satisfies (*) if and only if R is not maximally right bounded. A ring is called *right (left) duo* if every right ideal is two-sided, and a ring is called *right (left) quasi-duo* if every maximal right (left) ideal of is two-sided. Commutative rings and abelian regular rings are right duo, right duo rings are right quasi-duo, and right quasi-duo rings are maximally right bounded. The n by n full matrix ring over a division ring, with n any positive integer $\geqq 2$, is not right quasi-duo and does not satisfies (*); but it is maximally right bounded However the ring of row finite infinite matrices over a division ring, say R, satisfies (*) but is not maximally right bounded because there exist maximal right ideals of R that do not contain the nonzero proper ideal $\{f \in R \mid \text{rank}(f) \text{ is finite }\}$ of R. A ring R is said to satisfy *pm* if every prime ideal of R is maximal. Such rings are maximally right bounded, but there are rings which are maximally right bounded but do not satisfy pm as in [3, Example 3 3] In this note we also obtain direct proofs for the contrapositions of main results in [8]

We first take the contraposition of [8, Proposition 1] as follows.

PROPOSITION 1. *Given a ring R the following statements are equivalent:*

(1) R satisfies (*).

(2) There exists a right primitive ideal of R that is not maximal.

PROOF. (1)⇒(2). Since R satisfies (*), there exists a maximal right ideal M of R that does not contain a maximal ideal of R. But M contains a right primitive ideal of R which is the bound of M, say P. Thus P is not a maximal ideal of R.

(2)⇒(1). Let P be a right primitive ideal of R that is not maximal. There is a maximal right ideal of R whose bound is P, so R satisfies (*).

COROLLARY 2. *[8, Proposition 1] Given a ring R the following statements are equivalent:*

(1) R is a maximally right bounded ring.

(2) Every right primitive ideal of R is maximal.

PROOF. By Proposition 1

We next recall some properties of maximally right bounded rings in [8]. A ring R is called a *PI-ring* if R satisfies a polynomial identity with coefficients in the ring of integers.

LEMMA 3. *[8, Corollary 2, Corollary 3 and Lemma 4] Given a ring R we have the following statements:*

(1) If every right primitive factor ring of R is artinian then R is maximally right bounded

(2) If R is a PI-ring then R is maximally right bounded.

(3) If R is a division ring that is finite dimensional over its center then $R[x]$ is maximally right bounded.

(4) A semiprimitive maximally right bounded ring is a subdirect product of simple rings.

(5) If a ring R is maximally right bounded, then so is every homomorphic image of R.

Note that if given a ring R is a right primitive, then eRe is also a right primitive ring for every nonzero idempotent $e \in R$. The following is one of our main results in this note.

THEOREM 4. *Let R be a ring and $0 \neq e^2 = e \in R$. Suppose that $eIe \subsetneqq eRe$ for each proper ideal I of R. Then the following statements are equivalent:*

(1) R satisfies (*).

(2) eRe satisfies (*).

PROOF. (2)⇒(1) By [8, Lemma 7]

(1)⇒(2). We use the proof of [8, Theorem 8]. Let I be a maximal right ideal of R whose bound is P, such that P is not maximal Then P is a right primitive ideal of R. We will show that ePe is not a maximal ideal in eRe. For convenience, let $\overline{R} = R/P$, and $\bar{r} = r+P$ for all $r \in R$ Then $\overline{R}$ is a right primitive ring. Since $ePe = eRe \cap P$ and $ePe \neq eRe$ by hypothesis, we have $e \notin P$ and hence $\bar{e}$ is a nonzero idempotent in $\overline{R}$. Thus $\bar{e}\overline{R}\bar{e}$ is also a right primitive ring. Since $eRe/ePe \cong \bar{e}\overline{R}\bar{e}$, ePe is a right primitive ideal of eRe Now let Q be a maximal ideal of R that contains P (of course $P \subsetneq Q$). Then $ePe \subseteq eQe \subsetneqq eRe$ by hypothesis and eQe is maximal in eRe by Lemma 2.6. Assume $ePe = eQe$. Then $eQe = ePe \subseteq P$, and hence $(Re)Q(Re) = R(eQe) \subseteq RP = P$. Since P is right primitive and $e \notin P$, we get $Q \subseteq P$, a contradiction to the fact that $P \subsetneq Q$. Therefore $ePe \subsetneq eQe$ and this completes the proof.

COROLLARY 5. *[8, Theorem 8] Let R be a ring and $0 \neq e^2 = e \in R$. Suppose that $eIe \subsetneq eRe$ for each proper ideal I of R. Then the following statements are equivalent:*

(1) R is maximally right bounded.

(2) eRe is maximally right bounded.

We may compare the following result with [9, Proposition 2.1].

PROPOSITION 6. *For a ring R the following statements are equivalent:*

(1) R satisfies (*).

(2) Every n by n upper triangular matrix ring over R satisfies (*).

(3) Every n by n lower triangular matrix ring over R satisfies (*), where n is any finite (in this case assume $n \geq 2$) or an infinite cardinal number.

PROOF. We use the proofs of [8, Corollary 9]. (1)$\Rightarrow$(2). Let S be the n by n upper triangular matrix ring over R. Note that every right primitive ideal J of S is of the form. the (i, i)-entry of J is a right primitive ideal of R for some $i \in \{1, 2, \dots\}$. say P, and every other entry of J is R. By Proposition 1 and the condition (1), we may take a right primitive ideal P in R that is not a maximal ideal of R. So J is not maximal in S and this gives (2).

(2)$\Rightarrow$(1). Let e be the matrix such that $(1, 1)$-entry of e is 1_R and other entries of e are 0_R. Then $0 \neq e^2 = e \in S$ and $eSe \cong R$. So R satisfies (*) by the condition (2) and [8, Lemma 7].

We next obtain the equivalence (1)$\Leftrightarrow$(3) by the symmetry.

COROLLARY 7. *[8, Corollary 9] For a ring R the following statements are equivalent:*

(1) R is maximally right bounded.

(2) Every n by n upper triangular matrix ring over R is maximally right bounded.

(3) Every n by n lower triangular matrix ring over R is maximally right bounded, where n is any finite or an infinite cardinal number.

We denote the n by n full matrix ring over a ring R by $Mat_n(R)$ for any positive integer n.

LEMMA 8. *[8, Corollary 24] For a ring R and any positive integer n, the following statements are equivalent:*

(1) R is maximally right bounded.

(2) $Mat_n(R)$ is maximally right bounded.

By Lemma 8, we have the following equivalence for rings that satisfy (*).

COROLLARY 9. *For a ring R and any positive integer n, the following statements are equivalent:*

(1) R satisfies (*).

(2) $Mat_n(R)$ satisfies (*).

Therefore we have the following by Theorem 4, Corollary 9 and [1, Corollary 22.7].

COROLLARY 10. *Suppose that a ring R satisfies (*). Then for every finitely generated projective right R-module P, $End_R(P)$ also satisfies (*); especially the condition (*) is a Morita invariant property, where $End_R(P)$ is the endomorphism ring of P over R.*

Next we study the primitivity of polynomial rings over division rings First we observe the polynomial rings over rings satisfying (*).

PROPOSITION 11 *If a ring R satisfies (*), then $R[x]$ satisfies (*).*

PROOF. Notice first that $I + R[x]x$, with I a right primitive ideal of R. is also a right primitive ideal of $R[x]$. Since R satisfies (*), we may take I such that I is not a maximal ideal. So $I + R[x]x$ is also not a maximal ideal of $R[x]$ but a right primitive ideal of $R[x]$; hence $R[x]$ satisfies (*) by Proposition 1.

As the converse of Proposition 11, we may raise the following question.

Question. Does a ring R satisfy (*) if $R[x]$ satisfies (*)?

However the answer is negative by the following example.

EXAMPLE 12. Let $W = W_1[\mathbb{Q}]$ be the first Weyl algebra over the field $\mathbb{Q}$ of rational numbers, subject to $yx = xy + 1$, and let R be the right quotient division ring of W. Then the center of R is $\mathbb{Q}$, and since R is purely transcendental over $\mathbb{Q}$, it follows that $A = R \otimes_{\mathbb{Q}} \mathbb{Q}(t)$ is not a division ring by [5, Theorem 3. 21], where $\mathbb{Q}(t)$ is the quotient field of the polynomial ring $\mathbb{Q}[t]$ in an indeterminate t. Hence $A \neq R(t)$; so $R[t]$ is right primitive by [5, Theorem 3. 21], where $R[t]$ is the polynomial ring over R in t and $R(t)$ is the right quotient division ring of $R[t]$. Clearly R does not satisfy (*). But the zero ideal of $R[t]$ is right primitive which is not maximal. Therefore $R[t]$ satisfy (*) by Proposition 1.

The following is also one of our main results in this paper.

THEOREM 13. *For a simple ring R the following statements are equivalent:*
(1) $R[x]$ satisfies (*).
(2) $R[x]$ is right primitive.

PROOF. (2)$\Rightarrow$(1). Note that the zero ideal of $R[x]$ is always not maximal. Since $R[x]$ is right primitive by the condition, $R[x]$ satisfies (*) by Proposition 1.

(1)$\Rightarrow$(2). Suppose that the condition (1) holds. Then there is a right primitive ideal P of $R[x]$ that is not maximal by Proposition 1. Let M be a maximal ideal of $R[x]$ such that $P \subsetneq M$. Here assume $P \neq 0$. Then [8, Lemma 15] implies that P is generated by a nonzero central monic polynomial in $R[x]$ because R is simple by hypothesis, say $P = f(x)R[x]$. Also by [8, Lemma 15], $M = h(x)R[x]$ for some nonzero central monic polynomial $h(x) \in R[x]$. Since M contains P, $f(x) = h(x)g(x)$ for some $g(x) \in R[x]$ and so $P = f(x)R[x] = h(x)R[x]g(x)R[x]$. But P is right primitive (hence prime), so $M = h(x)R[x] \subseteq P$ (a contradiction to the fact that $P \subsetneq M$)

or $g(x)R[x] \subseteq P$. If $g(x)R[x] \subseteq P$, then $g(x) = f(x)m(x)$ for some $m(x) \in R[x]$ and so $f(x) = h(x)f(x)m(x) = f(x)h(x)m(x)$. It then follows that $h(x)m(x) = m(x)h(x) = 1_{R[x]}$ since $f(x)$ is monic; hence $M = R[x]$, a contradiction to the fact that M is a maximal ideal of $R[x]$. Consequently P must be the zero ideal and therefore $R[x]$ is right primitive.

By Theorem 13, we obtain the following result.

COROLLARY 14. *[8, Theorem 16] For a simple ring R the following statements are equivalent:*

(1) $R[x]$ is maximally right bounded.

(2) $R[x]$ is not right primitive.

We do not know whether the condition (*) is left-right symmetric. But if R is a division ring, then $R[x]$ satisfies (*) if and only if $R[x]$ satisfies the "left-handed" version of (*) as in the following.

COROLLARY 15. *Let R be a division ring. Then the following statements are equivalent:*

(1) $R[x]$ satisfies (*).

(2) $R[x]$ is right primitive.

(3) $R[x]$ is left primitive.

(4) $R[x]$ satisfies the left version of (*).

PROOF. By [8, Lemma 18] and Theorem 13.

Due to Jacobson [7], a ring is called *strongly right (left) bounded* if every nonzero right (left) ideal contains a nonzero ideal; and a ring is called *right (left) bounded* if every essential right (left) ideal contains a nonzero ideal. Strongly right bounded rings are clearly right bounded. In [4], we have that a ring R is right duo if and only if every factor ring of R is strongly right bounded. In the following arguments we obtain the connections among the preceding conditions, right duoness, maximally right boundedness and the condition (*).

LEMMA 16. *[6, Theorem 15.2] Let R be a simple Artinian ring. Then the following statements are equivalent:*

(1) $R[x]$ is right bounded.

(2) $R[x]$ is not right primitive.

A ring R is called right *Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is a well-known fact that R is a right Ore ring if and only if there exists the classical right quotient ring of R. Left case may be defined similarly. Given a division ring D, $D[x]$ is an Ore (i.e., both right and left Ore) domain, so every nonzero right (left) ideal is essential; hence $D[x]$ is strongly right (left) bounded if and only if it is right (left) bounded. Consequently we have the following results.

PROPOSITION 17. *Let R be a simple Artinian ring. Then the following statements are equivalent:*

(1) $R[x]$ is right bounded.

(2) $R[x]$ is not right primitive.

(3) $R[x]$ is maximally right bounded.

PROOF. By Corollary 14 and Lemma 16.

COROLLARY 18. *Let D be a division ring. Then the following statements are equivalent:*

(1) $D[x]$ is strongly right bounded.

(2) $D[x]$ is right bounded.

(3) $D[x]$ is not right primitive.

(4) $D[x]$ is maximally right bounded.

(5) The left versions of the statements (1)-(4) .

PROOF. By Corollary 15, Proposition 17 and the argument prior to Proposition 17.

There exists a division ring that does not satisfy the statements in Corollary 18. Let R be the Weyl division algebra over a field of characteristic zero. Then $R[x]$ is right primitive by [6, Theorem 15.16].

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