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COMMON FIXED POINTS FOR COMPATIBLE MAPPINGS OF TYPE(A) IN 2–METRIC SPACES

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Abstract. In this paper we obtain a criterion for the existence of a common fixed point of a pair of mappings in 2-metric spaces. Our result generalizes a number of fixed point theorems given by Imdad, Khan and Khan [1], Kahn and Fisher [2], Kubiak [3], Rhoades [5], and Singh, Tiwari and Gupta [6].

1. Introduction

Rhoades [5] and Singh, Tiwari and Gupta [6] obtained a few fixed point theorems for contractive type mappings in 2-metric spaces. Murthy, Chang, Cho and Sharma [4] introduced the concept of compatible mappings of type(A) in 2-metric spaces and proved common fixed point theorems for compatible mappings of type(A) in 2-metric spaces. Imdad, Khan and Khan [1], Khan and Fisher [2] and Kubiak [3] established some necessary and sufficient conditions which guarantee the existence of a common fixed point for a pair of continuous mappings in 2-metric spaces.

In this paper we establish a criterion for the existence of a common fixed point of a pair of mappings of 2-metric spaces. Our result generalizes the corresponding results of Imdad, Khan and Khan [1], Kahn and Fisher [2], Kubiak [3], Rhoades [5], and Singh, Tiwari and Gupta [6].

2. Preliminaries

Throughout this paper, N and ω denote the sets of positive and nonnegative integers, respectively. Let $R^+ = [0,\infty)$ and Φ the family of all functions $\varphi : (R^+)^5 \to R^+$ with the following properties:

(i) φ is upper semicontinuous, nondecreasing in each coordinate variable.

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(ii) $a(t) = \max\{\varphi(t,0,0,t,t), \varphi(t,t,t,2t,0), \varphi(t,t,t,0,2t)\} < t$ for all $t > 0$.

DEFINITION 2.1. Let f and g be mappings from a 2-metric spaces (X,d) into itself. f and g are said to be compatible of type(A) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n, a) = \lim_{n\to\infty} d(gfx_n, ffx_n, a) = 0$$

for all $a \in X$, whenever $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

DEFINITION 2.2. A mapping f from a 2-metric space (X,d) into itself is said to be continuous at $x \in X$ if for every sequence $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$, $\lim_{n\to\infty} d(fx_n, fx, a) = 0$. f is called continuous on X if it is so at all points of X.

LEMMA 2.1([4]). *Let f and g be compatible mappings of type(A) from a 2-metric spaces (X,d) into itself. If $ft = gt$ for some $t \in X$, then $fgt = ggt = gft = fft$.*

LEMMA 2.2([4]). *Let f and g be compatible mappings of type(A) from a 2-metric spaces (X,d) into itself. If f is continuous at some $t \in X$ and if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, then $\lim_{n\to\infty} gfx_n = ft$.*

LEMMA 2.3([7]). *For each $t > 0$, $a(t) < t$ if and only if $\lim_{n\to\infty} a^n(t) = 0$, where a^n denotes the n-times composition of a.*

3. Characterization of common fixed points

Our result is as follows.

THEOREM 3.1. *Let (X,d) be a complete 2-metric space with d continuous on X and let h and t be two mappings of X into itself. Then the following conditions are equivalent:*

(1) *h and t have a common fixed point;*

(2) *there exist $\varphi \in \Phi$, $f : X \to t(X)$ and $g : X \to h(X)$ satisfying $(a1)$, $(a2)$ annd $(a3)$:*

  (a1) *the pairs f, h and g, t are compatible,*

  (a2) *one of f, g, h and t is continuous,*

  (a3) $d(fx, gy, a) \le \varphi(d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), d(hx, gy, a), d(ty, fx, a))$

for all $x, y, a \in X$.

Proof. (1) $\Rightarrow$ (2). Let z be a common fixed point of h and t. Define $f : X \to t(X)$ and $g : X \to h(X)$ by $fx = gx = z$ for all $x \in X$. Then $(a1), (a2)$ and $(a3)$ hold for any $\varphi \in \Phi$.

(2) $\Rightarrow$ (1). Let x_0 be an arbitrary point in X. Since $f(X) \subset t(X)$ and $g(X) \subset h(X)$, there exisit sequences $\{x_n\}_{n\in\omega}$ and $\{y_n\}_{n\in\omega}$ in X satisfying $y_{2n} = tx_{2n+1} = fx_{2n}$, $y_{2n+1} = hx_{2n+2} = gx_{2n+1}$ for $n \in \omega$. Define $d_n(a) = d(y_n, y_{n+1}, a)$ for $a \in X$ and $n \in \omega$. We claim that for any $i, j, k \in \omega$

$$(3.1) \qquad d(y_i, y_j, y_k) = 0.$$

Suppose that $d_{2n}(y_{2n+2}) > 0$. Using $(a3)$, we have

$$\begin{aligned} d(fx_{2n+2}, gx_{2n+1}, y_{2n}) \le \varphi(&d(hx_{2n+2}, tx_{2n+1}, y_{2n}), d(hx_{2n+2}, fx_{2n+2}, y_{2n}),\\ &d(tx_{2n+1}, gx_{2n+1}, y_{2n}), d(hx_{2n+2}, gx_{2n+1}, y_{2n}), d(tx_{2n+1}, fx_{2n+2}, y_{2n})), \end{aligned}$$

which implies that

$$d_{2n}(y_{2n+2}) \le \varphi(0, d_{2n}(y_{2n+2}), 0, 0, 0) \le a(d_{2n}(y_{2n+2})),$$

which is a contradiction. Hence $d_{2n}(y_{2n+2}) = 0$. Similarly, we have $d_{2n+1}(y_{2n+3}) = 0$. Thus $d_n(y_{n+2}) = 0$ for all $n \in \omega$. Note that

$$(3.2) \quad d(y_n, y_{n+2}, a) \le d_n(y_{n+2}) + d_n(a) + d_{n+1}(a) = d_n(a) + d_{n+1}(a).$$

It follows fom $(a3)$ and (3.2) that

$$\begin{aligned} d_{2n+1}(a) &\le \varphi(d(hx_{2n+2}, tx_{2n+1}, a), d(fx_{2n+2}, hx_{2n+2}, a),\\ &\qquad d(gx_{2n+1}, tx_{2n+1}, a), d(hx_{2n+2}, gx_{2n+1}, a), d(hx_{2n+2}, gx_{2n+1}, a))\\ &\le \varphi(d_{2n}(a), d_{2n+1}(a), d_{2n}(a), 0, d_{2n}(a) + d_{2n+1}(a))\\ &\le a(\max\{d_{2n}(a), d_{2n+1}(a)\}). \end{aligned}$$

Suppose that $d_{2n+1}(a) > d_{2n}(a)$. Then $d_{2n+1}(a) \le a(d_{2n+1}(a)) < d_{2n+1}(a)$, which is a contradiction. Hence $d_{2n+1}(a) < d_{2n}(a)$ and so $d_{2n+1}(a) \le a(d_{2n}(a))$. Similarly, we have $d_{2n}(a) \le a(d_{2n-1}(a))$. That is, for all $n \in N$

$$(3.3) \qquad d_{n+1}(a) \le a(d_n(a)).$$

Let n, m be in ω. If $n \geq m$, then $0 = d_m(y_m) \geq d_n(y_m)$; if $n < m$, then

$$\begin{aligned} d_n(y_m) &\leq d_n(y_{m-1}) + d_{m-1}(y_n) + d_{m-1}(y_{n+1}) \\ &\leq d_n(y_{m-1}) + d_n(y_n) + d_n(y_{n+1}) \\ &\leq d_n(y_{m-1}) \leq d_n(y_{m-2}) \leq \Lambda \leq d_n(y_{n+1}) = 0. \end{aligned}$$

Thus, for any $n, m \in \omega$

$$d_n(y_m) = 0. \tag{3.4}$$

For all $i, j, k \in \omega$, we may without loss of generality assume that $i < j$. It follows from (3.4) that

$$\begin{aligned} d(y_i, y_j, y_k) &\leq d_i(y_j) + d_i(y_k) + d(y_{i+1}, y_j, y_k) = d(y_{i+1}, y_j, y_k) \\ &\leq d(y_{i+2}, y_j, y_k) \leq \Lambda \leq d(y_{j-1}, y_j, y_k) = d_{j-1}(y_k) = 0. \end{aligned}$$

Therefore (3.1) holds. In view of (3.3) and Lemma 2.3, we have

$$\lim_{n\to\infty} d_n(a) = 0. \tag{3.5}$$

In order to show that $\{y_n\}_{n\in\omega}$ is a Cauchy sequence, by (3.5), it is sufficient to show that $\{y_{2n}\}_{n\in\omega}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\omega}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and $a \in X$ such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ and $d(y_{2m(k)}, y_{2n(k)}, a) \geq \epsilon$.

For each even integer let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying the above inequality, so that

$$d(y_{2m(k)-2}, y_{2n(k)}, a) \leq \epsilon, \quad d(y_{2m(k)}, y_{2n(k)}, a) > \epsilon. \tag{3.6}$$

For each even integer $2k$, by (3.1) and (3.6) we have

$$\begin{aligned} \epsilon &< d(y_{2m(k)}, y_{2n(k)}, a) \\ &\leq d(y_{2m(k)-2}, y_{2n(k)}, a) + d(y_{2m(k)}, y_{2m(k)-2}, a) \\ &\qquad + d(y_{2m(k)}, y_{2n(k)}, y_{2m(k)-2}) \\ &\leq \epsilon + d(y_{2m(k)-2}, y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2m(k)-1}, a) \\ &\qquad + d(y_{2m(k)-1}, y_{2m(k)}, a) \\ &= \epsilon + d_{2m(k)-2}(a) + d_{2m(k)-2}(a) \end{aligned}$$

which implies that

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}, a) = \epsilon. \tag{3.7}$$

It follows from (3.6) that

$$\begin{aligned}0 &< d(y_{2n(k)}, y_{2m(k)}, a) - d(y_{2n(k)}, y_{2m(k)-2}, a)\\ &\le d(y_{2m(k)-2}, y_{2m(k)}, a) \le d_{2m(k)-2}(a) + d_{2m(k)-1}(a).\end{aligned}$$

Using (3.5) and (3.7), we have

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-2}, a) = \epsilon. \tag{3.8}$$

Note that

$$\begin{aligned}|d(y_{2n(k)}, y_{2m(k)-1}, a) - d(y_{2n(k)}, y_{2m(k)}, a)| &\le d_{2m(k)-1}(a)\\ &\quad + d_{2m(k)-1}(y_{2n(k)}),\\ |d(y_{2n(k)+1}, y_{2m(k)}, a) - d(y_{2n(k)}, y_{2m(k)}, a)| &\le d_{2n(k)}(a)\\ &\quad + d_{2n(k)}(y_{2m(k)}),\\ |d(y_{2n(k)+1}, y_{2m(k)-1}, a) - d(y_{2n(k)}, y_{2m(k)-1}, a)| &\le d_{2n(k)}(a)\\ &\quad + d_{2n(k)}(y_{2m(k)-1}).\end{aligned}$$

It is easy to see that

$$\begin{aligned}\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}, a) &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}, a)\\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}, a) = \epsilon.\end{aligned} \tag{3.9}$$

It follows from $(a3)$ that

$$\begin{aligned}d(y_{2m(k)}, y_{2n(k)+1}, a) &= d(fx_{2m(k)}, gx_{2n(k)+1}, a)\\ &\le \varphi(d(hx_{2m(k)}, tx_{2n(k)+1}, a), d(hx_{2m(k)}, fx_{2m(k)}, a),\\ &\quad d(tx_{2n(k)+1}, gx_{2n(k)+1}, a),\\ &\quad d(hy_{2m(k)}, gy_{2n(k)+1}, a), d(ty_{2n(k)+1}, fy_{2m(k)}, a))\\ &= \varphi(d(y_{2m(k)-1}, y_{2n(k)}, a), d_{2m(k)-1}(a), d_{2n(k)}(a),\\ &\quad d(y_{2m(k)-1}, y_{2n(k)+1}, a), d(y_{2n(k)}, y_{2m(k)}, a)).\end{aligned}$$

Letting $k \to \infty$, by (3.9), (3.7) and (3.5) we have

$$\epsilon \le \varphi(\epsilon, 0, 0, \epsilon, \epsilon) \le a(\epsilon) < \epsilon,$$

which is a contradiction. Therefore $\{y_{2n}\}_{n\in\omega}$ is a Cauchy sequence in X.

It follows from completeness of (X, d) that $\{y_n\}_{n\in\omega}$ converges to a point $u \in X$. Now, suppose that t is continuous. Since g and t are compatible of type(A) and $\{gx_{2n+1}\}_{n\in\omega}$ and $\{tx_{2n+1}\}_{n\in\omega}$ converge to the point u, by Lemma 2.2 we get that gtx_{2n+1}, $ttx_{2n+1} \to tu$ as $n \to \infty$. In virtue of $(a3)$ we have

$$d(fx_{2n}, gtx_{2n+1}, a) \le \varphi(d(hx_{2n}, ttx_{2n+1}, a), d(hx_{2n}, fx_{2n}, a), \\ d(ttx_{2n+1}, gtx_{2n+1}, a), d(hx_{2n}, gtx_{2n+1}, a), d(ttx_{2n+1}, fx_{2n}, a)).$$

Letting $n \to \infty$, we have

$$d(u, tu, a) \le \varphi(d(u, tu, a), 0, 0, d(u, tu, a), d(tu, u, a)) \le a(d(u, tu, a)),$$

which implies that $u = tu$. It follows from $(a3)$ that

$$d(fx_{2n}, gu, a) \le \varphi(d(hx_{2n}, tu, a), d(hx_{2n}, fx_{2n}, a), d(tu, gu, a), \\ d(hx_{2n}, gu, a), d(tu, fx_{2n}, a).$$

Letting $n \to \infty$, we have

$$d(u, gu, a) \le \varphi(0, 0, d(u, gu, a), d(u, gu, a), 0) \le a(d(u, gu, a)),$$

which implies that $u = gu$. It follows from $g(X) \subset h(X)$ that there exists $\nu \in X$ with $u = gu = h\nu$. From $(a3)$ we get that

$$d(f\nu, gu, a) \le \varphi(0, d(h\nu, f\nu, a), 0, 0, d(tu, f\nu, a)) \le a(d(u, f\nu, a)).$$

Therefore $u = f\nu$. Lemma 2.1 ensures that $fu = fh\nu = hf\nu = hu$. By $(a3)$ we obtain that

$$d(fu, gu, a) \le \varphi(d(fu, gu, a), 0, 0, d(fu, gu, a), d(fu, gu, a)) \\ \le a(d(fu, gu, a)).$$

Hence $u = fu$. That is, u is a common fixed point of f, g, h and t. Similarly, we complete the proof when f or g or h is continuous. This completes the proof.

REMARK 3.1. Theorem 3.1 generalizes Theorem 3.3 of Imdad, Khan and Khan [1], Theorem 2 of Khan and Fisher [2], Theorem 1 of Kubiak [3], Theorem 4 of Rhoades [5], Theorem 1 of Singh, Tiwari and Gupta [6].

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