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APPROXIMATION PROPERTY FOR HOLOMORPHIC FUNCTIONS ON OKA-WEIL DOMAINS

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1. Introduction

Let M be an n-dimensional complex analytic manifold, $f_1, f_2, \cdots, f_n$ be holomorphic functions on M and $F = (f_1, f_2, \cdots, f_n)$ An Oka-Weil domain $W \subset M$ is a Stein. If M is a Stein manifold, there exists a sequence of Oka-Weil domains W_k such that $W_k \nearrow M$ and $\overline{W_k}$ is compact and $\overline{W_k} \subset W_{k+1}$ for $k = 1, 2, \cdots$, (see Y. Katznelson [3]). For the Banach space, J. Mujica [4] extended the Oka-Weil approximation theorem, by the technique of polynomially convex set And S. Dineen [1] also obtained many results for the approximation theorem In [5], we obtain an approximation theorem for holomorphic functions in finite and infinite dimensional complex spaces. In this paper, we obtain some properties of cohomology groups for coherent analytic sheaves over W and approximation property on Oka-Weil domains.

2. Globally syzygetic sheaf

DEFINITION 2 1 Let M be an n-dimensional complex (analytic) manifold. An open set $W \subset M$ is an Oka-Weil domain if there exists

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a relatively compact open neighborhood $U \supset W$ and $f_1, f_2, \cdots, f_m \in \mathcal{H}(M)$ such that

(1) $W \subset \overline{W} \subset U$,

(2) $W = U \cap \{z \in M : |f_j(z)| < 1,\ 1 \leq j \leq m\}$,

(3) $F = (f_1, f_2, \cdots, f_m)$ is an injective non- singular mapping of W into the unit polydisc $P \subset \mathbf{C}^n$.

PROPOSITION 2.2. *[2] Let M be a Stein manifold and K be a holomorphically convex compact subset of M. If U is any neighborhood of K, there is an Oka-Weil domain W, defined by global functions, such that $K \subset W \subset \overline{W} \subset U$.*

Let K be a compact subset of M and $\mathcal{A}$ be an algebra of holomorphic functions on M. The $\mathcal{A}$-convex hull of K in M is defined to be the set

$$K(\mathcal{A}, M) = \{z \in M : \mathcal{H}(z)| \leq ||f||_K \text{ for all } f \in \mathcal{A}\}.$$

We say that K is $\mathcal{A}$ - convex in M if $K = K(\mathcal{A}, M)$.

THEOREM 2.3. *[2] Let M be a complex manifold, K be a compact subset of M and $\mathcal{A} \subset \mathcal{H}(M)$ be any subalgebra such that*

(1) $\mathcal{A}$ gives a local coordinates system at each point in M,

(2) $\mathcal{A}$ separates points in M,

(3) K is $\mathcal{A}$-convex.

Then any holomorphic function f in a neighborhood of K is approximated uniformly on K by a sequence of functions in $\mathcal{A}$.

THEOREM 2.4. *If M is a Stein manifold and if $U \subset M$ is an Oka-Weil domain, then the section $\Gamma(O_n^k, M) = \mathcal{H}(M) \oplus \cdots \oplus \mathcal{H}(M)$ restricted to U is dense in $\Gamma(O_n^k, U)$ in the compact uniform topology.*

PROOF. For simplicity, we prove that for $k = 1$. Let $\widetilde{K} = \hat{K}_{\mathcal{H}(M)} \cap U$. If $\mathcal{A} = \{f|_U : f \in \mathcal{H}(M)\}$, then $\widetilde{K}$ is on $\mathcal{A}$-convex compact set in U and Theorem 2.3 applies to the system $(U, \mathcal{A})$ to give $\mathcal{A}$ dense in $\mathcal{H}(U)$. If U is an Oka-Weil domain, U is a relatively compact union of components in $U \cap \{z \in M : |f_1(z)| < 1, \cdots, |f_n(z)| < 1\}$ for $f_1, f_2, \cdots, f_n \in \mathcal{H}(M)$. If $x \in \partial U$, then $|f_j(z)| \geq 1$ while $||f_j|| < 1$ for some index $1 \leq j \leq m$. Thus we have $z \notin \hat{K}_{\mathcal{H}(M)}$. Hence $\widetilde{K}$ is compact. We complete the proof.

We consider an Oka-Weil domain W and corresponding mapping $F = (f_1, f_2, \cdots, f_n) : W \longrightarrow P$, where P is a unit polydisc in C^m If $0 < \lambda < 1$, the inverse image under F of λP is itself an Oka-Weil domain under the mapping $\frac{1}{\lambda}F$ And the set $F^{-1}(\lambda \overline{P})$ is compact in W.

DEFINITION 2.5. Let $\mathcal{F}$ be a sheaf of O_n-modules over M and U be an open subset of M. If $\mathcal{F}$ is syzygetic over U, if $\mathcal{F}$ restricted to U has a finite free resolution as a sheaf of $O_n(U)$-modules. A sheaf $\mathcal{F}$ is syzygetic if for each $m \in M$ there exists a neighborhood N of m in U such that $\mathcal{F}$ is syzygetic over N.

LEMMA 2.6. *If $\mathcal{F}$ is a syzygetic sheaf of O_n-modules over an open polydisc $P \subset C^n$, then $H^p(P, \mathcal{F}) = 0$ for $p \geq 1$.*

PROOF From the assumption, there exists a free resolution

$$0 \longrightarrow O_n^{l_k}(P) \xrightarrow{\lambda_k} \cdots \xrightarrow{\lambda_1} O_n^{l_0}(P) \xrightarrow{\lambda_0} \mathcal{F} \longrightarrow 0.$$

Let $\mathcal{F}_k = Ker\lambda_k = Im\lambda_{k+1}$ for $k \geq 0$. If $\mathcal{F}_0$ is a subsheaf in $O_n^{l_0}$, we have exact sequences

$$0 \longrightarrow O_n^{l_k} \xrightarrow{\lambda_k} \cdots \xrightarrow{\lambda_2} O_n^{l_1} \xrightarrow{\lambda_1} \mathcal{F}_0 \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{F}_0 \xrightarrow{id} O_n^{l_0} \xrightarrow{\lambda_1} \mathcal{F} \longrightarrow 0.$$

Thus we have $H^p(P, \mathcal{F}_0) = 0$ for $p \geq 1$ And so, from the long exact sequence, we have $H^p(P, \mathcal{F}) = 0$ for $p \geq 1$.

THEOREM 2.7. *Let W be an Oka-Weil domain with mapping F $W \longrightarrow P$ in M and let $\mathcal{F}$ be a coherent sheaf of O_n-modules over W Then $\mathcal{F}$ is syzygetic on any Oka-Weil subdomain λW for $0 < \lambda \leq 1$*

PROOF. Since the sheaf $\mathcal{F}$ is coherent, $\mathcal{F}$ is syzygetic on the domain W. From Lemma 2.6, λW maps to an open polydisc with compact closure in P and $\mathcal{F}$ has a global free resolution over all small neighberhoods of $\overline{\lambda W}$. Let $\widetilde{W} = F(W) \subset P$ The $\widetilde{W}$ is closed and regularly imbedded in P If we give $\mathcal{F}$ a new projection map $F \circ \pi$, where $\pi : \mathcal{F} \longrightarrow W, \mathcal{F}$

can be regarded as a sheaf of abelian groups over $\widetilde{W}$. We make $\mathcal{F}$ a sheaf of O_m-modules as follows (m need not equal to n) : if $y \in \widetilde{W}$ and $F(x) = y$, then $a_y \in O_m(y)$ with a representative a near y acts on the germ $f_x \in \mathcal{F}_x$ with representative f near x to give the germ at x of $(a \circ F)f$. Define $\widetilde{\mathcal{F}}$ on P with $\widetilde{\mathcal{F}_y} = 0$ if $y \notin \widetilde{W}$, and $\widetilde{\mathcal{F}_y} = \mathcal{F}_y$ if $y \in \widetilde{W}$, regarding $\widetilde{\mathcal{F}}$ as a sheaf of O_m-modules. For $x \in \widetilde{W}$, the topology on $\widetilde{\mathcal{F}}$ is chosen so that a germ f_x has a basis of neighborhoods

$$N_U = \begin{cases} O_y, & \text{if } y \in U - \widetilde{W} \\ f_{F^{-1}(y)}, & \text{if } y \in U \cap \widetilde{W} \end{cases}$$

as U runs through a neighborhoodsbasis of x in P, where f is some representative of the germ near x in W. Since $\widetilde{\mathcal{F}}$ is coherent on P, $\widetilde{\mathcal{F}}$ is syzygetic over λP $(0 < \lambda < 1)$ and $\mathcal{F}$ is syzygetic over λW. Since $\widetilde{\mathcal{F}}$ is of finite type, there exists the relation sheaf of sections $\varphi_1, \varphi_2, \cdots, \varphi_s \in \Gamma(\widetilde{\mathcal{F}}, U)$ for open $U \subset P$. If $x \in U - \widetilde{W}$ then $R(\varphi_1, \cdots, \varphi_s)_y = O_m^s(y)$ for y near x, so $R(\varphi_1, \cdots, \varphi_s)$ has finite type at x. If $x \in U \cap \widetilde{W}$ let $\overline{\varphi_1}, \overline{\varphi_2}, \cdots, \overline{\varphi_s} \in \Gamma(\mathcal{F}, W \cap U)$ be the sections induced by $\varphi_1, \varphi_2, \cdots, \varphi_s$. Since $\mathcal{F}$ is a coherent sheaf, there exist sections $u_1, u_2, \cdots, u_l$ of $R(\overline{\varphi_1}, \overline{\varphi_2}, \cdots, \overline{\varphi_s})$ which generate $R(\overline{\varphi_1}, \overline{\varphi_2}, \cdots, \overline{\varphi_s})$ over some small neighborhood of x. These extend to sections $\widetilde{u_1}, \widetilde{u_2}, \cdots, \widetilde{u_l} \in \Gamma(O_m^s, V)$ defined in some neighborhood $V \subset U$ of x. If $\widetilde{u_j} = (\widetilde{u_1^j}, \widetilde{u_2^j}, \cdots, \widetilde{u_s^j})$ then $\Sigma\ \widetilde{u_k^j}\varphi_k \equiv 0$ on $V \cap \widetilde{W}$. Thus $\widetilde{u_j}$ is a section of $R(\varphi_1, \cdots, \varphi_s)$ over V. By making V smaller we can insure that the some of ideal sheaves $J = (J_{\widetilde{W}})^s = J_{\widetilde{W}} \oplus \cdots \oplus J_{\widetilde{W}}$ is generated by finitely many sections $g_1, g_2, \cdots, g_k \in \Gamma(J, V)$ and $J \subset R(\varphi_1, \cdots, \varphi_s)$. Hence $\{\widetilde{u_1}, \cdots, \widetilde{u_l}, g_1, \cdots, g_k\}$ generate the stalks of $R(\varphi_1, \cdots, \varphi_s)$ over V as O_m-modules. If $y \in V$ with $y = F(z)$ and if $h_y \in R(\varphi_1, \cdots, \varphi_s)_y$ with representative h, then $\overline{h} = (h|_{\widetilde{W}}) \circ F \in R(\overline{\varphi_1}, \cdots, \overline{\varphi_s})_y$ and there exist $\{a_1, a_2, \cdots, a_l\} \in O_n(y)$ such that $\overline{h} - \sum_{j=1}^{l} a_j u_j = 0$ near y If $\widetilde{a_1}, \cdots, \widetilde{a_l}$ are arbitrary extensions of the functions $a_j \circ F^{-1}$ to a neighborhood of y, then $h^* = h - \sum_{j=1}^{l} \widetilde{a_j}\widetilde{u_j} \equiv 0$ on $\widetilde{W}$ near y. Thus h^* belongs to the sum J.

References

[1] S Dineen, *Complex analysis in locally convex spaces*, North- Holland Math Studies 57, North - Holland Pub Co., Amsterdam, New York, Oxford 1981

[2] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables* Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1965.

[3] Y. Katznelson, *Lectures on several complex variables*, Yale University Press 1964.

[4] J Mujica, *Complex analysis in Banach spaces*, North - Holland Math Studies 120, Elsevier Science Publishers B. V , North - Halland, New York, Oxford, 1986

[5] K H. Shon, C J. Kang and S. M Kwon, *Approximation theorem for holomorphic functions in finite and infinite dimensional complex spaces*, East Asian Math. J , **15-1** (1999), 147-151.

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