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ESTIMATION IN AN EXPONENTIAL DISTRIBUTION WITH CENSORING

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1. Introduction

A random variable X has an exponential distribution if it has a probability density function(pdf) of the form:

$$f(x) = \frac{1}{\sigma} \exp\left(-\frac{x}{\sigma}\right) \ , \ x \geq 0 \ , \ \sigma > 0, \tag{1}$$

where σ is scale parameter.

Lloyd(1952) described a method of obtaining the best linear unbiased estimators(BLUEs) of the parameters of exponential distribution. using order statistics Gupta(1952) proposed estimation of the mean and standard deviation of a normal population from a censored sample The approximate maximum likelihood estimation method was first developed by Balakrishnan(1989a,b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution and the mean and standard deviation in the normal distribution with censoring Kang(1996) obtained the approximate maximum likelihood estimator(AMLE) for the scale parameter of the double exponential distribution based on Type-II censored samples and he showed that the proposed estimator is generally more efficient than the BLUE and the optimum unbiased absolute estimator. Some historical remarks and a good summary of the approximate maximum likelihood estimation may be found in Balakrishnan and Cohen(1991)

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In this paper, we derive the AMLE of the scale parameter in the one-parameter exponential distribution with pdf (1) based on the generalized censored sample which includes the Type-II censored sample.

We also obtain the asymptotic variance of the AMLE.

2. Preliminary

Consider one-parameter exponential distribution with density function (1) and cumulative distribution function (cdf)

$$F(x) = \begin{cases} 1 - \exp\left(-\frac{x}{\sigma}\right), & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{2}$$

Let us consider an experiment in which n exponential components are put to test simultaneously at time $x = 0$, and the failure times of there components are recorded.

We will consider the generalized censored sample which include the Type-II censored sample. Let

$$X_{a_0\, n} \leq X_{a_1\, n} \leq \cdots \leq X_{a_s\, n} \tag{3}$$

be the available censored sample from the exponential distribution with pdf (1), where a_i's are the integers such that

$$1 \leq a_0 < a_1 < \cdots < a_s \leq n.$$

3. Approximate estimation

We shall derive the AMLE of σ based on the censored sample in (3).

The likelihood function based on the censored sample in (3) is given by

(4)

$$L = \frac{n!}{\prod_{j=1}^{s}(a_j - a_{j-1} - 1)!} \times \prod_{j=1}^{s} \left[F(X_{a_j\, n}; \sigma) - F(X_{a_{j-1}\, n}; \sigma)\right]^{a_j - a_{j-1} - 1}$$

$$\times \prod_{j=0}^{s} f(X_{a_j\, n}; \sigma), \quad X_{a_j\, n} \geq 0,$$

which upon denoting $Z_{i:n} = X_{i:n}/\sigma$, can be written as

(5)
$$L = \frac{n!}{\prod_{j=1}^{s}(a_j - a_{j-1} - 1)!}\sigma^{-A} \times \prod_{j=1}^{s}\left[F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})\right]^{a_j - a_{j-1} - 1}$$
$$\times \prod_{j=0}^{s} f(Z_{a_j:n}), \quad Z_{a_j:n} \geq 0,$$

where $A = s + 1$ is the size of the censored sample (3), and $f(z)$ and $F(z)$ are the pdf and cdf of the standard exponential distribution, respectively.

Now, we will obtain the AMLE of the scale parameter. First, we differentiate the logarithm of the likelihood function (5) for σ as follows;

(6)
$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\{A + \sum_{j=1}^{s}(a_j - a_{j-1} - 1)$$
$$\left[\frac{f(Z_{a_j:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} Z_{a_j:n} - \frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \cdot Z_{a_{j-1}:n}\right]$$
$$+ \sum_{j=0}^{s} \frac{f'(Z_{a_j:n})}{f(Z_{a_j:n})} \cdot Z_{a_j:n}\} = 0$$

Equation (6) does not admit an explicit solution for σ. But since $\frac{f'(Z_{a_j:n})}{f(Z_{a_j:n})} = -1$, we can expand the function

$$H(Z_{a_{j-1}:n}, Z_{a_j:n}) = \frac{f(Z_{a_j:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})}$$

and

$$G(Z_{a_{j-1}:n}, Z_{a_j:n}) = -\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})}$$

appearing in (6) to Taylor series around the point $(\xi_{a_{j-1}}, \xi_{a_j})$, where $\xi_{a_j} = F^{-1}(p_{a_j}) = -\ln(q_{a_j})$(here, $p_i = \frac{i}{n+1}$, $q_i = 1-p_i$), $j = 1, 2, \ldots, s$, and then approximate it by

$$\frac{f(Z_{a_j\,n})}{F(Z_{a_j\,n}) - F(Z_{a_{j-1}\,n})} \cong \alpha + \beta Z_{a_j\,n} + \gamma Z_{a_{j-1}\,n}$$

and

$$\frac{-f(Z_{a_{j-1}\,n})}{F(Z_{a_j\,n}) - F(Z_{a_{j-1}\,n})} \cong \alpha^* + \beta^* Z_{a_{j-1}\,n} + \gamma^* Z_{a_j\,n}, \tag{7}$$

$$\alpha = \frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}}\left[1 + \xi_{a_j} + \frac{f(\xi_{a_j})\cdot\xi_{a_j} - f(\xi_{a_{j-1}})\cdot\xi_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\right],$$

$$\alpha^* = \frac{f(\xi_{a_{j-1}})}{p_{a_{j-1}} - p_{a_j}}\left[1 + \xi_{a_{j-1}} + \frac{f(\xi_{a_{j-1}})\cdot\xi_{a_{j-1}} - f(\xi_{a_j})\cdot\xi_{a_j}}{p_{a_{j-1}} - p_{a_j}}\right],$$

$$\beta = -\frac{f(\xi_{a_j})}{(p_{a_j} - p_{a_{j-1}})^2}\left[p_{a_j} - p_{a_{j-1}} + f(\xi_{a_j})\right],$$

$$\beta^* = -\frac{f(\xi_{a_{j-1}})}{(p_{a_{j-1}} - p_{a_j})^2}\left[p_{a_{j-1}} - p_{a_j} + f(\xi_{a_{j-1}})\right],$$

and

$$\gamma = \gamma^* = \frac{f(\xi_{a_j})\cdot f(\xi_{a_{j-1}})}{(p_{a_{j-1}} - p_{a_j})^2}.$$

Now making use of the approximate expression in (7), we obtain the approximate likelihood equation of (6) as follows:

(8)

$$\frac{\partial \ln L}{\partial\sigma} \cong \frac{\partial \ln L^*}{\partial\sigma} = -\frac{1}{\sigma}\{A + \sum_{j=1}^{s}(a_j - a_{j-1} - 1)[\alpha Z_{a_j\,n} + \alpha^* Z_{a_{j-1}\,n} + \beta Z_{a_j}{}^2 + \beta^* Z_{a_{j-1}}{}^2 + (\gamma+\gamma^*) Z_{a_j}\cdot Z_{a_{j-1}}] - \sum_{j=0}^{s} Z_{a_j}\} = 0.$$

Since $Z_{i:n} = \dfrac{X_{i,n}}{\sigma}$, we can derive the AMLE of σ as follows,

$$\hat{\sigma} = \frac{1}{2A}(-B + \sqrt{B^2 - 4AC}) \tag{9}$$

where

$$B = \sum_{j=1}^{s}(a_j - a_{j-1} - 1)\alpha X_{a_j,n} + \sum_{j=1}^{s}(a_j - a_{j-1} - 1)\alpha^* X_{a_{j-1},n} - \sum_{j=0}^{s} X_{a_j,n}$$

and

$$C = \sum_{j=1}^{s}(a_j - a_{j-1} - 1)[\beta X_{a_j,n}^2 + (\gamma + \gamma^*)X_{a_j,n}X_{a_{j-1},n} + \beta^* X_{a_{j-1},n}^2].$$

These proposed AMLEs admit it explicit estimator So we can easily estimate the scale parametor by using this estimator

We simulate the numerical values of $\hat{\sigma}$ by a Monte Carlo simulation method (MSE) for several censoring cases. These values are presented in Table 1.

4. Asymptotic properties

Since the AMLEs $\hat{\sigma}$ in (9) is the solutions of the approximate maximum likelihood equations (4), it immediately follows that $\hat{\sigma}$ is asymptotically normally distributed with mean σ and variance

$$1/E\{-d^2 \ln L^*/d\sigma^2\}$$

(See Kendall and Stuart (1973)). Now, from equation (4) we can obtain

$$E\left(-\frac{d^2 \ln L^*}{d\sigma^2}\right) = \frac{-(A + 2D + 3F)}{\sigma^2}, \tag{10}$$

where

$$D = \sum_{j=1}^{s}(a_j - a_{j-1} - 1)[\alpha E(Z_{a_j\,n}) + \alpha^* E(Z_{a_{j-1}\,n})] - \sum_{j=0}^{s} E(Z_{a_j\,n})$$

and

$$F = \sum_{j=1}^{s}(a_j - a_{j-1} - 1)[\beta E(Z^2_{a_j\,n}) + (\gamma + \gamma^*)E(Z_{a_j\,n})E(Z_{a_{j-1}\,n}) + \beta^* E(Z^2_{a_{j-1}\,n})].$$

From the equation (10), we can compute the asymptotic variance of the AMLE $\hat{\sigma}$ by using the following results (Govindarajulu (1966) and Rao et al (1991)).

$$E(Z_{i\,n}) = 2^{-n}\{\sum_{h=0}^{i-1}\binom{n}{h}S_1(i-h, n-h) - \sum_{h=i}^{n}\binom{n}{h}S_1(h-i+1, h)\}$$

$$E(Z^2_{i\,n}) = 2^{-n}\{\sum_{h=0}^{i-1}\binom{n}{h}[S_2(i-h, n-h) + S_1^2(i-h, n-h)] + \sum_{h=i}^{n}\binom{n}{h}[S_2(h-i+1, h) + S_1^2(h-i+1, h)]\},$$

where $S_k(i,n) = \sum_{l=n-i+1}^{n} 1/l^k, \quad k = 1, 2.$

Table 1. The MSE's of the AMLE of the scale parameter σ in an exponential distribution based on generalized censored sample.

(a) Full data

n	s	MSE($\sigma = 1.0$)	MSE($\sigma = 2.0$)
10	9	.09135	.35902

20	19	04762	18973
30	29	.03236	.12918
40	39	.02304	.09205
50	49	01905	.07614

(b) $X_{3\ n}, X_{5\ n}, \cdots, X_{2k+1\ n}$ are censors where k are $1, 2, \cdots, \frac{n}{2} - 1$

n	s	MSE($\sigma = 1\ 0$)	MSE($\sigma = 2.0$)
10	5	.16880	.67711
20	10	.11931	.47878
30	15	.09619	.38556
40	20	07963	.31902
50	25	.07715	.30895

(c) $X_{2\ n}, X_{4\ n}, \cdots, X_{2k\ n}$ are censors where k are $1, 2, \quad, \frac{n}{2} - 1$

n	s	MSE($\sigma = 1\ 0$)	MSE($\sigma = 2.0$)
10	5	.15113	.60583
20	10	.10946	.43930
30	15	.08996	.36065
40	20	07777	.31156
50	25	.07382	.29564

(d) $X_{2\ n}, X_{3\ n}$ are censors.

n	s	MSE($\sigma = 1\ 0$)	MSE($\sigma = 2.0$)
10	7	.10031	.39723
20	17	.05260	.20999
30	27	.03461	.13829
40	37	.02441	.09755

50	47	.01966	.07860

(e) $X_{2:n}, X_{3:n}, X_{4:n}, X_{5:n}$ are censors.

n	s	MSE($\sigma = 1.0$)	MSE($\sigma = 2.0$)
10	5	.10558	.41821
20	15	.05344	.21339
30	25	.03456	.13808
40	35	.02444	.09770
50	45	.02004	.08012

(f) $X_{n-2:n}, X_{n-1:n}$ are censors.

n	s	MSE($\sigma = 1.0$)	MSE($\sigma = 2.0$)
10	7	.12772	.50852
20	17	.06335	.25324
30	27	.04022	.16081
40	37	.02786	.11140
50	47	.02222	.08888

(g) $X_{n-4:n}, X_{n-3:n}, X_{n-2:n}, X_{n-1:n}$ are censors.

n	s	MSE($\sigma = 1.0$)	MSE($\sigma = 2.0$)
10	5	.14374	.57258
20	15	.06934	.27730
30	25	.04375	.17492
40	35	.02958	.11831
50	45	.02336	.09344

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