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# AN IDENTITY DUE TO RAO AND SARMA

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## 1. Introduction

The Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1), \tag{1}$$

which can be continued analytically to the whole $s$-plane except for a simple pole at $s = 1$ with its residue 1 by the contour integral representation (*cf.* Whittaker and Watson [7, p. 266]) or other various integral representations (see Erdélyi *et al.* [2, p. 33, Eqs. (12) to (15)])

Rao and Sarma [4] proved the following useful identity by making use of the above-mentioned method:

$$2\sum_{r=1}^{\infty} \frac{1}{r^a} \sum_{k=1}^{r} \frac{1}{k} = (a+2)\zeta(a+1) - \sum_{i=1}^{a-2} \zeta(a-i)\zeta(i+1), \tag{2}$$

where $a \in \mathbb{N} \setminus \{1\}$ and $\mathbb{N} := \{1, 2, 3, \ldots\}$. The identity (2) can easily be rewritten in an equivalent form.

$$2\sum_{q=1}^{\infty} \frac{1}{q^n} \left( \sum_{k=1}^{q-1} \frac{1}{k} \right) = n\zeta(n+1) - \sum_{k=1}^{n-2} \zeta(n-k)\zeta(k+1), \tag{3}$$

where the empty sum is understood (as usual) to be nil.

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## 2. A proof of the result and remarks

To give an elementary proof of (3), first consider

$$\begin{aligned}\sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1) &= \zeta(2)\zeta(n-1)+\zeta(3)\zeta(n-2)+\cdots+\zeta(n-1)\zeta(2)\\ &= \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\left\{\frac{1}{p^2}\cdot\frac{1}{q^{n-1}}+\frac{1}{p^3}\cdot\frac{1}{q^{n-2}}+\cdots+\frac{1}{p^{n-1}}\cdot\frac{1}{q^2}\right\}\\ &= \lim_{N\to\infty} S_N,\end{aligned}$$

where, for convenience,

(4)
$$S_N := \sum_{p=1}^{N}\sum_{q=1}^{N}\left\{\frac{1}{p^2}\cdot\frac{1}{q^{n-1}}+\frac{1}{p^3}\cdot\frac{1}{q^{n-2}}+\cdots+\frac{1}{p^{n-1}}\cdot\frac{1}{q^2}\right\} \quad (N\in\mathbb{N}).$$

Observing the series within braces in (4) is a geometric series, and taking care of the exceptional case $p=q$, the series (4) becomes

$$\sum_{q=1}^{N}\left\{\sum_{\substack{p=1\\p\neq q}}^{N}\frac{1}{p-q}\left(\frac{q^{1-n}}{p}-\frac{p^{1-n}}{q}\right)+(n-2)\frac{1}{q^{n+1}}\right\}. \tag{5}$$

Considering the inner summation part in (5) yields

(6)
$$\begin{aligned}&\sum_{q=1}^{N}\sum_{\substack{p=1\\p\neq q}}^{N}\frac{1}{p-q}\left(\frac{q^{1-n}}{p}-\frac{p^{1-n}}{q}\right)\\ &=\sum_{q=1}^{N}\sum_{\substack{p=1\\p\neq q}}^{N}\frac{1}{p-q}\cdot\frac{q^{1-n}}{p}+\sum_{p=1}^{N}\sum_{\substack{q=1\\q\neq p}}^{N}\frac{1}{q-p}\cdot\frac{p^{1-n}}{q}.\end{aligned}$$

Inverting the order of summation in the second double sum and considering the interchange of the dummy indices in the second sum,

(6) may be written as

$$\sum_{q=1}^{N}\sum_{\substack{p=1\\p\neq q}}^{N}\frac{q^{1-n}}{p(p-q)}+\sum_{p=1}^{N}\sum_{\substack{q=1\\q\neq p}}^{N}\frac{p^{1-n}}{q(q-p)}=2\sum_{q=1}^{N}\frac{1}{q^{n-1}}\sum_{\substack{p=1\\p\neq q}}^{N}\frac{1}{p(p-q)}. \tag{7}$$

Combining (4) through (7), we observe that

$$\begin{aligned}&\sum_{q=1}^{N}\sum_{p=1}^{N}\left\{\frac{1}{p^2}\cdot\frac{1}{q^{n-1}}+\frac{1}{p^3}\cdot\frac{1}{q^{n-2}}+\quad-\frac{1}{p^{n-1}}\quad\frac{1}{q^2}\right\}\\&=(n-2)\sum_{q=1}^{N}\frac{1}{q^{n+1}}+2\sum_{q=1}^{N}\frac{1}{q^{n-1}}\sum_{\substack{p=1\\p\neq q}}^{N}\frac{1}{p(p-q)}\end{aligned} \tag{8}$$

Now consider

$$\begin{aligned}-q\sum_{\substack{p=1\\p\neq q}}^{N}\frac{1}{p(p-q)}&=\sum_{\substack{p=1\\p\neq q}}^{N}\left(\frac{1}{p}-\frac{1}{p-q}\right)\\&=\sum_{p=1}^{q-1}\frac{1}{q-p}-\sum_{p=q+1}^{N}\frac{1}{p-q}+\sum_{p=1}^{N}\frac{1}{p}-\frac{1}{q}\\&=\sum_{p=1}^{q-1}\frac{1}{p}-\sum_{p=1}^{N-q}\frac{1}{p}+\sum_{p=1}^{N}\frac{1}{p}-\frac{1}{q}\\&=-\frac{1}{q}+\sum_{p=1}^{q-1}\frac{1}{p}+\sum_{p=N-q+1}^{N}\frac{1}{p},\end{aligned} \tag{9}$$

where we let $q-p=p'$ and $p-q=p'$ in the first and second summations of the second equality respectively, and then dropping the prime on $p$ we can easily get the third equality.

Setting (9) into (8) leads to

$$
\begin{aligned}
&\sum_{q=1}^{N}\sum_{p=1}^{N}\left\{\frac{1}{p^2}\cdot\frac{1}{q^{n-1}}+\frac{1}{p^3}\cdot\frac{1}{q^{n-2}}+\cdots+\frac{1}{p^{n-1}}\cdot\frac{1}{q^2}\right\} \\
&\quad = n\sum_{q=1}^{N}\frac{1}{q^{n+1}} - 2\sum_{q=1}^{N}\frac{1}{q^n}\left(\sum_{k=1}^{q-1}\frac{1}{k}\right) - 2\sum_{q=1}^{N}\frac{1}{q^n}\left(\sum_{k=N-q+1}^{N}\frac{1}{k}\right).
\end{aligned}
\tag{10}
$$

We finally consider

$$
0 \le \sum_{k=N-q+1}^{N}\frac{1}{k} = \frac{1}{N-q+1}+\frac{1}{N-q+2}+\cdots+\frac{1}{N} \le \frac{q}{N-q+1},
$$

and so

$$
\begin{aligned}
0 &< \sum_{q=1}^{N}\frac{1}{q^n}\left(\sum_{k=N-q+1}^{N}\frac{1}{k}\right) < \sum_{q=1}^{N}\frac{1}{q^{n-1}}\cdot\frac{1}{N-q+1} \\
&\le \sum_{q=1}^{N}\frac{1}{q(N-q+1)} \qquad (n \ge 2) \\
&= \frac{1}{N+1}\sum_{q=1}^{N}\left(\frac{1}{q}+\frac{1}{N-q+1}\right) = \frac{2}{N+1}\sum_{q=1}^{N}\frac{1}{q} \\
&< \frac{2}{N+1}(1+\log N) \to 0 \quad \text{as} \quad N \to \infty.
\end{aligned}
\tag{11}
$$

Taking the limit on (10) as $N \to \infty$ with (11) completes the proof of the identity (3).

An obvious special case of (3) when $n = 2$ becomes

$$
\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right). \tag{12}
$$

which was proven by many authors in various ways, among others, by Shen [5, p.1396] who proved by making use of the Stirling numbers and analyzing the well-known Gauss's summation theorem in the hypergeometric series ${}_2F_1\,(a,\,b\,;\,c;\,z)$ (*cf.* Gauss [3])

$$(13)\qquad {}_2F_1(a,b;\,c;\,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\quad(\Re(c-a-b)>0),$$

where $\Gamma$ is the familiar Gamma function whose Weierstrass canonical product form is:

$$(14)\qquad \Gamma(z)=\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}$$

and $\gamma$ denotes the Euler-Mascheroni constant defined by

$$(15)\qquad \gamma:=\lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k}-\log N\right)\cong 0.577\,215\,664\,901\,532\ldots .$$

We conclude this note by proving, in order to illustrate the usefulness of the above-employed elementary technique, a known identity

$$(16)\qquad \sum_{k=2}^{\infty}\frac{\zeta(k)}{k\;2^k}=\frac{1}{2}\log\pi-\frac{1}{2}\gamma,$$

which was shown by Choi and Srivastava [1, p. 109, Eq. (2.21)] who used the theory of the simple and double Gamma functions. First recall the Stirling's formula (see [6, p. 102, Entry (16.16)]):

$$(17)\qquad \lim_{N\to\infty}\left\{2\sum_{\ell=1}^{N}\log\ell-(2N+1)\log N+2N\right\}=\log(2\pi).$$

Now, starting with

$$\begin{aligned}S:=\sum_{k=2}^{\infty}\frac{\zeta(k)}{k\;2^k}&=\sum_{\ell=1}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k\;(2\ell)^k}\\&=-\lim_{N\to\infty}\sum_{\ell=1}^{N}\left\{\frac{1}{2\ell}+\log\left(1-\frac{1}{2\ell}\right)\right\},\end{aligned}$$

where the Maclaurin series expansion of $\log(1-x)$ was used. By considering (15), we find that

$$\begin{aligned} S &:= -\frac{\gamma}{2} + \frac{1}{2} \lim_{N\to\infty} \left[ 2\sum_{\ell=1}^{N} \log(2\ell) - \log N - 2\sum_{\ell=1}^{N} \log(2\ell-1) \right] \\ &= -\frac{\gamma}{2} + \frac{1}{2} \lim_{N\to\infty} \left[ 4N \log 2 - \log N + 4\sum_{\ell=1}^{N} \log \ell - 2\sum_{\ell=1}^{2N} \log \ell \right], \end{aligned}$$

the limit part of which, in virtue of (17), can easily be seen to become $\log \pi$. This completes an elementary proof of (16).

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