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HANDY CRITERIA FOR SIMPLICITY THEOREM

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1. Prerequisite and Introduction

H.Strade once constructed in his book [6] an important simplicity theorem for a finite-dimensional admissibly graded Lie algebra, but it was found by the authors recently that the finite-dimensionality condition is redundant, and more important, there are three equivalent criteria for Lie algebras under consideration to be simple.

In this paper, we would like to announce such facts and improve more or less the previously known proofs in this context.

We first introduce somewhat lengthy prerequisites required for understanding the simplicity theorem.

For an abelian group G, it is a *G-gradation* of a Lie algebra L that is specifically defined to be a family of subspaces $(L_g)_{g\in G}$ satisfying that $L = \oplus_{g\in G} L_g$ and $L_g L_h \subset L_{g+h}$. We refer to the pair $\langle L, (L_g)_{g\in G}\rangle$ as a *$G -$ graded Lie algebra*;

we call any element of L_g a homogeneous element of degree g; if $L_g^{[p]} \subset L_{pg}$ $\forall g \in G$, we call *restricted* the G-gradation of a restricted Lie algebra $\langle L, [p]\rangle$.

Suppose that $L = \oplus_{i=-r}^{s} L_i$ is $\mathbb{Z}-$graded with $r, s \geq 1$; then L is said to be *admissibly graded* if $L_{-r} = C_L(L_{-1})$.

A (descending) filtration $(L_{(n)})_{n\in\mathbb{Z}}$ of a restricted Lie algebra $\langle L, [p]\rangle$ is said to be *restricted* in case that $L_{(n)}^{[p]} \subset L_{(pn)}$ $\forall n \in \mathbb{Z}$.

Suppose that L is a Lie algebra with a maximal subalgebra $L_{(0)}$ and that $L_{(-1)}$ is a minimal $L_{(0)}$-module containing $L_{(0)}$ properly; define for $n > 0$,

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$$L_{(n)} := \{x \in L_{(n-1)} \mid xL_{(-1)} \subset L_{(n-1)}\}$$

$$L_{(-n)} := L^n_{(-1)} + L_{(-n+1)}.$$

It is then known that

$(L_{(n)})_{n\in\mathbb{Z}}$ becomes an exhaustive filtration of L enjoying the properties as follows:

(*) $\forall x \in L_{(i)} \setminus L_{(i+1)}$ for $i \geq 0$, $\exists y \in L_{(-1)}$ satisfying $xy \notin L_{(i)}$.

(**) $\forall i < 0$, $L_{(i-1)} = L_{(-1)}L_{(i)} + L_{(i)}$.

(***) A faithful irreducible representation of $L_{(0)}/L_{(1)}$ on $L_{(-1)}/L_{(0)}$ is induced from the ad representation.

Furthermore the filtration becomes restricted if, in addition, $L_{(0)}$ contains a CSA of L and L is restricted.

Now we call any exhaustive filtration satisfying (*)-(***) a *standard filtration.*

LEMMA (1.1). *Suppose that $L = \oplus^s_{i=-r} L_i$ is any $\mathbb{Z}-$graded (not necessarily finite-dimensional) Lie algebra over F; we have then*

(i) *$L^- := \sum_{i\leq -1} L_i$, $L^+ := \sum_{i\geq 1} L_i$ are nilpotent subalgebras of L.*

(ii) *If $L \neq L_0$ and L is semisimple, then $s, r \geq 1$.*

Proof. The proofs are all the same as those in Lemma 2.1, chap 3 in [6].

For a given (descending) filtration $(L_{(n)})_{n\in\mathbb{Z}}$, we put $L_i := L_{(i)}/L_{(i+1)} \forall i \in \mathbb{Z}$ and endow the vector space $\oplus_{i\in\mathbb{Z}} L_i =: \mathrm{gr}(L)$ with the structure of a Lie algebra by letting : $\forall x \in L_{(i)}$, $\forall y \in L_{(j)}$, $(x + L_{(i+1)})(y + L_{(j+1)}) := xy + L_{(i+j+1)}$.

The $\mathbb{Z}-$graded Lie algebra $\langle \mathrm{gr}(L), (L_i)_{i\in\mathbb{Z}}\rangle$ thus obtained shall be referred to as the *graded Lie algebra associated with* $\langle L, L_{(i)_{i\in\mathbb{Z}}}\rangle$.

It is easy to see that properties (*)$-$(***) yield naturally the following properties on the gradations $(L_i)_{i\in\mathbb{Z}}$ of $\mathrm{gr}(L)$.

(*)′ For $x \in L_i \setminus \{0\}$ and $i \geq 0$, $\exists y \in L_{-1}$ satisfying $xy \neq 0$.

(**)′ $\forall i < 0$, $L_i = (L_{-1})^{-i}$.

(***)′ L_{-1} becomes a faithful irreducible L_0-module.

(*)′ and (***)′ being straightforward, we will show (**)′ by downward induction on i : First, (**)′ holds obviously for $i = -1$. Secondly, we assume that $L_i = (L_{-1})^i$ holds for $i < 0$.

Thirdly, we have to show $L_{i-1} = (L_{-1})^{-i+1}$.

But our inductive assumption ensures that $L_{(i)}/L_{(i+1)} = (L_{(-1)}/L_{(0)})^{-i}$ $= L_{(-1)}^{-i} + L_{(i+1)}/L_{(i+1)}$ iff $L_{(i)} = L_{(-1)}^{i} + L_{(i+1)}$, so that we obtain $L_{i-1} = L_{(i-1)}/L_{(i)} = L_{(-1)}L_{(i)} + L_{(i)}/L_{(i)}$ (by **) $= L_{(-1)}^{-i+1} + L_{(i)}/L_{(i)}$ $= (L_{(-1)}/L_{(0)})^{-i+1} = (L_{-1})^{-i+1}$.

The following proposition is easily to be shown using our Lemma (1.1) and the proof exhibited in proposition (3.5), chap 3 in [6].

PROPOSITION (1.2). *Suppose that* $L = \oplus_{i=-r}^{s} L_i$ *is simple and* $\mathbb{Z}$*−graded but not necessarily finite-dimensional ; the following assertions then hold :*

(i) L_s *and* L_{-r} *are irreducible* L_0*−modules.*
(ii) $L_0 L_s = L_s$ *and* $L_0 L_{-r} = L_{-r}$.
(iii) $C_{L_{s-1}}(L_1) = 0,\ L_1 L_{s-1} = L_s$.
(iv) $C_L(L^-) = L_{-r}$ *and* $C_L(L^+) = L_s$.

PROPOSITION (1.3). *Suppose that* $L = \oplus_{i=-r}^{s} L_i$ *be a simple* $\mathbb{Z}$*−graded Lie algebra enjoying* $(**)'$ *; we then obtain:*
(i) *L becomes admissibly graded.*
(ii) $L_i = L_{-1}L_{i+1}$ *for any* $-r \leq i \leq s-1$.

Proof. See proposition 3.6, chap 3 in [6].

2. Generalized simplicity theorem

Simplicity theorem in 3.3 of [6] shall be generalized and other equivalent criteria shall be expressly exhibited in this section.

Conventions and notations are mainly those in §1 throughout this section.

THEOREM (2.1) (GENERALIZED SIMPLICITY THEOREM). *The following statements* (1) (2) (3) *concerning simplicity of Lie algebras are equivalent:*

(1) $L = \oplus_{i=-r}^{s} L_i$ *is a nontrivial graded Lie algebra,* i.e. $L \neq L_0$; (**) *and* (***) *hold for not necessarily finite dimensional Lie algebra L which is simple.*

(2) *As regards* $L = \oplus_{i=-r}^{s} L_i$, *we have*
(a) $s, r \geq 1$,
(b) $L = \oplus_{i=-r}^{s} L_i$ *is an admissibly graded Lie algebra,*
(c) $C_L(L^+) \subset \sum_{i \geq -1} L_i$,

(d) L_{-1} *is an irreducible* L_0*−module,*
(e) $L_{-1}L_{i+1} = L_i$ *for* $-r \leq i \leq s-1$,
(f) $L_0 L_s = L_s$.

(3) $L = \oplus_{i=-r}^{s} L_i$ *is a nontrivial graded Lie algebra,* i.e. $L \neq L_0$ *and* $(*)' - (****)'$ *hold for L which is not necessarily a finite dimensional simple Lie algebra,*

where $(****)'$ *states that* $\forall i < 0$ *and* $\forall x \in \bar{L}_i \setminus \{0\} \subset \bar{L} := L/M(L)$, *which inherits the gradation from L satisfying* $(*)' - (***)'$ *with* $M(L)$ *the sum of all ideals of L that are contained in* L^-, *there exists* $j > 0$ *satisfying* $x\bar{L}_j \neq (0)$.

Sketch of Proof. (1) $\Rightarrow$ (2) : First, noting that L being simple implies L being semisimple and that $L \neq L_0$, we have $s, r \geq 1$ by our Lemma (1.1). (b) is easily obtained by (a) and our proposition (1.3) (i). (c) is obtained by our proposition (1.2) (iv). (d) is a straightforward consequence of $(***)'$, (e) is easily obtained by our proposition (1.3) (ii). (f) is also immediate by virtue of our proposition (1.2) (ii).

(2) $\Rightarrow$ (1) : an immediate consequence of Theorem 3.7, chap 3 in [6].

(2) $\Rightarrow$ (3) is easily obtained modifying slightly the simplicity theorem and its proof (ibid.).

(3) $\Rightarrow$ (2) is obvious by virtue of (1) $\Rightarrow$ (2).

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