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THE STABILITY OF A GENERALIZED CAUCHY FUNCTIONAL EQUATION

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Abstract. We prove the stability of a generalized Cauchy functional equation of the form ;

$$f(a_1x + a_2y) = b_1f(x) + b_2f(y) + w.$$

That is, we obtain a partial answer for the open problem which was posed by the Th.M Rassias and J. Tabor on the stability for a generalized functional equation.

1. Introduction

The stability of the Cauchy functional equation ;

$$f(x + y) - f(x) - f(y) = 0$$

was originally raised by S.M.Ulam[5] and proved by D. H. Hyers[2]. There are many papers about the generalization of the Cauchy functional equation[1, 2, 3]. In this paper we consider a generalized Cauchy functional equation of the form

$$f(a_1x + a_2y) = b_1f(x) + b_2(y) + w.$$

In[4], Th. M. Rassias and J. Tabor have asked about the stability of a generalized Cauchy functional equation in the following sense ;

$$\|f(a_1x + a_2y + v) - b_1f(x) - b_2f(y) - w\| \le \theta(\|x\|^p + \|y\|^p),$$

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where $a_1, a_2, b_1, b_2 \neq 0, \theta, p \in R$, v and w are fixed elements of Banach space X and Y respectively and $f : X \longrightarrow Y$ a function. Under some conditions we obtain answers for the open problem which was posed by the Th. M. Rassias and J. Tabor [4] on the stability for a generalized Cauchy functional equation.

2. Results

THEOREM 1. *Let X and Y be Banach spaces and $w \in Y$ fixed. Assume that $f : X \longrightarrow Y$ and $\varphi : X \times X \longrightarrow R$ are mappings such that*

$$\|f(a_1x + a_2y) - b_1f(x) - b_2f(y) - w\| < \varphi(x, y)$$

for fixed $a_1, a_2, b_1, b_2 \in R$ with $a_1 + a_2 \neq 0$ and every $x, y \in X$ and assume that $\sum_{n=1}^{\infty} |b_1 + b_2|^{n-1}\varphi(B^n(x), B^n(x))$ converges, where $B(x) = \frac{1}{a_1+a_2}x$ and $|b_1 + b_2|^n \varphi(B^n(x), B^n(y))$ converges to 0 as $n \longrightarrow \infty$ for every $x, y \in X$.

Then there is a unique mapping $g : X \longrightarrow Y$ such that for every $x, y \in X$

$$g(a_1x + a_2y) = b_1g(x) + b_2g(y) + w$$

and for every $x \in X$,

$$\|f(x) - g(x)\| \leq \sum_{n=1}^{\infty} |b_1 + b_2|^{n-1}\varphi(B^n(x), B^n(x)).$$

Proof. Let $y = x$. Then we have

$$\|f((a_1 + a_2)x) - (b_1 + b_2)f(x) - w\| \leq \varphi(x, x).$$

Let $z = (a_1 + a_2)x$. Since $(a_1 + a_2) \neq 0$, $x = \frac{1}{a+b}z$.
Let $J(s) = (b_1 + b_2)s + w$. Thus

$$\|f(z) - Jf(B(z))\| \leq \varphi(B(z), B(z)), \qquad z \in X$$

and so

$$\left\|f(B(z)) - Jf(B^2(z))\right\| \leq \varphi(B^2(z), B^2(z)), \qquad z \in X.$$

Now

$$\left\|Jf(B(z)) - J^2f(B^2(z))\right\| \leq |b_1 + b_2|\,\varphi(B^2(z), B^2(z))$$

and so

$$\left\|J^2 f(B^2(z)) - J^3 f(B^3(z))\right\| \le |b_1 + b_2|^2 \varphi(B^3(z), B^3(z)).$$

By induction we have

$$\left\|J^n f(B^n(z)) - J^{n-1} f(B^{n-1}(z))\right\| \le |b_1 + b_2|^{n-1} \varphi(B^n(z), B^n(z))$$

for every $z \in X$. Let $g_n(x) = J^n f(B^n(x))$.

Since $\sum_{n=1}^{\infty} |b_1 + b_2|^{n-1} \varphi(B^n(x), B^n(x))$ converges, $\{g_n(x)\}$ is a Cauchy sequence. Thus we can define a function $g(x)$ from X to Y by

$$g(x) = \lim_{n\to\infty} g_n(x).$$

Let $G(s_1, s_2) = a_1 s_1 + a_2 s_2 + w$. Then $G(s, s) = J(s)$ and

$$\begin{aligned} J(G(s_1, s_2)) &= (b_1 + b_2)(b_1 s_1 + b_2 s_2 + w) + w \\ &= b_1((b_1 + b_2)s_1 + w) + b_2((b_1 + b_2)w) + w \\ &= b_1 J(s_1) + b_2 J(s_2) + w \\ &= G(J(s_1), J(s_2)). \end{aligned}$$

Now we have

$$\begin{aligned} &\|g(a_1 x + a_2 y) - (b_1 g(x) + b_2 g(y) + w)\| \\ &\quad = \|g(a_1 x + a_2 y) - G(g(x), g(y))\| \\ &\quad = \lim_{n\to\infty} \|J^n f(B^n(a_1 x + a_2 y)) - G(J^n f(B^n(x)), J^n f(B^n(y)))\| \\ &\quad = \lim_{n\to\infty} \|J^n f(B^n(a_1 x + a_2 y)) - J^n G(f(B^n(x)), f(B^n(y)))\| \\ &\quad \le \lim_{n\to\infty} |b_1 + b_2|^n \|f(B^n(a_1 x + a_2 y)) - G(f(B^n(x)), f(B^n(y)))\| \\ &\quad = \lim_{n\to\infty} |b_1 + b_2|^n \|f(a_1 B^n(x) + a_2 B^n(y)) \\ &\qquad\qquad - b_1 f(B^n(x)) - b_2 f(B^n(y)) - w\| \\ &\quad \le \lim_{n\to\infty} |b_1 + b_2|^n \varphi(B^n(x), B^n(y)) \\ &\quad = 0. \end{aligned}$$

Thus we have

$$g(a_1 x + a_2 y) = b_1 g(x) + b_2 g(y) + w.$$

For every $x \in X$,

$$\begin{aligned}\|f(x) - g(x)\| &= \lim_{n\to\infty} \|f(x) - J^n f(B^n(x))\| \\ &\leq \lim_{n\to\infty} (\|f(x) - Jf(B(x))\| + \|Jf(B(x)) - J^2 f(B^2(x))\| \\ &+ \cdots + \|J^{n-1} f(B^{n-1}(x)) - J^n f(B^n(x))\| \\ &\leq \sum_{n=1}^{\infty} |b_1 + b_2|^{n-1} \varphi(B^n(x), B^n(x)).\end{aligned}$$

Assume h is a solution of

$$h(a_1 x + a_2 y) = b_1 h(x) + b_2 h(y) + w$$

and

$$\|h(x) - f(x)\| \leq \sum_{n=i}^{\infty} |b_1 + b_2|^{i-1} \varphi(B^i(x), B^i(x).$$

Consider $y = x$ and $z = (a_1 + a_2)z$. Then

$$\begin{aligned}h(z) &= Jh(B(z)) \\ &= J^2 h(B^2(z)) \\ &\cdots\cdots\cdots \\ &= J^n h(B^n(z)).\end{aligned}$$

Thus we have

$$\begin{aligned}\|h(z) - g(z)\| &= \lim_{n\to\infty} \|J^n h(B^n(z)) - J^n f(B^n(z))\| \\ &= \lim_{n\to\infty} |b_1 + b_2|^n \, \|h(B^n(z)) - f(B^n(z))\| \\ &= \lim_{n\to\infty} |b_1 + b_2|^n \sum_{i=1}^{n} |b_1 + b_2|^{i-1} \varphi(B^{i+n}(z), B^{i+n}(z)) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} |b_1 + b_2|^{n+i-1} \varphi(B^{i+n}(z), B^{i+n}(z)) \longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$ And so $h(z) = g(z)$ for every $x \in X$.

COROLLARY 2. *Let X and Y be Banach spaces and $w \in Y$ fixed. Assume that $f : X \longrightarrow Y$ be a mapping such that*

$$\|f(a_1x + a_2y) - b_1f(x) - b_2f(y) - w\| < M$$

for $a_1, a_2, b_1, b_2 \in R$ with $a_1 + a_2 \neq 0, |b_1 + b_2| < 1$ and every $x, y \in X$.

Then there is a unique mapping $g : X \longrightarrow Y$ such that for every $x, y \in X$

$$g(a_1x + a_2y) = b_1g(x) + b_2g(y) + w$$

and for every $x \in X$

$$\|f(x) - g(x)\| \leq \frac{1}{1 - |b_1 + b_2|} \quad .$$

Proof. By theorem 1 with $\varphi(x, y) = M$, we complete the proof of the corollary.

COROLLARY 3. *Let X and Y be Banach spaces such that*

$$\|f(ax + by) - af(x) - bf(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for every $x, y \in X$, $0 \neq |a + b| < 1$ and $p < 1$ in R.

Then there is a unique function $g : X \longrightarrow Y$ such that $g(ax + by) = ag(x) + bg(y)$ for every $x, y \in X$

and

$$\|f(x) - g(x)\| \leq \frac{2\theta \|x\|^p}{|a + b|^p - |a + b|} \quad .$$

Proof. Let $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ and $B(x) = \frac{1}{a+b}x$. By Theorem 1, there is a unique function $g : X \longrightarrow Y$ such that

$$g(ax + by) = ag(x) + bg(y)$$

and

$$\begin{aligned}\|f(x) - g(x)\| &\leq \sum_{n=1}^{\infty} |a + b|^{n-1}\varphi(B^n(x), B^n(x)) \\ &= \frac{2\theta \|x\|^p}{|a + b|} \sum_{n=1}^{\infty} (|a + b|^{1-p})^n \\ &= \frac{2\theta \|x\|^p}{1 - |a + b|^{1-p}} \cdot \frac{1}{|a + b|^p} \\ &= \frac{2\theta \|x\|^p}{|a + b|^p - |a + b|} \quad .\end{aligned}$$

THOEREM 4. *Let X and Y be Banach spaces and $w \in Y$ fixed. Assume that $f : X \longrightarrow Y$ and $\varphi : X \times X \longrightarrow R$ are mappings such that*

$$\|f(a_1x + a_2y) - b_1f(x) - b_2f(y) - w\| < \varphi(x, y)$$

for fixed $a_1, a_2, b_1, b_2 \in R$ with $a_1 + a_2 \neq 0$ and every $x, y \in X$ and assume that
$\sum_{n=0}^{\infty} \frac{1}{|b_1+b_2|^n}\varphi(G^{n-1}(x), G^{n-1}(x))$ converges, where $G(x) = (a_1+a_2)x$ and $\frac{1}{|b_1+b_2|^n}\varphi(G^n(x), G^n(y))$ converges to 0 as $n \longrightarrow \infty$.

Then there is a unique mapping $g : X \longrightarrow Y$ such that for every $x, y \in X$

$$g(a_1x + a_2y) = b_1g(x) + b_2g(y) + w$$

and for every $x \in X$

$$\|f(x) - g(x)\| \leq \sum_{n=1}^{\infty} \frac{1}{|b_1 + b_2|^n}\varphi(G^{n-1}(x), G^{n-1}(x)).$$

Proof. Let $y = x$. Then we have

$$\|f((a_1 + a_2)x) - (b_1 + b_2)f(x) - w\| \leq \varphi(x, x).$$

Let $G(x) = (a_1 + a_2)x$, $L(x) = \frac{1}{b_1+b_2}(x - w)$ and $H(x, y) = b_1 + b_2y + w$. Then we have

$$\begin{aligned} H(L(x), L(y)) &= b_1L(x) + b_2L(y) + w \\ &= \frac{1}{b_1 + b_2}(b_1x + b_2y) \\ &= L(H(x, y)). \end{aligned}$$

For every $x \in X$

$$\left\|\frac{1}{|b_1 + b_2|}(f(G(x)) - w) - f(x)\right\| \leq \frac{1}{|b_1 + b_2|}\varphi(x, x)$$

and so

$$\|L(f(G(x))) - f(x)\| \leq \frac{1}{|b_1 + b_2|}\varphi(x, x).$$

Thus for every $x \in X$

$$\begin{aligned}\left\|L^2(f(G^2(x))) - L(f(G(x)))\right\| &\leq \frac{1}{|b_1 + b_2|}\left\|L(f(G^2(x))) - f(G(y))\right\| \\ &\leq \frac{1}{|b_1 + b_2|^2}\varphi(G(x), G(x)).\end{aligned}$$

By induction, for every $x \in X$ we have

$$\begin{aligned}&\left\|L^n f(G^n(x)) - L^{n-1} f(G^{n-1}(x))\right\| \\ &\leq \frac{1}{|b_1 + b_2|^n}\varphi(G^{n-1}(x), G^{n-1}(x)).\end{aligned}$$

Let $g_n(x) = L^n(f(G^n(x)))$, for each $x \in X$. Since

$$\sum_{n=1}^{\infty} \frac{1}{|b_1 + b_2|^n}\varphi(G^{n-1}(x), G^{n-1}(x))$$

converges, $\{g_n(x)\}$ is a Cauchy sequence for every $x \in X$. Thus we can define a function $g(x)$ from X to Y by

$$g(x) = \lim_{n\to\infty} g_n(x).$$

Now we show that g is a solution of a generalized Cauchy equation.

Thus we get

$$\begin{aligned}&\|g(a_1x + a_2y) - b_1g(x) - b_1g(y) - w\| \\ &= \lim_{n\to\infty} \|L^n f(G^n(a_1x + a_2y)) - H(L^n f(B^n(x)), L^n f(B^n(y)))\| \\ &= \lim_{n\to\infty} \|L^n f(a_1G^n(x) + a_2G^n(y)) - L^n H f(B^n(x), f(B^n(y))\| \\ &= \lim_{n\to\infty} \frac{1}{|b_1 + b_2|^n}\varphi(G^n(x), G^n(y)) \\ &= 0.\end{aligned}$$

Therefore, $g(a_1x + a_2y) = b_1g(x) + b_1g(y) + w$ for every $x \in X$.

$$\begin{aligned}&\|f(x) - g(x)\| \\ &= \lim_{n\to\infty} \|f(x) - L^n f(G^n(x))\| \\ &\leq \lim_{n\to\infty} \|f(x) - Lf(G(x))\| \\ &+ \left\|Lf(G(x)) - L^2 f(G^2(x))\right\| \\ &+ \cdots + \left\|L^{n-1}(f(G^{n-1}(x))) - L^n(f(G^n(x)))\right\| \\ &= \sum_{n=1}^{\infty} \frac{1}{|b_1 + b_2|^n}\varphi(G^{n-1}(x), G^{n-1}(x)).\end{aligned}$$

Assume that h is an another solution of

$$h(a_1x+a_2y)=b_1h(x)+b_2h(y)+w$$

and

$$\|h(x)-f(x)\|\le\sum_{n=1}^{\infty}\frac{1}{|b_1+b_2|^n}\varphi(G^{n-1}(x),G^{n-1}(x)).$$

Let $y=x$. Then $L(h(G(x)))=h(x)$, and

$$L^2(h(G^2(x)))=L(h(G(x)))=h(x).$$

By induction we obtain $h(x)=L^n(h(G^n(x)))$. Thus we have

$$\begin{aligned}
&\|h(x)-g(x)\|\\
&=\lim_{n\to\infty}\|L^nh(G^n(x))-L^nf(G^n(x))\|\\
&=\lim_{n\to\infty}\frac{1}{|b_1+b_2|^n}\|h(G^n(x))-f(G^n(x))\|\\
&=\lim_{n\to\infty}\frac{1}{|b_1+b_2|^n}\sum_{i=1}^{\infty}|b_1+b_2|^i\varphi(G^{i-1}(G^n(x)),G^{i-1}(G^n(x)))\\
&=\lim_{n\to\infty}\sum_{i=1}^{\infty}|b_1+b_2|^{n+i}\varphi(G^{n+i-1}(x),G^{n+i-1}(x))\\
&=\lim_{n\to\infty}\sum_{i=1}^{\infty}|b_1+b_2|^{i+1}\varphi(G^n(x),G^n(x))\\
&=0.
\end{aligned}$$

for every $x\in X$

Therefore we get $h(x)=g(x)$ for every $x\in X$ and so we complete the proof.

COROLLARY 5. *Let X and Y be Banach spaces and $w\in Y$ fixed. Assume that $f:X\longrightarrow Y$ be a mapping such that*

$$\|f(a_1w+a_2y)-b_1f(x)-b_2f(y)-w\|<M$$

for $a_1,a_2,b_1,b_2\in R$ with $|b_1+b_2|>1$ and $x,y\in X$.

Then there is a unique mapping $g:X\longrightarrow Y$ such that for every $x,y\in X$

$$g(a_1x+a_2y)=b_1g(x)+b_2g(y)+w$$

and for every $x \in X$

$$\|f(x) - g(x)\| \le \frac{|b_1 + b_2|}{|b_1 + b_2| - 1} .$$

Proof. By Theorem 4 with $\varphi(x, y) = M$, we complete the proof.

COROLLARY 6. *Let* X *and* Y *be Banach spaces such that*

$$\|f(ax + by) - af(x) - bf(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for every $x, y \in X$, $|a + b| > 1$ *and* $p < 1$ *in* R.
Then there is a unique function $g : X \longrightarrow Y$ *such that*

$$g(ax + by) = ag(x) + bg(y)$$

for every $x, y \in X$
and

$$\|f(x) - g(x)\| \le \frac{2\theta \|x\|^p}{|a + b| - |a + b|^p} .$$

Proof. Let $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ and $G(x) = (a + b)x$. By Theorem 4, there is a unique function $g : X \longrightarrow Y$ such that

$$g(ax + by) = ag(x) + bg(y)$$

and

$$\begin{aligned}
\|f(x) - g(x)\| &\le \sum_{n=1}^{\infty} \frac{1}{|a + b|^n} \varphi(G^{n-1}(x), G^{n-1}(x)) \\
&= \sum_{n=1}^{\infty} \frac{1}{|a + b|^n} 2\theta(|a + b|^{n-1})^p \|x\|^p \\
&= \frac{2\theta \|x\|^p}{|a + b|^p} \sum_{n=1}^{\infty} (|a + b|^{-(1-p)})^n \\
&= \frac{2\theta \|x\|^p}{|a + b| - |a + b|^p} .
\end{aligned}$$

Thus we complete the proof of corollary.

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