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PRIME IDEALS IN LIPSCHITZ ALGEBRAS OF FINITE DIFFERENTIABLE FUNCTIONS

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Abstract. Lipschitz Algebras $\text{Lip}(X, \alpha)$ and $\text{lip}(X, \alpha)$ were first studied by D. R. Sherbert in 1964. B. Pavlovic in 1995 shown that in these algebras, the prime ideals containing a given prime ideal form a chain. In this paper, we show that the above property holds in $\text{Lip}^n(X, \alpha)$ and $\text{lip}^n(X, \alpha)$, the Lipschitz algebras of finite differentiable functions on a perfect compact place set X.

0. Introduction

Let (X, d) be a compact metric space, and take α with $0 < \alpha \leq 1$. Then $\text{Lip}(X, \alpha)$ is the complex algebra of bounded complex-valued function f on X such that

$$P_\alpha(f) = \sup\left\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in X, x \neq y\right\}$$

is finite, and for $0 < \alpha < 1$, $\text{lip}(X, \alpha)$ is the subalgebra of functions f on X such that

$$\frac{|f(x) - f(y)|}{d^\alpha(x, y)} \longrightarrow 0 \quad \text{as} \quad d(x, y) \longrightarrow 0.$$

For $f \in \text{lip}(X, \alpha)$. Set $\|f\|_\alpha = \|f\|_X + P_\alpha(f)$, where $\|f\|_X$ the is uniform norm on X. Then $(\text{Lip}(X, \alpha), \|\cdot\|_\alpha)$ and $(\text{lip}(X, \alpha), \|\cdot\|_\alpha)$ are Banach function algebras on X with the Lipschitz norm $\|\cdot\|_\alpha$. These algebras were first studied by sherbert [4].

Throughout this paper, we assume that $0 < \alpha \leq 1$ for $\text{Lip}(X, \alpha)$ and $0 < \alpha < 1$ for $\text{lip}(X, \alpha)$, and from now on, X will denote a perfect compact plane set.

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A complex-valued function f on X is called differentiable at $z_0 \in X$ if

$$f'(z_0) = \lim \left\{ \frac{f(z) - f(z_0)}{z - z_0} : z \in X, z \longrightarrow z_0 \right\}$$

exists, and f is called differentiable on X if it is differentiable at each point of X.

The algebra of functions f on X whose derivatives up to order n exist and for each $k (0 \leq k \leq n)$, $f^{(k)} \in \mathrm{Lip}(X, \alpha)$ is denoted by $\mathrm{Lip}^n(X, \alpha)$. The algebra $\mathrm{lip}^n(X, \alpha)$ is defined in a similar way. For f in $\mathrm{Lip}^n(X, \alpha)$ or in $\mathrm{lip}^n(X, \alpha)$, let

$$\|f\|_{\alpha,n} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\alpha}{k!} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X + P_\alpha(f^{(k)})}{k!}.$$

If X is a perfect compact plane set such that $D^1(X)$ [1] is complete, then for $n \geq 0$, $(\mathrm{Lip}^n(X, \alpha), \|\cdot\|_{\alpha,n})$ and $(\mathrm{lip}^n(X, \alpha), \|\cdot\|_{\alpha,n})$ are natural Banach function algebras on X. These algebras were first studied by T. G. Honary and H. Mahyar [2].

1. Prime Ideals in $\mathrm{Lip}^n(X, \alpha)$ and $\mathrm{lip}^n(X, \alpha)$

Our goal in this section is theorem which shows that the prime ideals of $\mathrm{Lip}^n(X, \alpha)$ (or $\mathrm{lip}^n(X, \alpha)$) containing a given prime ideal form a chain (with respect to the inclusion order).

DEFINITION 1.1. Let X be a perfect compact plane set. We define, $\mathrm{Lip}^n_{\mathbb{R}}(X, \alpha) := \{f \in \mathrm{Lip}^n(X, \alpha) : f \text{ is real}\}$ and $\mathrm{lip}^n_{\mathbb{R}}(X, \alpha) := \{f \in \mathrm{lip}^n(X, \alpha) : f \text{ is a real}\}$. It is easy to see that, $\mathrm{Lip}^n_{\mathbb{R}}(X, \alpha)$ (respectively, $\mathrm{lip}^n_{\mathbb{R}}(X, \alpha)$), are real subspaces of $\mathrm{Lip}^n(X, \alpha)$ (respectively, $\mathrm{lip}^n(X, \alpha)$).

The following definition is a similar definition with [3, Definition 4.3].

DEFINITION 1.2. Let $A = \mathrm{Lip}^n_{\mathbb{R}}(X, \alpha)$ or $A = \mathrm{lip}^n_{\mathbb{R}}(X, \alpha)$. Let $\preceq$ be a partial order on A, and assume $\preceq$ is compativle with addition, i.e., $\forall f, g, h \in A$,

$$f \preceq g \Longrightarrow f + h \preceq g + h.$$

A subspace $I \subseteq A$ is $\preceq$-convex if $\forall f, g \in A$, $0 \preceq f \preceq g$ and $g \in I$ implies $f \in I$.

Denote by $\leq$ the usual pointwise order on $A = \mathrm{Lip}^n_{\mathbb{R}}(X, \alpha)$ or $A = \mathrm{lip}^n_{\mathbb{R}}(X, \alpha)$, i.e., $\forall f, g \in A$, $f \leq g$ if and only if $\forall t \in X$, $f(t) \leq g(t)$.

LEMMA 1.3. *Let $A = Lip^n_{\mathbb{R}}(X, \alpha)$, or $A = lip^n_{\mathbb{R}}(X < \alpha)$. Let $P \subseteq A$ be a prime ideal. Then P is $\leq$-convex.*

Proof. Suppose $0 \leq f \leq g$ and $g \in P$, $f \in A$. Since P is prime it is enough to show that $f^{4^n} \in P$. We do this by induction on n, by showing that $h_n \in A$, where

$$h_n(t) := \begin{cases} 0 & \text{if} \quad g(t) = 0 \\ \frac{f^{4^n}(t)}{g(t)} & \text{if} \quad g(t) \neq 0. \end{cases}$$

Assume first that $n = 1$, and claim that $h_1 \in A$. We first show that $h_1 \in D^1_{\mathbb{R}}(X)$. Evidently, only points t_0, where $g(t_0) = 0$, but when $g(t) \neq 0$ in every neighborhood of t_0, need examination. If $t_k \longrightarrow t_0$ with $g(t_k) \neq 0$, then $|h_1(t_k)| = |f^4(t_k)/g(t_k)| = |f(t_k)/g(t_k)||f^s(t_k)| \leq |f^3(t_k)| \longrightarrow 0$ so h_1 is continuous. As $0 \leq f$ and $g(t_0) = f(t_0) = 0$, t_0 is a local minimum for f so $f'(t_0) = 0$; consequently

$$\begin{aligned} 0 \leq \lim_{t \to t_0\ (g(t) \neq 0)} \left| \frac{h_1(t) - h_1(t_0)}{t - t_0} \right| &= \lim_{t \to t_0\ (g(t) \neq 0)} \left| \frac{f(t) f^2(t)}{g(t)} \cdot \frac{f(t)}{t - t_0} \right| \\ \leq \lim_{t \to t_0\ (g(t) \neq 0)} \left| f^2(t) \cdot \frac{f(t)}{t - t_0} \right| &= f^2(t_0)|f'(t_0)| = 0. \end{aligned}$$

Therefore,

$$h_1'(t) = \begin{cases} 0 & \text{if} \quad g(t) = 0 \\ \frac{4f^3(t)f'(t)}{g(t)} - \frac{f^4(t)g'(t)}{g^2(t)} & \text{if} \quad g(t) \neq 0. \end{cases}$$

Now, we show that h' is continuous. Let $t_k \longrightarrow t_0$. If $g(t_{k_n}) = 0$ for some subsequence $\{t_{k_n}\}$, then by what has been shown $h'(t_{k_n}) = 0 = h'(t_0)$. On the other hand, if $g(t_k) \neq 0$ for all k, then

$$\begin{aligned} |h_1'(t_k)| &= \left| 4\frac{f^3(t_k)f'(t_k)}{g(t_k)} - \frac{f^4(t_k)g'(t_k)}{g^2(t_k)} \right| \\ &\leq 4|f^2(t_k)||f'(t_k)| + |f^2(t_k)||g'(t_k)| \longrightarrow 0. \end{aligned}$$

Thus $h_1 \in D^1_{\mathbb{R}}(X)$. Since by [2] $D^1_{\mathbb{R}}(X) \subseteq \mathrm{Lip}(X, \alpha)$ or $\mathrm{lip}(X, \alpha)$, we have $h_1 \in \mathrm{Lip}(X, \alpha)$ or $\mathrm{lip}(X, \alpha)$.

It remains to see that $h_1' \in \mathrm{Lip}_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}_{\mathbb{R}}(X,\alpha)$. Suppose first that $g(t)\neq 0$, and $g(s)\neq 0$. Then

$$
\begin{aligned}
\frac{|h_1'(t)-h_1'(s)|}{|t-s|^\alpha} &= \frac{|\frac{4f^3(t)f'(t)}{g(t)} - \frac{f^4(t)g'(t)}{g^2(t)} - \frac{4f^3(s)f'(s)}{g(s)} + \frac{f^4(s)g'(s)}{g^2(s)}|}{|t-s|^\alpha}\\
&= \frac{|\frac{4f^3(t)f'(t)}{g(t)} - \frac{f^4(t)g'(t)}{g^2(t)} - \frac{4f^3(s)f'(s)}{g(s)} + \frac{f^4(s)g'(s)}{g^2(s)} + \frac{4f^3(t)f'(s)}{g(t)} - \frac{4f^3(t)f'(s)}{g(t)}}{|t-s|^\alpha}\\
&+ \frac{\frac{4f^2(t)f(s)f'(s)}{g(t)} - \frac{4f^2(t)f(s)f'(s)}{g(t)} + \frac{4f(t)f^2(s)f'(s)}{g(t)} - \frac{4f(t)f^2(s)f'(s)}{g(t)}}{|t-s|^\alpha}\\
&+ \frac{\frac{f^2(t)}{g^2(t)}f^2(t)g'(s) - \frac{f^2(t)}{g^2(t)}f^2(t)g'(s) + \frac{f^2(s)}{g^2(s)}f^2(t)g'(s) - \frac{f^2(s)}{g^2(s)}f^2(t)g'(s)|}{|t-s|^\alpha}\\
\le& 4|\frac{f^3(t)}{g(t)}|\frac{|f'(t)-f'(s)|}{|t-s|^\alpha} + 4|\frac{f^2(t)f'(s)}{g(t)}|\frac{|f(t)-f(s)|}{|t-s|^\alpha} + 4|\frac{f(t)f(s)f'(s)}{g(t)}|\\
&\times \frac{|f(t)-f(s)|}{|t-s|^\alpha} + 4|f^2(s)f'(s)|\frac{|f(s)/g(s) - f(t)/g(t)|}{|t-s|^\alpha}\\
&+ \frac{f^4(t)}{g^2(t)}\frac{|g'(t)-g'(s)|}{|t-s|^\alpha} + \frac{f^2(s)}{g^2(s)}\frac{|f^2(s)-f^2(t)||g'(s)|}{|t-s|^\alpha}\\
&+ \frac{|f^2(t)||g'(s)|}{g^2(t)}\frac{|f^2(t)g^2(s)-f^2(s)g^2(t)|}{g^2(s)|t-s|^\alpha}\\
\le& 4\|f^2\|_X\frac{|f'(t)-f'(s)|}{|t-s|^\alpha} + 4\|f\|_X\|f'\|_X\frac{|f(t)-f(s)|}{|t-s|^\alpha}\\
&+ 4\|f\|_X\|f'\|_X\frac{|f(t)-f(s)|}{|t-s|^\alpha} + \frac{4|f^2(s)f'(s)|}{|g(t)g(s)|}\cdot\frac{|f(s)g(t)-f(t)f(s)|}{|t-s|^\alpha}\\
&+ \|f^2\|_X\frac{|g'(t)-g'(s)|}{|t-s|^\alpha} + + \|g'\|_X\frac{|f(s)-f(t)||f(s)+f(t)|}{|t-s|^\alpha}\\
&+ \|g'\|_X\frac{|f^2(t)g^2(s)+f^2(t)g^2(t)-f^2(t)g^2(t)-f^2(s)g^2(s)|}{|t-s|^\alpha g^2(s)}\\
\le& 4\|f^2\|_X\frac{|f'(t)-f'(s)|}{|t-s|^\alpha} + 4\|f\|_X\|f'\|_X\frac{|f(t)-f(s)|}{|t-s|^\alpha}\\
&+ 4\|f\|_X\|f'\|_X\frac{|f(t)-f(s)|}{|t-s|^\alpha} + \frac{4|f^2(s)f'(s)|}{|g(t)g(s)|}\frac{|f(s)-f(t)||g(t)|}{|t-s|^\alpha}\\
&+ \frac{4|f^2(s)f'(s)|}{|g(t)g(s)|}\cdot\frac{|f(t)||g(t)-g(s)|}{|t-s|^\alpha} + \|g'\|_X\|g^2\|_X\frac{|f^2(t)-f^2(s)|}{|t-s|^\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|f^2\|_X \frac{|g'(t) - g'(s)|}{|t-s|^\alpha} + \|g'\|_X \frac{|f(s) - f(t)||f(s) + f(t)|}{|t-s|^\alpha} \\
&+ \|g'\|_X \frac{|f^2(t)g^2(s) + f^2(t)g^2(t) - f^2(t)g^2(t) - f^2(s)g^2(s)|}{|t-s|^\alpha g^2(s)} \\
\le & 4\|f^2\|_X \frac{|f'(t) - f'(s)|}{|t-s|^\alpha} + 8\|f\|_X\|f'\|_X \frac{|f(t) - f(s)|}{|t-s|^\alpha} \\
&+ 4\|f\|_X\|f'\|_X \frac{|f(t) - f(s)|}{|t-s|^\alpha} + 4\|f\|_X\|f'\|_X \frac{|g(t) - g(s)|}{|t-s|^\alpha} \\
&+ \|f^2\|_X \frac{|g'(t) - g'(s)|}{|t-s|^\alpha} + 2\|g'\|_X\|f\|_X \frac{|f(t) - f(s)|}{|t-s|^\alpha} \\
&+ 2\|g'\|_X\|f^2\|_X\|g\|_X \frac{|g(s) - g(t)|}{|t-s|^\alpha} \\
&+ 2\|f\|_X\|g'\|_X\|g^2\|_X \frac{|f(s) - f(t)|}{|t-s|^\alpha} \; |t-s| \to 0 \longrightarrow 0 \\
&\begin{cases}
\le 4\|f^2\|_X P_\alpha(f') + 12\|f\|_X\|f'\|_X P_\alpha(f) + 4\|f\|_X\|f'\|_X P_\alpha(g) \\
\quad + \|f^2\|_X P_\alpha(g') + 2\|g'\|_X\|f\|_X P_\alpha(f) + 2\|g'\|_X\|f^2\|_X\|g\|_X P_\alpha(g) \\
\quad + 2\|f\|_X\|g'\|_X\|g^2\|_X P_\alpha(f) \quad \text{when } A = \mathrm{Lip}^1_{\mathbb{R}}(X,\alpha) \\
\longrightarrow 0 \text{ as } |t-s| \longrightarrow 0 \quad \text{when } A = \mathrm{lip}^1_{\mathbb{R}}(X,\alpha).
\end{cases}
\end{aligned}
$$

Now suppose that $g(t) \neq 0$, but $g(s) = 0$. Then $f'(s) = g'(s) = 0$, $h_1'(s) = 0$, and we have

$$
\begin{aligned}
\frac{|h_1^{'(t)} - h_1'(s)|}{|t-s|^\alpha} &= \frac{|4f^3(t)f'(t)/g(t) - f^4(t)g'(s)/g^2(t)|}{|t-s|^\alpha} \\
&\le \frac{\left|\frac{4f^3(t)f'(t)}{g(t)}\right|}{|t-s|^\alpha} + \frac{\left|\frac{|f^4(t)g'(t)}{g^2(t)|}\right|}{|t-s|^\alpha} \\
&= \frac{\left|\frac{4f^3(t)f'(t)}{g(t)} - \frac{4f^3(t)f'(s)}{g(t)|}\right|}{|t-s|^\alpha} + \frac{\left|\frac{f^4(t)g'(t)}{g^2(t)} - \frac{f^2(t)g'(s)}{g^2(t)}\right|}{|t-s|^\alpha} \\
&= \frac{\frac{4f^3(t)}{g(t)}|f'(t) - f'(s)|}{|t-s|^\alpha} + \frac{\frac{f^4(t)}{g^2(t)}|g'(t) - g'(s)|}{|t-s|^\alpha} \\
&\le 4\|f^2\|_X \frac{|f'(t) - f'(s)|}{|t-s|^\alpha} + \|f^2\|_X \frac{g'(t) - g'(s)|}{|t-s|^\alpha} \\
&\begin{cases}
\le 4\|f^2\|_X P_\alpha(f') + \|f^2\|_X P_\alpha(g') & \text{when } A = \mathrm{Lip}^1_{\mathbb{R}}(X,\alpha) \\
\longrightarrow 0 \text{ as } |t-s| \longrightarrow 0 & \text{when } A = \mathrm{lip}^1_{\mathbb{R}}(X,\alpha).
\end{cases}
\end{aligned}
$$

Thus $h_1 \in A$ and since $f^4 = h_1 g$ and $g \in P$ we have shown that f^4, hence $f \in P$.

And now the inductive step: suppose $f, g \in Lip^{n+1}_{\mathbb{R}}(X,\alpha)$, or $\mathrm{lip}^{n+1}_{\mathbb{R}}(X,\alpha)$, $g \in P$ and $0 \leq f \leq g$. Similar with the case $n = 1$, $h'_{n+1}(t) = 0$ whenever $g(t) = 0$. Moreover, if $g(t) \neq 0$, then

$$\begin{aligned} h'_{n+1}(t) &= 4^{n+1}\frac{f^{4^{n+1}-1}}{g}f' - \frac{f^{4^n+1}}{g^2}g' \\ &= 4^{n+1}f^{4^{n+1}-1-4^n}.f' - h_n^4 g^2 g' \end{aligned}$$

From the expressions for h'_{n+1} we see that $h'_{n+1} \in \mathrm{Lip}^n_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}^n_{\mathbb{R}}(X,\alpha)$, by inductive hypothesis, hence that $h_{n+1} \in \mathrm{Lip}^{n+1}_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}^{n+1}_{\mathbb{R}}(X,\alpha)$ and since clearly have the equation $h_{n+1}g = f^{4^{n+1}}$ it follows that $f \in P$.

Definition 1.4. Let $A = \mathrm{Lip}^n_{\mathbb{R}}(X,\alpha)$ or $A = \mathrm{lip}^n_{\mathbb{R}}(X,\alpha)$, P a prime ideal in A. We define

$$K(P) = \{f \in \mathrm{Lip}_{\mathbb{R}}(X,\alpha) \ \text{ or } \ \mathrm{lip}_{\mathbb{R}}(X,\alpha) : \ \exists g \in P \text{ and } \ q \in \mathbb{N} : |f|^q \leq g\}.$$

Lemma 1.5. *$K(P)$ is a $\leq$-convex ideal in $Lip_{\mathbb{R}}(X,\alpha)$ or $lip(X,\alpha)$.*

Proof. Convexity $K(P)$ is immediate from the definition. If $f_1 \in \mathrm{Lip}(X,\alpha)$ or $\mathrm{lip}(X,\alpha)$ and $f_2 \in K(P)$ so that $|f_2|^k \leq g$ for suitable $k \in \mathbb{N}$ and $g \in P$, then $|f_1 f_2|^k \leq \|f_1\|^k_X |f_2|^k \leq \|f_1\|^k_X g$ so it remains only to observe that $K(P)$ is a subspace: if $f_1, f_2 \in K(P)$ and $|f_j|^{n_j} \leq g_j$, $j = 1, 2$

with $g_j \in P$, then

$$\begin{aligned}
|f_1+f_2|^{n_1+n_2} &\le (|f_1|+|f_2|)^{n_1+n_2} \\
&= \sum_{j=0}^{n_1+n_2} \binom{n_1+n_2}{j} |f|^j |f_2|^{n_1+n_2-j} \\
&= \sum_{j=0}^{n_1} \binom{n_1+n_2}{j} |f_1|^j |f_2|^{n_1+n_2+j} \\
&+ \sum_{j=n_1+1}^{n_1+n_2} \binom{n_1+n_2}{j} |f_1|^j \cdot |f_2|^{n_1+n_2-j} \\
&\le \sum_{j=0}^{n_1} \binom{n_1+n_2}{j} \|f_1\|_X^j \|f_2\|_X^{n_1-j} g_2 \\
&+ \sum_{j=n_1+1}^{n_1+n_2} \binom{n_1+n_2}{j} \|f_1\|_X^{j-n_1} g_1 \|f_2\|^{n_1+n_2-j} \\
&\le N(g_1+g_2)
\end{aligned}$$

$$N = \max\{\sum_{j=0}^{n_1} \binom{n_1+n_2}{j} \|f_1\|_X^j \|f_2\|_X^{n_1-j},$$

$$\sum_{j=n+1}^{n_1+n_2} \binom{n_1+n_2}{j} \|f_1\|_X^{j-n_1} \|f_2\|_X^{n_1+n_2-j}\}.$$

For a real valued function h, denote by h^+ the function

$$h^+ = h \vee 0 = \begin{cases} h(t) & \text{if } h(t) \ge 0 \\ 0 & \text{if } h(t) < 0. \end{cases}$$

And let $h^- = h^+ - h$. Obviously, for $f \in \mathrm{Lip}_{\mathbb{R}}(X,\alpha)$ (respectively $\mathrm{lip}_{\mathbb{R}}(X,\alpha)$), $f^+, f^- \in \mathrm{Lip}_{\mathbb{R}}(X,\alpha)$ (respectively $\mathrm{lip}_{\mathbb{R}}(X,\alpha)$), and $\|f^+\|_\alpha \le \|f\|_\alpha$, $\|f^-\|_\alpha \le \|f\|_\alpha$.

The following definition is as [3, Definition 4.5].

DEFINITION 1.6. Let $A = \mathrm{Lip}_{\mathbb{R}}^n(X,\alpha)$ or $A = \mathrm{lip}_{\mathbb{R}}^n(X,\alpha)$. Let P be a prime ideal in A. Let $f, g \in A$. We say that $f \le_P g$ if $(g-f)^- \in K(P)$. And that $f \sim_P g$ if and only if $f \le_P g$ and $g \le_P f$.

LEMMA 1.7. *$\leq_P$ is a partial order, compatible with addition in $Lip_{\mathbb{R}}(X,\alpha)$, or $lip_{\mathbb{R}}(X,\alpha)$, and $f \sim_P g$ if and only if $f-g \in K(P)$.*

Proof. Transitivity:

$$f_1 \leq_P f_2 \leq_P f_3 \Longleftrightarrow (f_2-f_1)^-, (f_3-f_2)^- \in K(P).$$

Since

$$\begin{aligned} 0 \leq (f_3-f_1)^- &= (f_3-f_2+f_2-f_1)^- \\ &\leq (f_3-f_2)^- + (f_2-f_1)^- \in K(P) \end{aligned}$$

the claim is a consequence of the convexity of $K(P)$. The rest is trivial.

The following elementary observation about differentiability of f^+ and f^- is quite significant in the rest of the development.

LEMMA 1.8. *Let $A = Lip^n_{\mathbb{R}}(X,\alpha)$ or $A = lip^n_{\mathbb{R}}(X,\alpha)$ and $f \in A$. Then $(f^{\pm})^k \in A$ for $k \geq n+1$ and $[(f^{\pm})^k]' = k(f^{\pm})^{k-1}f'$ for $k \geq 2$.*

Proof. We prove this for f^+. The argument is by induction. We begin by showing that $(f^+)^2 \in \mathrm{Lip}^1_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}^1_{\mathbb{R}}(X,\alpha)$ and $((f^+)^2)' = 2f^+f'$. If $f(t_0) \neq 0$, then f has constant sign in a neighborhood of t_0 and if follows immediately that $(f^+)^2$ is $D^1_{\mathbb{R}}(X)$ in this neighborhood. If $(f^+)^2 = f^+f$, we get

$$\lim_{t\to t_0} \frac{(f^+(t))^2-(f^+(t_0))^2}{t-t_0} = \lim_{t\to t_0} \frac{f^+(t)f(t)}{t-t_0} = 0.$$

In either case it is obvious that we get the equation $((f^+)^2)' = 2f^+f$ and since f^+ is continuous, we have shown that $(f^+)^2 \in D^1_{\mathbb{R}}(X)$. Therefore, by [2], $(f^+)^2 \in \mathrm{Lip}_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}_{\mathbb{R}}(X,\alpha)$. It remains to see that $[(f^+)^2]' \in \mathrm{Lip}_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}_{\mathbb{R}}(X,\alpha)$. We have

$$\begin{aligned} &\frac{|[(f^+)^2]'(t)-[(f^+)^2]'(s)|}{|t-s|^\alpha} = \frac{|2f^+(t)f'(t)-2f^+(s)f'(s)|}{|t-s|^\alpha} \\ &= \frac{|2f^+(t)f'(t)-2f^+(t)f'(s)+2f^+(t)f'(s)-2f^+(s)f'(s)|}{|t-s|^\alpha} \\ &\leq \frac{2|f^+(t)||f'(t)-f'(s)|}{|t-s|^\alpha} + \frac{2|f'(s)||f^+(t)-f^+(s)|}{|t-s|^\alpha} \\ &\begin{cases} \leq 2\|f\|_X P_\alpha(f') + 2\|f'\|_X P_\alpha(f) & \text{when } A = \mathrm{Lip}^1_{\mathbb{R}}(X,\alpha) \\ \longrightarrow 0 \text{ as } |t-s| \longrightarrow 0 & \text{when } A = \mathrm{lip}^1_{\mathbb{R}}(X,\alpha). \end{cases} \end{aligned}$$

Thus $(f^+)^2 \in \mathrm{Lip}^1_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}^1_{\mathbb{R}}(X,\alpha)$.

Now suppose $n \geq 2$, and suppose we know that $(f^+)^k \in \mathrm{Lip}^m_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}^m_{\mathbb{R}}(X,\alpha)$ and $((f^+)^k)' = k(f^+)^{k-1}f'$ if $m < k \leq n$ and $f \in \mathrm{Lip}^m_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}^m_{\mathbb{R}}(X,\alpha)$. Fix $m \leq n$, fix $f \in \mathrm{Lip}^m_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}^m_{\mathbb{R}}(X,\alpha)$, consider $(f^+)^{n+1} = (f^+)^n f$, and compute

$$((f^+)^{n+1})' = (f^+)^n f' + n(f^+)^{n-1}f'f = (n+1)(f^+)^n f'.$$

Since $(f^+)^n$ and f' both belong to $\mathrm{Lip}^{m-1}_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}^{m-1}_{\mathbb{R}}(X,\alpha)$, this shows that $(f^+)^{n+1} \in \mathrm{Lip}^m_{\mathbb{R}}(X,\alpha)$ or $\mathrm{lip}^m_{\mathbb{R}}(X,\alpha)$.

PROPOSITION 1.9. *Let $A = Lip^n_{\mathbb{R}}(X,\alpha)$ or $A = lip^n_{\mathbb{R}}(X,\alpha)$. Then order $\leq_P$ has the following properties*

(i) *P is $\leq_P$-convex.*

(ii) *A/P is totally ordered by the order $\leq_P$ induced by $\leq_P$.*

(iii) *Any prime ideal in A/P is convex with respect to $\leq_P$.*

Proof. (i) We begin by noting that $K(P) \cap A = P$. Since $\supseteq$ is obvious we check $\subseteq$: if $f \in K(P) \cap A$, then there is an integer q such that $|f|^q \leq g$ for suitable $g \in P$. But then $f^{2q} \leq g^2$ and since P is $\leq_P$-convex, $f^{2q} \in P$. As P is prime this shows that $f \in P$.

Clearly $K(P)$ is $\leq_P$-convex: if $0 \leq_P f \leq_P g$ in $\mathrm{Lip}(X,\alpha)$ or $\mathrm{lip}(X,\alpha)$ and $g \in K(P)$, then $(g-f)^- \in K(P)$ and since $f^-, g^- \in K(P)$ we get that $g^+ \in K(P)$ so $f = f - g + g$ and this implies $0 \leq f^+ \leq (f-g)^+ + g^+ = (g-f)^- + g^+ \in K(P)$ from which we conclude that $f = f^+ - f^- \in K(P)$.

As $P = K(P) \cap A$ this settles (i). It also shows that A/P is partially ordered by $\leq_P$.

Now let $f \in A$. By 1.8, $(f^+)^{n+1}$ and $(f^-)^{n+1}$ are in A and since

$$(f^+)^{n+1}(f^-)^{n+1} = 0,$$

it follows that $(f^+)^{n+1}$ or $(f^-)^{n+1} \in P$ and hence $f^+ \in K(P)$ or $f^- \in K(P)$.

Consequently, for any $f \in A$ we conclude that $f \geq_P 0$ (i.e., $f^- \in K(P)$) or $f \leq_P 0$ (i.e., $f^+ \in K(P)$) and $f \sim_P 0$ iff $f \in K(P) \cap A = P$.

We have shown that $\leq_P$ is a total order on A/P. Finally, let $I \subset A/P$ be a prime ideal and suppose

$$0 \leq_P a + P \leq_P b + P \quad \text{with } b + P \in I.$$

Pulling back to A we suppose $0 \leq_P a \leq_P b$ and $b \in$prime ideal $P' \supset P$ where $P' + P = I$ in A/P. $a^- \in K(P), (b-a)^- \in K(P)$, and $b \in P'$, so $0 \leq a^+ = (a-b+b)^+ \leq (a-b)^+ + b^+ = (b-a)^- + b^+ \in K(P) + K(P') = K(P')$. Hence $a = a^+ - a^- \in K(P')$. i.e., $a \in K(P') \cap A = P'$ and we are finished.

THEOREM 1.10. *Let $A = Lip^n(X, \alpha)$ or $A = lip^n(X, \alpha)$. Let P be a prime ideal in A. Then, the prime ideals of A/P form a chain with respect to the inclusion order.*

Proof. Consider first $A_{\mathbb{R}} = \mathrm{Lip}^n_{\mathbb{R}}(X, \alpha)$ or $A_{\mathbb{R}} = \mathrm{lip}^n_{\mathbb{R}}(X, \alpha)$, i.e., suppose P is a prime in $A_{\mathbb{R}}$. We must show that if P_1 and P_2 are primes in $A_{\mathbb{R}}$. We must show that if P_1 and P_2 are primes in $A_{\mathbb{R}}/P$ and if $P_1 \not\subseteq P_2$, then $P_2 \subseteq P_1$. Let $g \in P_1 \backslash P_2$ and let $f \in P_2$. By changing the signs, if necessary we may assume $0 \leq_P g$ and $0 \leq_P f$. Since $\leq_P$ is total and $f \sim_P g$ means $g \in P_2$, we must have $0 \leq_P g \leq_P f$ or $0 \leq_P f \leq_P g$. By 1.9 the fomer would mean $g \in P_2$, so the latter must hold. But then $f \in P_1$, i.e., $P_2 \subseteq P_1$. This proves the theorem for $A_{\mathbb{R}}$.

Suppose then that P is a prime in A and that P_1 and P_2 are primes in A/P. Now, if $I \subset A$ is any ideal let

$$I_{\mathbb{R}} = ReI \subset A_{\mathbb{R}}.$$

Obviously, $I_{\mathbb{R}}$ is a linear space and if $f \in I_{\mathbb{R}}$, $g \in A_{\mathbb{R}}$, then there is $f_1 \in I$ with $Ref_1 = f$; since $g \in A$ and $Re(f_1 g) = gRef_1 = gf$, we see that $I_{\mathbb{R}}$ is an ideal. Also if I is prime and $f_1, f_2 \in A_{\mathbb{R}}$ so that $f_1 f_2 \in I_{\mathbb{R}}$, then $f_1 \in I \cap A_{\mathbb{R}}$ or $f_2 \in I \cap A_{\mathbb{R}}$ or $f_2 \in I \cap A_{\mathbb{R}}$. This means $I_{\mathbb{R}}$ is prime. In our situation we conclude that $ReP_1 \subseteq ReP_2$; Say (by the real result already proved), and consequently $f \in P_1$ implies $Ref \in P_2$; since if $\in P_1$ we obtain $Re(if) \in P_2$, i.e., $-Imf \in P_2$. Hence $P_1 \subseteq P_2$ and we have the general result.

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