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EINSTEIN WARPED PRODUCT SPACES

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Abstract. We study Einstein warped product spaces. As a result, we prove the following: if M is an Einstein warped product space with base a compact 2-dimensional surface, then M is simply a Riemannian product space.

0. Introduction and Preliminaries

Let $B = (B^m, g_B)$ and $F = (F^k, g_F)$ be two Riemannian manifolds. We denote by π and σ the projections of $B \times F$ onto B and F, respectively. For a positive smooth function f on B the warped product $M = B \times_f F$ is the product $M = B \times F$ furnished with metric tensor g defined by $g = \pi^* g_B + f^2 \sigma^* g_F$, where $(^*)$ denotes pull back. The function f will be called the warping function.

The notion of warped product $B \times_f F$ generalizes that of a surface of revolution. It was introduced in [3] for studying manifolds of negative curvature.

Obviously, the Riemannian product $M = B \times F$ is Einstein if B and F are Einstein with the same scalar curvarures. A warped product $B \times_f F$ with a constant warping function f can be considered as a Riemannian product.

In search for a new compact Einstein space in ([2], p. 265), A. L. Besse asked the following:

"Does there exist a compact Einstein warped product with nonconstant warping function? "

In this article, we give a negative partial answer as follows(cf. [1], p. 127)

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THEOREM 1. *Let $M = B \times_f F$ be an Einstein warped product space. If B is a compact 2-dimensional Riemannian surface, then the warped product is simply a Riemannian product.*

For warped products with 1-dimensional base, see Theorem 9.110 in [2].

In [2] A. L. Besse state, without proof, a theorem(Theorem 9.119, due to R.S. Palais, C. L. Terng, A. Derdzinski) from which we may deduce Theorem 1. But we couldn't find a proof of the theorem anywhere. So we give an elementary proof of Theorem 1.

1. Proof of Theorem 1

We denote by Ric^B, Ric^F the lifts to M of Ricci curvatures of B and F, respectively. Then we have the following([7]):

PROPOSITION 2. *The Ricci curvature* Ric *of the warped product $M = B \times_f F$ with $k = \dim F$ satisfies*
(1) $\mathrm{Ric}(X,Y) = \mathrm{Ric}^B(X,Y) - \frac{k}{f}H^f(X,Y)$,
(2) $\mathrm{Ric}(X,V) = 0$,
(3) $\mathrm{Ric}(V,W) = \mathrm{Ric}^F(V,W) - g(V,W)f^\#$, $f^\# = \frac{-\Delta f}{f} + (k-1)\frac{g_B(\nabla f, \nabla f)}{f^2}$,
for any horizontal vectors X, Y and any vertical vectors V, W, where H^f and Δf denote the Hessian of f and the Laplacian of f given by $-tr(H^f)$, respectively.

Hence the Einstein equations become

COROLLARY 3. *The warped product $M = B \times_f F$ is Einstein(with* $\mathrm{Ric} = \lambda g$*) if and only if*
(1.1) $\mathrm{Ric}_B = \lambda g_B + \frac{k}{f}H^f$,
(1.2) (F, g_F) *is Einstein(with* $\mathrm{Ric}_F = \mu g_F$*),*
(1.3) $-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu$.

Now we give a proof of Theorem 1. Let (B, g_B) be a compact 2-dimensional Riemannian surface. By Theorem 1 in [5], we may assume that $\lambda > 0$. Since B is of 2-dimensional, the Ricci tensor satisfies $\mathrm{Ric}_B = Kg_B$, where K denotes the Gaussian curvature of B. Hence (1.1) becomes

$$H^f = \frac{f}{k}(K - \lambda)g_B. \tag{1.4}$$

Suppose that the warping function f is nonconstant. Then (1.4) shows that if p, q denote the minimum and maximum points of f, then $(B - \{p, q\}, g_B)$ is isometric with a warped product metric(Theorem 21 of [6])

$$ds^2 = dt^2 + f'(t)^2 d\theta^2 \tag{1.5}$$

on $(a, b) \times S^1$, where $f = f(t)$ and $f'(t) = \frac{df}{dt}$. Obviously, we have

$$f'(a) = f'(b) = 0. \tag{1.6}$$

Since the metric (1.5) extends to a C^∞ Riemannian metric on B, we may assume that ([2], p.269 or [6], p. 123)

$$f''(a) = -f''(b) = 1. \tag{1.7}$$

Note that $\Delta f = -2f''(t)$ in the metric (1.5). Hence (1.3) becomes

$$2f(t)f''(t) + (k-1)f'(t)^2 + \lambda f(t)^2 = \mu. \tag{1.8}$$

Hereafter, we assume that the dimension k of the fibre F is greater than or equal to 2. Integrating (1.8), we get

$$f'(t)^2 = \frac{\mu}{k-1} - \frac{\lambda}{k+1} f(t)^2 + \nu f(t)^{1-k}, \tag{1.9}$$

and hence

$$f''(t) = -\frac{\lambda}{k+1} f(t) - \frac{k-1}{2} \nu f(t)^{-k}, \tag{1.10}$$

where ν is a constant.

Now if we put

$$\begin{aligned} g(x) &= \frac{\mu}{k-1} - \frac{\lambda}{k+1} x^2 + \nu x^{1-k} \\ &= x^{1-k}\left(-\frac{\lambda}{k+1} x^{k+1} + \frac{\mu}{k-1} x^{k-1} + \nu\right), \end{aligned} \tag{1.11}$$

then we have $f'(t)^2 = g(f(t))$ and $f''(t) = \frac{1}{2} g'(f(t))$. If A, B denote the minimum $f(a) = f(p)$ and maximum $f(b) = f(q)$ of f, respectively, then (1.6) and (1.7) imply

$$g(A) = 0, \quad g''(A) = 2, \tag{1.12}$$

and

$$g(B) = 0, \quad g''(B) = -2. \tag{1.13}$$

From (1.11) and (1.12) we get

$$\nu = \frac{-2}{k^2 - 1}(\sqrt{1 + \mu\lambda} + k)A^k, \tag{1.14}$$

and

$$A = \frac{1}{\lambda}(\sqrt{1 + \mu\lambda} - 1). \tag{1.15}$$

And from (1.11) and (1.13) we obtain

$$B = \frac{1}{\lambda}(\sqrt{1 + \mu\lambda} + 1). \tag{1.16}$$

Since $g(B) = 0$, from (1.11), (1.14), (1.15) and (1.16) we see that the positive constant $y = \sqrt{1 + \mu\lambda}$ is a positive zero of the following polynomial:

$$h_k(y) = (k-1)(y+1)^{k+1} - (k+1)(y^2-1)(y+1)^{k-1} + 2(y+k)(y-1)^k. \tag{1.17}$$

It can be easily shown that $h_k(y)$ is a polynomial of degree $k - 2$ which can be expanded as follows:

$$h_k(y) = 8 \sum_{j=1}^{[\frac{k-1}{2}]} j \binom{k+1}{2j+1} y^{k-2j},$$

where [.] denotes the Gaussian integer function. Since all the coefficients of $h_k(y)$ are positive, it cannot have a positive zero. This contradiction completes the proof of Theorem 1 in case $k \geq 2$. If $k = 1$, then a similar argument to the above proves the theorem.

Added in Proof. Just before this article comes to be published, Prof. R. S. Palais sent me the preliminary version([8]) of an unpublished article (jointly with C. L. Terng and A. Derdzinski). It only has the statements of their results from which we may deduce Theorem 9.119 in [2]. But his treatments seem to be different from mine. I would like to express my gratitude to Prof. Palais for his kindness.

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