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ON SUBMODULES INDUCING PRIME IDEALS OF ENDOMORPHISM RINGS

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Abstract In this paper, for any ring R with an identity, in order to study prime ideals of the endomorphism ring $End_R(M)$ of left $R-$module ${}_RM$, meet-prime submodules, prime radical, sum-prime submodules and the prime socle of a module are defined Some relations of the prime radical, the prime socle of a module and the prime radical of the endomorphism ring of a module are investigated It is revealed that meet-prime(or sum-prime) modules and semi-meet-prime(or semi-sum-prime) modules have their prime, semi-prime endomorphism rings, respectively.

0. Introduction

For an associative ring R and any left $R-$module ${}_RM$, its endomorphism ring $End_R(M)$ will act on the right side of ${}_RM$. in other words. ${}_RM_{End_R(M)}$ will be studied mainly. Thus the composite of endomorphisms preserves the order such that the composite $fg : M \to M$ of $f : M \to M$ and $g : M \to M$ defined by $mfg = (mf)g$ for every $m \in M$.

For any submodule $N \leq {}_RM$, we have ideals:

$$(N : M) = \{r \in R \mid rM \subseteq N\} \trianglelefteq R$$

$$Ann_R(N) = (0 : N) = \{r \in R \mid rN = 0\} \trianglelefteq R$$

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of R and a left and a right ideals:

$$I^N = \{f \in End_R(M) \mid Imf = Mf \leq N\} \trianglelefteq_l End_R(M)$$

$$I_N = \{f \in End_R(M) \mid N \leq kerf\} \trianglelefteq_r End_R(M)$$

of the endomorphism ring $End_R(M)$, briefly denoted by S. For any subset J of S, let $ImJ = \sum_{f\in J} Imf = \sum_{f\in J} Mf$ and $kerJ = \cap_{f\in J} kerf$ be the sum of images of endomorphisms in J and the intersection of kernels in J, respectively. Also we call N an *open* submodule if $N = N^o$, where $N^o = \sum_{Imf\leq N} Imf,\ f \in S$, is the sum of images of endomorphisms contained in N and call N a *closed* submodule if $N = \overline{N}$, where $\overline{N} = \cap_{N\leq kerf} kerf,\ f \in S$, is the intersection of kernels of endomorphisms containing N.

Here is the definition of a prime submodule that McCasland and Moore set up in their paper "Prime Submodules", 1992 [4].

For a proper submodule $P \leq {}_RM$, if $rm \in P (r \in R$ and $m \in M)$ implies that either $m \in P$ or $r \in (P : M)$, then P will be said to be *prime* in M.

LEMMA 0.1. *[4] Let R be any ring and M any $R-$module. Then a submodule $N \leq M$ is prime if and only if $P = (N : M)$ is a prime ideal of R and the $(R/P)-$module M/N is torsion-free.*

Comparing the definition of prime submodule with the following definition of meet-prime submodule, it looks very different. But for any integral domain R with identity if ${}_RM$ is a multiplication module([3]), it follows immediately from the Lemma 0.1 and the Corollary 3.3 that P is a prime submodule of ${}_RM \Longleftrightarrow P$ is meet-prime in ${}_RM$ defined in the next section §1. Also it follows immediately from the Lemma 0.1 and the Corollary 3.7 that P is a prime submodul e of ${}_RM \Longleftrightarrow P$ is sum-prime in ${}_RM$ defined in the next section §3.

1. Meet-Prime Submodules

It isn't easy to see the structure of prime ideals of R, S and the structures of prime submodule of ${}_RM$. In addition, there are operations $+$ and $\cap$ on the family of all submodules of ${}_RM$. Using the fact that under these $+$ and $\cap$, the family of all submodules of ${}_RM$ is closed. from the structure of submodules there may be some methods to find prime ideals of S and those of R.

The following definition is one of methods to see relations between submodules of ${}_RM$ and prime ideals of the ring R and the endomorphism ring S.

DEFINITION 1.1. For a submodule $P \leq {}_RM$ of a left $R-$module ${}_RM$, we will say that P is a meet-prime submodule of M if it satisfies the following conditions

For any open submodules $A, B \leq M$ with $P^o + A \neq M$ or $P^o + B \neq M$,

(1) if $A \cap B \leq P$, then $A \leq P$ or $B \leq P$,
(2) if $(P \cap A \cap B)^o \neq 0$, then $A \leq P$ or $B \leq P$,
(3) if $P \cap A = 0$, then $A = 0$ or $P + A = M$.

Trivially every module M is meet-prime in ${}_RM$.

For example, we have that any prime ideal $\langle p \rangle$ with prime p of a commutative integer ring $\mathbb{Z}$ is a meet-prime submodule of a left $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Z}$. Clearly the zero submodule of any simple module is meet-prime.

REMARK 1.2. Every meet-prime submodule is not maximal, in general.

Since the non-maximal submodule $p\mathbb{Z}[x] \leq {}_{\mathbb{Z}[x]}\mathbb{Z}[x]$(for prime p) is meet-prime.

PROPOSITION 1.3. *For any left $R-$module ${}_RM$, we have the following.*

(1) *For distinct nonzero open meet-prime submodules P and Q of a left $R-$module ${}_RM$, it follows that $P + Q = M$.*
(2) *For a submodule $P \leq {}_RM$, P^o is meet-prime if and only if P is meet-prime.*

PROOF. (1) Assume $P + Q \neq M$. Since $P \cap (P + Q) = P \neq 0$ and $Q \cap (P + Q) = Q \neq 0$, the meet-primenesses of P and Q and the openness of $P + Q$ implies $P = Q$. Therefore we have $P + Q = M$.

(2) Since for any open submodule $U \leq M$, $U \leq P \Longleftrightarrow U \leq P^o$ and $(P^o)^o = P^o$. The proof is easy from the definition of *meet-prime* submodule.

REMARK 1.4. In a $\mathbb{Z}$–left module ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ every proper submodule is meet-prime since ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ has a unique zero open submodule of it, in other words, no nontrivial submodule is open in ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ telling that 0 is a unique proper open meet-prime submodule.

The following are some criterior of meet-prime submodules of a module.

A submodule $P \leq {}_RM$ is maximal among open submodules whenever $K \leq M$ is open such that $P \leq K$, then $P = K$ or $K = M$ follows

LEMMA 1.5. If a submodule $P \leq {}_RM$ is maximal among open submodules, then P is meet-prime

PROOF. From the maximality of open submodule $P = P^o$ among the open submodules of M, the proof is easy.

It is well-known that not every module has a maximal submodule.

COROLLARY 1.6. *For any left R–module ${}_RM$ we have the following.*

(1) *If $P \leq {}_RM$ is any maximal submodule of M, then P is a meet-prime submodule of M.*
(2) *If $I^P = J \lhd S$ is a maximal ideal of S, then P is a meet-prime submodule of M.*
(3) *For an ideal $J \lhd S$, if the ideal I^{MJ} is a maximal ideal of S, then MJ is a fully invariant open meet-prime submodule.*

PROOF. The proof is established easily.

2. Sum-Prime Submodules

As a dual way of meet-primeness of submodules of ${}_RM$, the following definition is one of methods to see relations between submodules of ${}_RM$, prime ideals of the ring R, and the endomorphism ring S.

DEFINITION 2.1. For a submodule $P \leq {}_RM$ of a left R−module ${}_RM$, we will say that P is a sum-prime submodule of M if it satisfies the following conditions:

For any *closed* submodules $A, B \leq M$ with $\overline{P} \cap A \neq 0$ or $\overline{P} \cap B \neq 0$

(1) if $P \leq A + B$, then $P \leq A$ or $P \leq B$,
(2) if $\overline{P + A + B} \neq M$, then $P \leq A$ or $P \leq B$.
(3) if $P + A = M$, then $A = M$ or $P \cap A = 0$

Trivially the zero submodule 0 is sum-prime in ${}_RM$.

For example, we have that a prime ideal $\overline{2}\mathbb{Z}_6 = \overline{4}\mathbb{Z}_6 \trianglelefteq {}_{\mathbb{Z}_6}\mathbb{Z}_6$ of a commutative ring $\mathbb{Z}_6$ is a sum-prime submodule of a left $\mathbb{Z}_6$−module ${}_{\mathbb{Z}_6}\mathbb{Z}_6$.

REMARK 2 2 The submodule $\{\overline{0}, \overline{1/p}, \overline{2/p}, \cdots, \overline{(p-1)/p}\}$ is a sum-prime submodule of ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ with a prime number p However the submodule $\{\overline{0}, \overline{1/p}, \overline{2/p}, \cdots, \overline{(p-1)/p}, \cdots, \overline{1/p^n}, \overline{2/p^n}, \cdots, \overline{(p-1)/p^n}\}(n \in \mathbb{N}, n \geq 2)$ is not a sum-prime submodule of it.

Every sum-prime submodule is not minimal. in general. Each non-minimal submodule $n\mathbb{Z} \leq {}_{\mathbb{Z}}\mathbb{Z}(0 \neq n \in \mathbb{Z})$ of an integer module ${}_{\mathbb{Z}}\mathbb{Z}$ is sum-prime.

PROPOSITION 2 3 *For any left R−module ${}_RM$, we have the following:*

(1) *For distinct proper closed sum-prime submodules P and Q of a left R−module ${}_RM$, it follows that $P \cap Q = 0$.*
(2) *For a submodule $P \leq {}_RM$, $\overline{P}$ is sum-prime if and only if P is sum-prime.*

PROOF

(1) Assume $P \cap Q \neq 0$. Since $P + (P \cap Q) = P \neq M$ and $Q + (P \cap Q) = Q \neq M$, the sum-primenesses of P and Q and the closedness of $P \cap Q$ implies $P = Q$ Therefore we have $P \cap Q = 0$
(2) Since for any closed submodule $F \leq M$, $P \leq F \iff \overline{P} \leq F$ and $\overline{\overline{P}} = \overline{P}$. The proof is easy from the definition of sum-prime submodule.

REMARK 2.4. In a $\mathbb{Z}$−left module ${}_{\mathbb{Z}}\mathbb{Z}_p \oplus \mathbb{Z}_p$ for prime p the closed sum-prime submodules which are not fully invariant are $0 \oplus \mathbb{Z}_p$ and $\mathbb{Z}_p \oplus 0$. And ${}_{\mathbb{Z}}\mathbb{Z}_p \oplus \mathbb{Z}_p$ is a unique closed fully invariant sum-prime submodule.

The following are some criterion of sum-prime submodules of a module.

A submodule $P \leq {}_RM$ is minimal among closed submodules whenever $K \leq M$ is closed such that $K \leq P$, then $P = K$ or $K = 0$ follows.

LEMMA 2.5. *If a submodule $P \leq {}_RM$ is minimal among closed submodules, then P is sum-prime.*

PROOF. From the minimality of closed submodule $P = \overline{P}$ among the closed submodules of M. the proof is easy.

It is well-known that not every module has a minimal submodule.

COROLLARY 2.6. *For any left $R-$module ${}_RM$ we have the following.*

(1) *If $P \leq {}_RM$ is any minimal submodule of M, then P is a sum-prime submodule of M.*
(2) *If $I_P = J \lhd S$ is a minimal ideal of S, then P is a sum-prime submodule of M.*
(3) *For an ideal $J \lhd S$, if the ideal I_{kerJ} is a minimal ideal of S, then $kerJ = \bigcap_{f \in J} kerf$ is a closed fully invariant sum-prime submodule.*

PROOF. The proof is established easily.

3. Meet-Prime, Sum-Prime Submodules, and Prime Ideals

For any fully invariant submodule $N \leq M$ we have two-sided ideals $(N : M) \trianglelefteq R$ and $I^N \trianglelefteq S$. Thus it may be found relations between meet-prime submodules of ${}_RM$ and prime ideals of S(or R). Here are some results.

PROPOSITION 3.1. *If P is a fully invariant meet-prime submodule of ${}_RM$, then I^P is a prime ideal of S.*

Proof Since P is meet-prime if and only if P^o is meet-prime by (2) of the Proposition 1.3 it suffices to show that for any fully invariant open meet-prime submodule $P \leq M, I^P$ is a prime ideal of S. Assume that P is a fully invariant open meet-prime submodule

Let $IJ \subseteq I^P = \{f \in S \mid Mf \leq P\}$ for subsets $I, J \subsetneq S = End_R(M)$.

(I) First if $P = 0$, then $MIJ = 0$. If $MI \cap MJ = 0 \leq P = 0$, then $MI = 0$ or $MJ = 0$ follows from the meet-primness of $P = 0$ Or if $MI \cap MJ \neq 0$, then $P \cap (MI \cap MJ) = 0$ and by (3) of the definition 1.1 we have $MI = MJ = M$ which cann't make $MIJ = 0$. Therefore $I = 0$ or $J = 0$ follows. Hence $I^P = 0$ is a prime ideal of S.

(II) Secondly let P be a nonzero submodule of M. Then we have the following cases (i) and (ii) to be concerned. Here let $A = MJ$ and $B = P + MI$ as open submodules of ${}_RM$.

(i) for the case of $P + B = M$, from the fully invariantness of P it follows that

$$MJ = (P + MI)J \leq PJ + MIJ \leq P + P = P. \qquad (*)$$

Hence we have that $J \subseteq I^P$.

(ii) for the case of $P + B \neq M$, we have to consider the following cases (a) and (b).

(a) if $(P \cap A \cap B)^o = P \cap A \cap B \neq 0$, then from (2) of the definition of meet-prime submodule P, it follows that

$$A = MJ \leq P \text{ or } B = P + MI \leq P, \text{ implying that } I \subseteq I^P \text{ or } J \subseteq P.$$

(b) if $(P \cap A \cap B)^o = P \cap A \cap B = 0$, then $P \cap (P + A \cap B) = P \neq 0$ with $P + A \cap B \neq M$ (otherwise, $P + A \cap B = M$ implies $P + B = M$ contradicted to $P + B \neq M$) Then by the meet-primeness of P we have that $P + A \cap B \leq P$ Hence $A \cap B \leq P$ and $A \leq P$ or $B \leq P$ follows from (1) of the definition 1.1 Therefore $I \subseteq I^P$ or $J \subseteq I^P$ for any $IJ \subseteq I^P$ and for $I, J \subsetneq S$. Therefore the ideal I^P is a prime ideal of S.

REMARK 3.2. The fully invariantness of a meet-prime submodule P of $_RM$ inducing a prime ideal I^P of S is essential. For example, on a $\mathbb{Z}$–module $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ there are meet-prime submodules $2\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 0$, the one is fully invariant and the other is not, which induce a prime ideal $I^{2\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3} \trianglelefteq' S$ and a non-prime ideal $I^{\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 0} \trianglelefteq S$.

The converse of the Proposition 3.1 is not true in general. For example. the trivial ideal $0 = I^0 \trianglelefteq' End_{\mathbb{Z}[x]}(\mathbb{Z}[x])$ is a prime ideal of $End_{\mathbb{Z}[x]}(\mathbb{Z}[x])$.

COROLLARY 3.3. *If P is a meet-prime submodule of $_RM$ over a commutative ring R with identity, then $(P : M)$ is a prime ideal of R.*

PROOF. Since R is a commutative ring with identity it follows immediately that for any $r \in R$, $rM = M\rho(r) \leq M$ is an open submodule of $_RM$, where $\rho(r) : M \to M$ is defined by $m\rho(r) = rm$ for each $m \in M$ And hence $(P : M)M$ is also an open submodule of M.

For any $rs \in (P : M)$ with $r, s \in R$, let $A = rM$ and $B = (P : M)M + sM$. Then by the same method of the proof of the Proposition 3.1 it follows that $r \in (P : M)$ or $s \in (P : M)$. Hence $(P : M)$ is a prime ideal of R.

COROLLARY 3.4. *If P is a fully invariant meet-prime submodule of $_RM$ over a commutative ring R with identity, then $(P : M)$ and the ideal I^P are prime ideals of R and S, respectively*

PROOF. By the Proposition 3.1 and the Corollary 3 3 it is proved

For any fully invariant submodule $N \leq M$ we have two-sided ideals $I_N \trianglelefteq S$ and $(0 : M) = Ann_R(K) \trianglelefteq R$. Thus it certainly can be found relations between sum-prime submodules of $_RM$ and prime ideals of S and R. Here is some result.

PROPOSITION 3.5. *If P is a fully invariant sum-prime submodule of $_RM$, then I_P is a prime ideal of S.*

PROOF. Since P is sum-prime if and only if $\overline{P}$ is sum-prime by (2) of the Proposition 2.3 it suffices to show that for any fully invariant closed sum-prime submodule P, I_P is a prime ideal of S. Assume that P is a fully invariant closed sum-prime submodule of M.

Let $IJ \subseteq I_P = \{f \in S \mid P \leq kerf\}$ for subsets $I, J \subseteq S = End_R(M)$.

(I) If $P = M$, then $kerIJ = M$. If $M = kerI + kerJ$, then $M \leq kerI$ or $M \leq kerJ$ from the sum-primeness of $P = M$. If $M \neq kerI + kerJ$, then by (3) of the definition 2.1 it follows that $kerI + kerJ = M$ contradicted. Therefore $I_M = 0$ is a prime ideal of S.

(II) For $P \neq M$, let $A = P \cap kerI$ and $B = P \cap kerJ$ as closed submodules of ${}_RM$.

The cases of (i) $P \cap B = 0$ and (ii) $P \cap B \neq 0$ must be considered

(i) for $P \cap B = 0$, we have $P = 0$, then from the equation

$$0 = P \geq P \cap kerIJ \geq \cap_{h \in I} h^{-1}(P \cap kerJ) = \cap_{h \in I} h^{-1}(0) = kerI \quad (*)$$

we have that $kerI = 0$ Thus $I \subseteq I_P = I_0$ follows

(ii) for the case of $P \cap B \neq 0$. $P \cap B = P \cap kerJ \neq 0$ and $P + kerI + P \cap kerJ \leq kerIJ$.

(a) If $kerIJ \neq M$, then from (2) of the sum-prime submodule P it follows that $P \leq kerI$ or $P \leq P \cap kerJ$. Hence I or $J \subseteq I_P$.

(b) If $kerIJ = M$. Then the following cases :

(α) $P + kerI + P \cap kerJ = P + kerI = M$ or

(β) $P + kerI + P \cap kerJ = P + kerI \neq M$

are concerned. Since P is a sum-prime submodule, with (β) we have that $P \leq kerI$ or $P \leq P \cap kerJ \leq kerJ$. Thus $I \subseteq I_P$ or $J \subseteq I_P$ follows

It remains to deal with (α) $P + kerI = M$ Then by the sum-primeness of P we have that $kerI = M$ or $P \cap (kerI + P \cap kerJ) = 0$, respectively. Thus $I = 0 \subseteq I_P$ only follows. Otherwise if $P \cap (kerI + P \cap kerJ) = 0$, then $P \cap kerI = 0 = P \cap kerJ$ implies $P \not\leq kerIJ = M$ which contradicts to $P \leq kerIJ = M$.

Therefore the ideal I_P is a prime ideal of S.

REMARK 3.6. The fully invariantness of a sum-prime submodule P inducing a prime ideal I_P of S is essential. For example, on $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ there are sum-prime submodules $4\mathbb{Z}_8 \oplus 0 \oplus 0$ and $0 \oplus \mathbb{Z}_3 \oplus 0$, the one is fully invariant and the other is not, which induce a prime ideal $I_{4\mathbb{Z}_8\oplus 0\oplus 0} \trianglelefteq S$ and a non-prime ideal $I_{0\oplus\mathbb{Z}_3\oplus 0} \trianglelefteq S$.

The converse of the Proposition 4.1 is not true in general. For example, the trivial ideal $0 = I_M \trianglelefteq End_{\mathbb{Z}}(\mathbb{Z}(p^\infty))$ is a prime ideal of $End_{\mathbb{Z}}(\mathbb{Z}(p^\infty))$, however the trivial submodule $\mathbb{Z}(p^\infty) \leq {}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ is not sum-prime.

COROLLARY 3.7. *If P is a sum-prime submodule of ${}_RM$ over a commutative ring R with identity, then $Ann_R(P) = (0 : P)$ is a prime ideal of R.*

PROOF. Since R is a commutative ring with identity it follows immediately that for any $r \in R$, $rM = M\rho(r) \leq M$ is an open submodule of ${}_RM$, where $\rho(r) : M \to M$ is defined by $m\rho(r) = rm$ for each $m \in M$. And hence $ker\rho(0 : P) = \cap_{r\in(0:P)} ker\rho(r)$ is also a closed submodule of M.

For $rs \in (0 : P)$ with $r, s \in R$, let $A = ker\rho(r)$ and $B = ker\rho(0 : P) \cap ker\rho(s)$. Then by the same method of the proof of the Proposition 3.5 it follows that $r \in (0 : P)$ or $s \in (0 : P)$. Hence $(0 : P)$ is a prime ideal of R.

COROLLARY 3.8. *For a commutative ring R with identity, if P is a fully invariant sum-prime submodule of ${}_RM$, then $(0 : P)$ and the ideal I_P are prime ideals of R and S, respectively.*

PROOF. By the Proposition 3.5 and the Corollary 3.7 it is proved.

4. (Semi-)Meet-Prime Modules and (Semi-)Prime Rings

In order to see relations among semi-prime rings, semi-meet-prime modules and semi-prime endomorphism rings of left modules, and relations among the prime radicals of rings, endomorphism rings and the prime radicals of modules, we need a new definition as follows

DEFINITION 4.1. For a left R-module ${}_RM$, we define the prime radical $rad(M) = \cap_{\alpha\in\Lambda} P_\alpha$ to be the intersection of all fully invariant meet-prime submodules $P_\alpha \leq M$ for $\alpha \in \Lambda$. Consequently, if $rad(M) =$

$\cap P_\alpha$, then we have an open submodule $rad(M) = \cap P_\alpha = \cap P_\alpha^o$ of M by (2) of the Proposition 1.3.

A left R-module ${}_RM$ is said to be semi-meet-prime if the prime radical $rad(M) = 0$ i.e., the intersection $\cap P_\alpha = 0$ for all fully invariant meet-prime submodules $P_\alpha \leq M$.

THEOREM 4 2. *If a left R-module ${}_RM$ is semi-meet-prime, then its endomorphism ring S is semi-prime.*

PROOF. Let $rad(M) = \cap P_\alpha$ be the prime radical of ${}_RM$ for each fully invariant meet-prime submodule P_α of ${}_RM$. Assume that ${}_RM$ is semi-meet-prime.

Then $rad(M) = 0 = \cap P_\alpha, P_\alpha$ is a fully invariant meet-prime submodule of M. Thus the prime radical $rad(S) \leq \cap I^{P_\alpha} = I^{\cap P_\alpha} = I^{rad(M)} = I^0 = 0$ must be zero. Therefore S is semi-prime

THEOREM 4 3. *For a commutative ring R with identity, if a faithful left R-module ${}_RM$ is semi-meet-prime, then the ring R and its endomorphism ring S are semi-prime.*

PROOF. The relation $(\cap P_\alpha : M) = \cap(P_\alpha : M)$ implies that
$0 = (0 : M) = (rad(M) : M) = \cap(P_\alpha : M) \geq rad(R)$ since M is faithful.

Recall that a ring is said to be prime if its zero ideal is prime. Thus we can define in the same way, if ${}_RM$ has a zero meet-prime submodule of it. ${}_RM$ will be called a meet-prime module.

THEOREM 4.4 *For a commutative ring R with identity, if there is at least one faithful meet-prime module, say ${}_RM$, then R is a prime ring. And so is S.*

PROOF. Let ${}_RM$ be a left faithful meet-prime R-module. Since 0 is a fully invariant meet-prime submodule of a faithful module M we have that
$0 = (0 \cdot M) \trianglelefteq R$ and $0 = I^0 \trianglelefteq S$ are prime ideals of R and S. respectively. Hence R and S are prime rings.

THEOREM 4.5. *If a left R–module $_RM$ is meet-prime, then S is a prime ring.*

PROOF. Since $I^0 = 0 \trianglelefteq S$ is a prime ideal of S by the Proposition 2.1 it follows immediately that S is a prime ring.

PROPOSITION 4.6. *For a commutative ring R with identity, if a left R–module $_RM$ is meet-prime, then $Ann_R(M) = (0 : M)$ is a prime ideal of R and S is a prime ring.*

PROOF. It is an immediate consequence of the Theorem 3 3 and the meet-primeness of the prime radical $rad(M) = 0$.

PROPOSITION 4.7. *For a commutative ring R with identity, if a left faithful R–module $_RM$ is meet-prime, then R and S are prime rings.*

PROOF. It is clear.

APPLICATION 4.8. *Clearly semi-meet-primeness of $_{\mathbb{Z}[x]}\mathbb{Z}[x]$ implies that its endomorphism ring is semi-prime.*

For any indexed family of $\{M_\alpha\}_{\alpha\in\Gamma}$ of R–modules $M_\alpha(\alpha \in \Gamma)$, a direct product $\prod_{\alpha\in\Gamma} M_\alpha$(a direc sum $\oplus_{\alpha\in\Gamma} M_\alpha$) is said to be of *invariant* factor modules $M_\alpha(\alpha \in \Gamma)$ if each homomorphism group

$$Hom_R(M_\alpha, M_\beta) = \{f \mid f \quad M_\alpha \to M_\beta \text{ is an } R\text{-homomorphism}\}$$

is a trivial additive group for each $\alpha \neq \beta$ in Γ, that is, zero.

PROPOSITION 4.9. *If $\prod_{\alpha\in\Gamma} M_\alpha$ is a direct product of semi-meet-prime invariant factor R–modules M_α $(\alpha \in \Gamma)$, then $\prod_{\alpha\in\Gamma} M_\alpha$ is also semi-meet-prime. Hence S is semi-prime.*

PROOF. Considering the canonical projections(here they are endomorphisms) $\pi_\iota : \prod M_\alpha \to M_\iota$ defined by $(x_\alpha)\pi_\iota = x_\iota$ for each $(x_\alpha) \in \prod M_\alpha$, from the fact that
for every fully invariant meet-prime submodule $P_\iota \leq M_\iota$, it follows that every preimage $\pi_\iota^{-1}(P_\iota)$ is meet-prime in $\prod M_\alpha$ since $\prod M_\alpha$ is a direct product of invariant factors. The rest of the proof can be established easily.

COROLLARY 4.10 *If $\prod_{\alpha\in\Gamma} M_\alpha$ is a direct product of meet-prime invariant factor $R-$modules M_α, then $\prod_{\alpha\in\Gamma} M_\alpha$ is also semi-meet-prime. Moreover S is semi-prime.*

PROOF. The proof is an immediate consequence of the Proposition 5.9.

COROLLARY 4.11 *For a commutative ring R with identity, if a left semi-simple $R-$module ${}_RM$ is a direct sum of simple invariant factor modules , then its endomorphism ring S is semi-prime. Additionally, if ${}_RM$ is faithful, then R is also a semi-prime ring*

PROOF. The proof is elementary by the Theorem 4.3 and the Proposition 4.9 since every a direct sum of semi-simple invariant factor modules is semi-meet-prime.

5. (Semi-)Sum-Prime Modules and (Semi-)Prime Rings

In order to see relations among semi-prime rings, semi-sum-prime modules and semi-prime endomorphism rings of left modules, and relations among the prime radicals of rings, endomorphism rings and the prime socles of modules, we need a new definition as follows

DEFINITION 5.1. For a left $R-$module ${}_RM$, we define the prime socle

$soc(M) = \sum_{\alpha\in\Lambda} P_\alpha$ to be the sum of all fully invariant sum-prime submodules $P_\alpha \leq M$ for $\alpha \in \Lambda$. Consequently, if $soc(M) = \sum P_\alpha$, then we have a closed submodule $soc(M) = \sum P_\alpha = \sum \overline{P_\alpha}$ of M by (2) of the Proposition 2.3

A left $R-$module ${}_RM$ is said to be semi-sum-prime if the prime socle $soc(M) = M$ i.e , the sum $\sum P_\alpha = M$ for all fully invariant sum-prime submodules $P_\alpha \leq M$

THEOREM 5.2 *If a left $R-$module ${}_RM$ is semi-sum-prime, then its endomorphism ring S is semi-prime.*

PROOF. Let $soc(M) = \sum P_\alpha$ be the prime socle of ${}_RM$ for each fully invariant sum-prime submodule P_α of ${}_RM$. Assume that ${}_RM$ is semi-sum-prime.

Then $soc(M) = M = \sum P_\alpha$, P_α is a fully invariant sum-prime submodule of M. Thus the prime radical $rad(S) \leq \cap I_{P_\alpha} = I_{\sum P_\alpha} = I_{soc(M)} = I_M = 0$ must be zero. Therefore S is semi-prime.

Recall that a ring is said to be prime if its zero ideal is prime. Thus we can define in the dual way, if ${}_RM$ has a nonzero trivial sum-prime submodule M of itself M, ${}_RM$ will be called a sum-prime module.

THEOREM 5.3. *For a commutative ring R with identity, if a faithful left $R-$module ${}_RM$ is semi-sum-prime, then the ring R and its endomorphism ring S are semi-prime.*

PROOF. The relation $(0 : \sum P_\alpha) = \cap(0 : P_\alpha)$ implies that
$0 = (0 : soc(M)) = \cap(0 : P_\alpha) \geq rad(R)$ since M is faithful

THEOREM 5.4. *For a commutative ring R with identity, if there is at least one faithful sum-prime module, say ${}_RM$, then R is a prime ring. And so is S.*

PROOF. Let ${}_RM$ be a left faithful sum-prime $R-$module. Since M is a fully invariant sum-prime submodule of a faithful module M we have that
$0 = (0 : M) \trianglelefteq R$ and $0 = I_M \trianglelefteq S$ are prime ideals of R and S. respectively. Hence R and S are prime rings

THEOREM 5.5. *If a left $R-$module ${}_RM$ is sum-prime, then S is a prime ring.*

PROOF. Since $I_M = 0 \trianglelefteq S$ is a prime ideal of S by the Proposition 4.1 it follows immediately that S is a prime ring.

PROPOSITION 5.6. *For a commutative ring R with identity, if a left $R-$module ${}_RM$ is sum-prime, then $Ann_R(M) = (0 : M)$ is a prime ideal of R and S is a prime ring.*

PROOF. It is an immediate consequence of the Theorem 5 3 and the sum-primeness of the prime socle $soc(M) = M$.

PROPOSITION 5.7. *For a commutative ring R with identity, if a left faithful $R-$module ${}_RM$ is sum-prime, then R and S are prime rings.*

PROOF. It is clear.

APPLICATION 5.8. *Clearly semi-sum-primeness of a left $\mathbb{Z}-$module $\oplus_p \mathbb{Z}_p$ (prime number p) implies that its endomorphism ring is semi-prime.*

PROPOSITION 5.9 *If $\oplus_{\alpha\in\Gamma} M_\alpha$ is a direct sum of semi-sum-prime invariant factor $R-$modules M_α $(\alpha \in \Gamma)$, then $\oplus_{\alpha\in\Gamma} M_\alpha$ is also semi-sum-prime. Hence S is semi-prime.*

PROOF Considering the canonical injections(here they are $R-$homor phisms) $\iota_\alpha : M_\alpha \to \oplus M_\gamma$ defined by $x_\alpha \iota_\alpha = \oplus x_\gamma$ for each $x_\gamma = 0_\gamma$ if $\gamma \neq \alpha$, $x_\gamma = x_\alpha$ if $\gamma = \alpha$, there is an $R-$endomorphism $\oplus\iota_\alpha : \oplus M_\gamma \to \oplus M_\gamma$. ¿From the fact that
for every fully invariant sum-prime submodule $P_\alpha \leq M_\alpha$, it follows that every image $\iota_\alpha(P_\alpha)$ is sum-prime in $\oplus M_\alpha$ since $\oplus M_\alpha$ is a direct sum of invariant factors. The rest of the proof can be established easily

COROLLARY 5.10. *If $\oplus_{\alpha\in\Gamma} M_\alpha$ is a direct sum of sum-prime invariant factor $R-$modules M_α, then $\oplus_{\alpha\in\Gamma} M_\alpha$ is also semi-sum-prime. Moreover S is semi-prime.*

PROOF. The proof is an immediate consequence of the proposition 6.9.

COROLLARY 5.11. *For a commutative ring R with identity, if a left semi-simple $R-$module ${}_RM$ is a direct sum of simple invariant factor modules , then its endomorphism ring S is semi-prime. Additionally, if ${}_RM$ is faithful, then R is also a semi-prime ring.*

PROOF. The proof is elementary by the Theorem 6.3 and the Proposition 6.9 since every a direct sum of semi-simple invariant factor modules is semi-meet-prime.

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