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INFINITE SERIES ASSOCIATED WITH PSI AND ZETA FUNCTIONS

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Abstract. We evaluate some interesting families of infinite series expressed in terms of the Psi (or Digamma) and Zeta functions by analyzing the well-known identity associated with ${}_3F_2$ due to Watson. Some special cases are also considered.

1. Introduction and Preliminaries

Shen [9] investigated the connections between the Stirling numbers s(n,k) of the first kind and the Riemann Zeta funnction $\zeta(n)$ by analyzing the well-known Gauss summation formula [3, pp. 2] for the hypergeometic series:

$$\sum_{n=k}^{\infty} \frac{s(n,k)}{n \cdot n!} = \zeta(k+1) \quad ;$$

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \right\}^2 = \frac{11}{4}\zeta(4) .$$

In this line Choi et al. evaluated many interesting families of infinite series by analyzing known identities involving generalized hypergeometric series (see [4], [5]). In this paper we also show that certain interesting classes of infinite series can be evaluated by analyzing a well-known identy for ${}_3F_2$ due to Watson. For this purpose we introduce some definitions and their properties among them. The generalized hypergeometric function

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with p numerator and q denominater parameters is defined by

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\,; \\ \beta_1,\ldots,\beta_q\,;\end{matrix}\; z\right] = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by, for any complex number α,

$$(1.2)\qquad (\alpha)_n := \begin{cases} 1 & (n=0), \\ \alpha(\alpha+1)\ldots(\alpha+n-1) & (n\in\mathbf{N}:=\{1,2,3,\ldots\}), \end{cases}$$

which can also be rewritten in the following equivalent form :

$$(1.3)\qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

by using the well-known Gamma function Γ whose Weierstrass canonical product form is

$$(1.4)\qquad \Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^{-1}e^{z/n}\right\},$$

γ being the Euler-Mascheroni constant defined by

$$(1.5)\qquad \gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right) \cong 0.577\,215\,664\,901\,532\ldots.$$

The Psi (or Digamma) function is defined as the logarithmic derivative of the Gamma function:

$$(1.6)\qquad \psi(z) := \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

We recall here some well-known properties of the ψ−function (see [6]): For a positive integer n,

$$(1.7)\qquad \begin{aligned} &\psi(1) = -\gamma; \qquad \psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2; \\ &\psi(z+n)-\psi(z) = \sum_{k=0}^{n-1}\frac{1}{z+k}. \end{aligned}$$

The Polygamma functions are defined by (see [6, pp. 41])

$$\psi^{(n)}(z) := \begin{cases} \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) & (n \in \mathbf{N}) \\ \psi(z) & (n = 0) \end{cases} \tag{1.8}$$

from which it is easy to show that

$$\psi^{(n)}(z) = (-1)^{n+1} n! \zeta(n+1, z) \ (\mathrm{n} \in \mathbf{N}), \tag{1.9}$$

where $\zeta(z, a)$ is the generalized (or Hurwitz) Zeta function defined by

$$\zeta(z, a) = \sum_{k=0}^{\infty} (k+a)^{-z} \ \ (\mathrm{Re}(z) > 1;\ a \neq 0, -1, -2, \ldots), \tag{1.10}$$

which can be continued analytically to the whole z-plane except for a simple pole at $z = 1$ with residue 1 and $\zeta(z, 1) = \zeta(z)$.

It is not difficult to derive the following results (see [11, pp. 265 - 275]):

$$\begin{aligned} &\zeta(z) = \frac{1}{1-2^{-z}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^z} = \frac{1}{2^z - 1} \zeta\left(z, \frac{1}{2}\right) \ \ (\mathrm{Re}(z) > 1); \\ &\zeta(z, a) = \zeta(z, n+a) + \sum_{k=0}^{n-1} (k+a)^{-z} \ \ (n \in \mathbf{N}). \end{aligned} \tag{1.11}$$

The Stirling numbers of the first kind s(n,k) can be defined by the following equation (see [5, pp. 204-215]):

$$(z)_n = z(z+1)\cdots(z+n-1) = \sum_{k=0}^{n} (-1)^{n+k} s(n, k) z^k. \tag{1.12}$$

It is not difficult to see also that

$$\begin{aligned} &\frac{(-1)^{n+1} s(n, 1)}{n!} = \frac{1}{n} \ \text{ and } \ \frac{(-1)^n s(n, 2)}{n!} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k}; \\ &\frac{(-1)^n s(n, 3)}{n!} = \frac{1}{2n} \left\{ \Big(\sum_{k=1}^{n-1} \frac{1}{k}\Big)^2 - \sum_{k=1}^{n-1} \frac{1}{k^2} \right\}. \end{aligned} \tag{1.13}$$

Using (1.3) in the left member of (1.12) and differentiating each side of the resulting identity with respect to z successively, we obtain the following finite sums involving s(n,k):

$$\sum_{k=1}^{n}(-1)^{n+k}\, k\, s(n,k)z^{k-1} = (z)_n[\psi(z+n)-\psi(z)];$$

$$(1.14) \qquad \sum_{k=2}^{n}(-1)^{n+k}\, k(k-1)\, s(n,k)z^{k-2} = (z)_n\Big[\{\psi(z+n)-\psi(z)\}^2 + \psi^{(1)}(z+n)-\psi^{(1)}(z)\Big].$$

2. Series Derivable from Watson's Summation Theorems

First recall Watson's theorem [3, pp. 16] : For $\mathrm{Re}(2c-a-b) > -1$,

$$(2.1) \qquad {}_3F_2\left[\begin{matrix} a, b, c\,; \\ \frac{1}{2}(a+b+1), 2c\,; \end{matrix}\; 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\Gamma(\frac{1}{2}-\frac{1}{2}b+c)}.$$

We show how Watson's formula (2.1) can be applied in order to evaluate the sums of certain classes of infinite series expressed in terms of the Psi (or Digamma) and the Zeta functions. In fact, we will prove the following interesting classes of infinite series:

$$(2.2) \qquad \sum_{n=1}^{\infty}\frac{(b)_n(c)_n}{n(2c)_n(\frac{1+b}{2})_n} = \frac{1}{2}\Big\{\psi\Big(\frac{1}{2}+\frac{1}{2}b\Big)+\psi\Big(\frac{1}{2}+c\Big) - \psi\Big(\frac{1}{2}-\frac{1}{2}b+c\Big)-\psi\Big(\frac{1}{2}\Big)\Big\};$$

$$(2.3) \qquad \sum_{n=1}^{\infty}\frac{(b)_n(c)_n}{n(2c)_n(\frac{1+b}{2})_n}\Big\{\sum_{k=1}^{n}\Big(\frac{1}{k}-\frac{1}{2k+b-1}\Big)-\frac{1}{n}\Big\}$$
$$= \frac{1}{8}\Big[\Big\{\psi\Big(\frac{1}{2}+\frac{1}{2}b\Big)+\psi\Big(\frac{1}{2}+c\Big)-\psi\Big(\frac{1}{2}-\frac{1}{2}b+c\Big)-\psi\Big(\frac{1}{2}\Big)\Big\}^2 + \zeta(2,\frac{1}{2}+\frac{1}{2}b)+\zeta(2,\frac{1}{2}-\frac{1}{2}b+c)-\zeta(2,\frac{1}{2})-\zeta(2,\frac{1}{2}+c)\Big].$$

Consider the parameter a in (2.1) as a variable z and let

(2.4)
$$\begin{aligned} g(z) :&= 1+\sum_{n=1}^{\infty}\frac{(b)_n(c)_n}{n!(2c)_n}\frac{(z)_n}{(\frac{1}{2}z+\frac{1}{2}b+\frac{1}{2})_n} \\ &= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}z+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}z-\frac{1}{2}b+c)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}z)\Gamma(\frac{1}{2}-\frac{1}{2}z+c)\Gamma(\frac{1}{2}+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}b+c)}. \end{aligned}$$

Using (1.7), (1.12), (1.13), and (1.14), we obtain
(2.5)
$$(z)_n = (n-1)!\left[z+\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)z^2+\frac{1}{2}\left\{\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)^2-\sum_{k=1}^{n-1}\frac{1}{k^2}\right\}z^3+\cdots\right]$$

and

$$\begin{aligned} \left(\frac{z+b+1}{2}\right)_n &= \left(\frac{1+b}{2}\right)_n\left[1+\frac{1}{2}\left(\psi\left(\frac{1+b}{2}+n\right)-\psi\left(\frac{1+b}{2}\right)\right)z\right. \\ &\quad +\frac{1}{8}\left\{\left(\psi\left(\frac{1+b}{2}+n\right)-\psi\left(\frac{1+b}{2}\right)\right)^2-\psi^{(1)}\left(\frac{1+b}{2}+n\right)\right. \\ &\quad \left.\left.-\psi^{(1)}\left(\frac{1+b}{2}\right)\right\}z^2+\cdots\right]. \end{aligned}$$

Expanding the first member of (2.4) in terms of the Stirling numbers with the help of (2.5), we have

$$g(z) = 1+\sum_{n=1}^{\infty}\frac{(2b)_n(c)_n}{n(2c)_n(\frac{1+b}{2})_n}(z+\alpha_1z^2+\alpha_2z^3+\cdots), \tag{2.6}$$

where

$$\alpha_1 = \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{(2k+b-1)}\right)-\frac{1}{n},$$

$$\alpha_2 = \frac{1}{8}\left(\sum_{k=1}^{n}\frac{1}{2k+b-1}-\sum_{k=1}^{n-1}\frac{2}{k}\right)^2+\frac{1}{2}\sum_{k=1}^{n}\left\{\frac{1}{(2k+b-1)^2}-\frac{1}{k^2}\right\}+\frac{1}{2n^2}.$$

Now, to expand the right member of (2.4) in the Maclaurin series, we let
(2.7)
$$g(z) := \sum_{n=0}^{\infty}a_nz^n = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}z+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}z-\frac{1}{2}b+c)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}z)\Gamma(\frac{1}{2}-\frac{1}{2}z+c)\Gamma(\frac{1}{2}+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}b+c)}$$

which, upon setting z=0, yields $a_0 = g(0) = 1$. The logarithmic derivative of (2.7) with respect to z yields

$$\begin{aligned}\frac{g'(z)}{g(z)} =&\frac{1}{2}\psi\Big(\frac{1}{2}+\frac{1}{2}z+\frac{1}{2}b\Big)-\frac{1}{2}\psi\Big(\frac{1}{2}-\frac{1}{2}z-\frac{1}{2}b+c\Big)\\ &-\frac{1}{2}\psi\Big(\frac{1}{2}+\frac{1}{2}z\Big)+\frac{1}{2}\psi\Big(\frac{1}{2}-\frac{1}{2}z+c\Big)\end{aligned} \tag{2.8}$$

Let

$$\frac{g'(z)}{g(z)} := \sum_{n=0}^{\infty} c_n z^n. \tag{2.9}$$

Considering (2.8) and (1.9), we readily obtain
(2.10)

$$\begin{aligned}c_0 &= \frac{1}{2}\Big\{\psi\Big(\frac{1}{2}+\frac{1}{2}b\Big)-\psi\Big(\frac{1}{2}-\frac{1}{2}b+c\Big)-\psi\Big(\frac{1}{2}\Big)+\psi\Big(\frac{1}{2}+c\Big)\Big\},\\ c_n &= \frac{(-1)^{n+1}}{2^{n+1}}\Big\{\zeta(n+1,-\frac{1}{2}+\frac{1}{2}b)+(-1)^{n+1}\zeta(n+1,\frac{1}{2}-\frac{1}{2}b+c)\\ &-\zeta(n+1,\frac{1}{2})+(-1)^n\zeta(n+1,\frac{1}{2}+c)\Big\} \quad (n\in\mathbf{N}).\end{aligned}$$

From (2.7) and (2.9) we find that

$$\begin{aligned}g'(z) &= \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n = \frac{g'(z)}{g(z)}g(z)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}a_{n-k}c_k\right)z^n,\end{aligned}$$

which, upon equating coefficients of z^n, yields

$$(n+1)a_{n+1} = \sum_{k=0}^{n} a_{n-k}c_k \qquad (n\in\mathbf{N}). \tag{2.11}$$

Note that

$$a_1 = \frac{g'(0)}{1!} = \frac{g'(0)}{g(0)} = c_0. \tag{2.12}$$

From (2.8), (2.9), and (2.10), we obtain

$$
\begin{aligned}
a_2 = \frac{1}{8}\Big[\Big\{\psi\Big(\frac{1}{2}+\frac{1}{2}b\Big)-\psi\Big(\frac{1}{2}-\frac{1}{2}b+c\Big)+\psi\Big(\frac{1}{2}+c\Big)-\psi\Big(\frac{1}{2}\Big)\Big\}^2 \\
+\zeta(2,\frac{1}{2}b+\frac{1}{2})+\zeta(2,\frac{1}{2}-\frac{1}{2}b+c)-\zeta(2,\frac{1}{2})-\zeta(2,\frac{1}{2}+c)\Big].
\end{aligned} \tag{2.13}
$$

Finally comparing the coefficients of z^n in (2.6) and (2.7) with the aid of (2.11), (2.12)and (2.13) immediately reaches at our desired results (2.2) and (2.3). It is easy to rewrite (2.2) as in the following equivalent form:
(2.14)

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix} b+1, c+1, 1, 1\,; \\ \frac{b+3}{2},\ 2c+1,\ 2\,; \end{matrix} 1\right] = \frac{b+1}{2b}\Big\{\psi\Big(\frac{1}{2}+\frac{1}{2}b\Big) \\
-\psi\Big(\frac{1}{2}-\frac{1}{2}b+c\Big)-\psi\Big(\frac{1}{2}\Big)+\psi\Big(\frac{1}{2}+c\Big)\Big\}.
\end{aligned}
$$

Setting b=1 in (2.2) and (2.3) with the aid of (1.7) and (1.11), we obtain the following interesting special cases:

$$
\sum_{n=1}^{\infty}\frac{(c)_n}{n(2c)_n}=\frac{1}{2}\left\{\psi\Big(\frac{1}{2}+c\Big)-\psi(c)+2\log 2\right\}; \tag{2.15}
$$

(2.16)

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{(c)_n}{n(2c)_n}\left\{\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{n}\right\} \\
=\frac{1}{8}\Big[\Big\{\psi\Big(\frac{1}{2}+c\Big)-\psi(c)+2\log 2\Big\}^2+\zeta(2,c)-\zeta(2,\frac{1}{2}+c)-2\zeta(2)\Big],
\end{aligned}
$$

which, for c=1, in view of (1.7), (1.9) and (1.10), yields

$$
\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\Big(\sum_{k=1}^{n}\frac{1}{k}\Big)=\zeta(2), \tag{2.17}
$$

with which it is interesting to compare the known result (see [5, pp. 22, Eq. (2.23)]):

$$
\sum_{n=1}^{\infty}\frac{1}{n2^{n-1}}\Big(\sum_{k=1}^{n}\frac{1}{k}\Big)=\zeta(2). \tag{2.18}
$$

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