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DERIVED LIMITS OF INVERSE SYSTEMS OVER (PRE)ORDERED SETS

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Abstract. After considering an equivalence relation on a directed preordered set, we construct an isomorphism between derived limits of inverse systems indexed by the directed (pre)ordered sets.

1. Introduction

The derived limit functor $\lim^n(-), n \geq 0$ was defined by J. E. Roos and G. Nöbeling independently and simultaneously. The first derived limit is an important algebraic tool in the computation of phantom maps. C. A. McGibbon [6] wrote a good book on derived limits and phantom maps. C. A. McGibbon and R. Steiner [7] introduced some questions about the first derived limits of inverse limits and phantom maps. Mathematicians have studied the properties of the derived limits and found the desirable exact sequences with respect to several functors [2,3]. S. Araki and Z. I. Yoshimura [1] showed that if H is an additive (reduced) cohomology theory on arbitrary CW-complexes, then $E_2^{n,m} = \lim^n(H^m(X_\gamma))$. M. Huber and W. Meier [2] proved that $\ker(\theta : H^n(X) \to \lim^0(H^n(X_\gamma)))$ is isomorphic to the group $\mathrm{Pext}(F_{n-1}(X), A)$, where F_* is a homology theory of finite type and $\lim^n(H^m(X_\gamma)) = 0$ for all $n \geq 2$. In 1993, L. Mdzinarishvili and E. Spanier [8] established the long exact sequence for the derived limit functor $\lim^n(-), n \geq 0$ with respect to a cohomology module $\bar{H}^*(X; A)$. S. Mardešić [4, Section 10] proved that there is an isomorphism between strong homology groups of inverse systems over a directed preordered set and an ordered set.

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The purpose of this paper is to apply those results to the derived limits of inverse systems indexed by the directed (pre)ordered sets.

2. An isomorphism between derived limits

Let $\mathfrak{G} = (G_\gamma, g_{\gamma\gamma'}, \Gamma)$ be an inverse system of abelian groups G_γ and group homomorphisms $g_{\gamma\gamma'} : G_{\gamma'} \to G_\gamma$, $\gamma \leq \gamma'$ over a directed preordered set Γ. Let $\Gamma^n, n \geq 0$ be the set of all increasing sequences $\bar\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n), \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_n, \gamma_i \in \Gamma$ and let $\bar\gamma_j = (\gamma_0, \cdots, \gamma_{j-1}, \gamma_{j+1}, \cdots, \gamma_n) \in \Gamma^{n-1}$, $0 \leq j \leq n$ be obtained from $\bar\gamma \in \Gamma^n$ by deleting the j-th coordinate γ_j.

Throughout this paper, $\prod$ means the direct product. And for each $\bar\gamma = (\gamma_0, \cdots, \gamma_n) \in \Gamma^n$, we associate an object $G_{\bar\gamma}$ by the object G_{γ_0} of the first index γ_0 of $\bar\gamma \in \Gamma^n$ in a category of abelian groups, i.e., $G_{\bar\gamma} = G_{\gamma_0}$.

We define n-cochain groups $C^n(\mathfrak{G})$ of $\mathfrak{G}$ by

$$C^n(\mathfrak{G}) = \prod_{\bar\gamma = (\gamma_0, \cdots, \gamma_n) \in \Gamma^n} G_{\bar\gamma}, \ n \geq 0.$$

Let $pr_{\bar\gamma} : C^n(\mathfrak{G}) \to G_{\bar\gamma}$ be a projection. If y is an element of $C^n(\mathfrak{G})$, then we denote the element $y_{\bar\gamma}$ of $G_{\bar\gamma}$ by

$$y_{\bar\gamma} = pr_{\bar\gamma}(y).$$

The coboundary operator $\delta^n : C^{n-1}(\mathfrak{G}) \to C^n(\mathfrak{G}) (n \geq 1)$ is defined by

$$(\delta^n y)_{\bar\gamma} = g_{\gamma_0\gamma_1}(y_{\bar\gamma_0}) + \sum_{j=1}^{n} (-1)^j y_{\bar\gamma_j},$$

where $y \in C^{n-1}(\mathfrak{G})$. For $n = 0$, we put $\delta^0 = 0 : 0 \to C^0(\mathfrak{G})$. Then we have a cochain complex

$$(C^*(\mathfrak{G}), \delta) : 0 \xrightarrow{\delta^0} C^0(\mathfrak{G}) \xrightarrow{\delta^1} C^1(\mathfrak{G}) \to \cdots \to C^{n-1}(\mathfrak{G}) \xrightarrow{\delta^n} C^n(\mathfrak{G}) \to \cdots.$$

DEFINITION 2.1. [9,10] The n-th *derived limit* $\lim^n(\mathfrak{G})$ of $\mathfrak{G}$ is defined by

$$\lim{}^n(\mathfrak{G}) = \ker(\delta^{n+1})/\mathrm{im}(\delta^n).$$

We can see that $\lim^0(\mathfrak{G})$ is equal to the inverse limit $\lim(\mathfrak{G})$ of the inverse system $\mathfrak{G}$.

Let $\mathfrak{G} = (G_\gamma, g_{\gamma\gamma'}, \Gamma)$ and $\mathfrak{H} = (H_\omega, h_{\omega\omega'}, \Omega)$ be inverse systems of abelian groups and group homomorphisms, where Ω is another directed set. We say that $f = \{\varphi, f_\omega : \omega \in \Omega\} : \mathfrak{G} \to \mathfrak{H}$ is a *rigid system map* [5,10] from $\mathfrak{G}$ to $\mathfrak{H}$ if $\varphi : \Omega \to \Gamma$ is an increasing function, $f_\omega : G_{\varphi(\omega)} \to H_\omega, \omega \in \Omega$ is a homomorphism and the following diagram

$$\begin{array}{ccc} G_{\varphi(\omega)} & \xleftarrow{g_{\varphi(\omega)\varphi(\omega')}} & G_{\varphi(\omega')} \\ {\scriptstyle f_\omega}\big\downarrow & & \big\downarrow{\scriptstyle f_{\omega'}} \\ H_\omega & \xleftarrow{h_{\omega\omega'}} & H_{\omega'} \end{array}$$

is commutative for any $\omega \leq \omega'$ in Ω . We can make a category inv-$\mathfrak{C}$ of inverse systems and rigid system maps in a category $\mathfrak{C}$. The rigid system map is called a *level system map* provided $\Gamma = \Omega$ and φ is an identity map on Γ. It is easy to see that the category $\mathfrak{C}^\Gamma$ of the inverse systems and the level system maps is not full subcategory but subcategory of inv-$\mathfrak{C}$.

If $\gamma_0 \leq \gamma_1$ and $\gamma_1 \leq \gamma_0$, in the directed preordered set Γ, then we put $\gamma_0 \sim \gamma_1$. We can easily check that $\sim$ is an equivalence relation on $(\Gamma, \leq)$. Let Γ' be a subset of Γ which contains only one element from every equivalence class of Γ with respect to $\sim$. Then the set $(\Gamma', \leq)$ is a directed ordered set. We can construct an inverse system $\mathfrak{G}' = (G_\gamma, g_{\gamma\gamma'}, \Gamma')$ over the directed ordered set Γ'. We now define a rigid system map $r = (r_\gamma, i) : \mathfrak{G} \to \mathfrak{G}'$ by the inclusion map $i : \Gamma' \to \Gamma$ and by the identity maps $r_\gamma = 1_{G_\gamma} : G_\gamma \to G_\gamma$. If we define maps

$$r^\sharp : C^\sharp(\mathfrak{G}) \to C^\sharp(\mathfrak{G}')$$

and

$$s^\sharp : C^\sharp(\mathfrak{G}') \to C^\sharp(\mathfrak{G})$$

by

$$(r^\sharp x)_{(\gamma_0,\cdots,\gamma_n)} = x_{(\gamma_0,\cdots,\gamma_n)}, \; (\gamma_0, \cdots, \gamma_n) \in \Gamma'^n$$

and

$$(s^\sharp y)_{(\gamma_0,\cdots,\gamma_n)} = g_{\gamma_0\gamma_0'}(y_{(\gamma_0',\cdots,\gamma_n')}), \; (\gamma_0, \cdots, \gamma_n) \in \Gamma^n$$

respectively, where γ_i' is the only element of Γ' such that $\gamma_i' \sim \gamma_i, i = 0, 1, \cdots, n$, then we have the following:

LEMMA 2.2. *The maps $r^\sharp$ and $s^\sharp$ are cochain maps.*

Proof. If $y \in C^{n-1}(\mathfrak{G}')$ and $\bar{\gamma} = (\gamma_0, \cdots, \gamma_n) \in \Gamma^n$, then

$$
\begin{aligned}
(\delta s^\sharp y)_{\bar{\gamma}} &= g_{\gamma_0\gamma_1}(s^\sharp y)_{\bar{\gamma}_0} + \sum_{j=1}^{n}(-1)^j (s^\sharp y)_{\bar{\gamma}_j} \\
&= g_{\gamma_0\gamma_1}(g_{\gamma_1\gamma_1'}(y_{(\gamma_1',\cdots,\gamma_n')})) + \sum_{j=1}^{n}(-1)^j g_{\gamma_0\gamma_0'}(y_{\bar{\gamma}_j'}) \\
&= g_{\gamma_0\gamma_0'}(g_{\gamma_0'\gamma_1'}(y_{(\gamma_1',\cdots,\gamma_n')})) + \sum_{j=1}^{n}(-1)^j g_{\gamma_0\gamma_0'}(y_{\bar{\gamma}_j'}) \\
&= g_{\gamma_0\gamma_0'}[g_{\gamma_0'\gamma_1'}(y_{\bar{\gamma}_0'}) + \sum_{j=1}^{n}(-1)^j y_{\bar{\gamma}_j'}] \\
&= g_{\gamma_0\gamma_0'}(\delta y)_{(\gamma_0',\cdots,\gamma_n')} \\
&= (s^\sharp \delta y)_{\bar{\gamma}}
\end{aligned}
$$

showing that $s^\sharp$ is a cochain map. We can easily check that $r^\sharp$ is also a cochain map.

The following theorem is due to S. Mardešić's result [4, Section 10, Theorem 3] about strong homology groups. We apply the case of strong homology groups to the one of derived limits of inverse systems.

THEOREM 2.3. *The homomorphism $r^* : \lim^n(\mathfrak{G}) \to \lim^n(\mathfrak{G}')$ is an isomorphism.*

Proof. For all $y \in C^n(\mathfrak{G}')$ and $\bar{\gamma} = (\gamma_0, \cdots, \gamma_n) \in {\Gamma'}^n$, we have

$$
\begin{aligned}
(r^\sharp s^\sharp y)_{\bar{\gamma}} &= (s^\sharp y)_{\bar{\gamma}} \\
&= g_{\gamma_0\gamma_0'}(y_{\bar{\gamma}'}) \\
&= y_{\bar{\gamma}} \quad (\bar{\gamma} \in {\Gamma'}^n).
\end{aligned}
$$

Thus $r^\sharp s^\sharp = 1_{C^\sharp(\mathfrak{G}')}$ (an identity on $C^\sharp(\mathfrak{G}')$). To construct a cochain homotopy between $s^\sharp r^\sharp$ and the identity $1_{C^\sharp(\mathfrak{G})}$ on $C^\sharp(\mathfrak{G})$, we define a map

$$D : C^{n+1}(\mathfrak{G}) \to C^n(\mathfrak{G})$$

by

$$(Dx)_{(\gamma_0,\cdots,\gamma_n)} = \sum_{k=0}^{n}(-1)^k x_{(\gamma_0,\cdots,\gamma_k,\gamma_k',\cdots,\gamma_n')}$$

for $x \in C^{n+1}(\mathfrak{G})$ and $\bar{\gamma} = (\gamma_0, \cdots, \gamma_n) \in \Gamma^n$. In the case of $n = 0$ or 1, we can easily check that D is a cochain homotopy. We focus on the case $n \geq 2$. First we compute the following:

(A)
$$\begin{aligned}(\delta Dx)_{\bar{\gamma}} &= g_{\gamma_0\gamma_1}(Dx)_{\bar{\gamma}_0} + \sum_{j=1}^{n}(-1)^j(Dx)_{\bar{\gamma}_j} \\ &= g_{\gamma_0\gamma_1}\sum_{k=1}^{n}(-1)^{k+1}x_{(\gamma_1,\cdots,\gamma_k,\gamma'_k,\cdots,\gamma'_n)} \\ &+ \sum_{k=1}^{n-1}\sum_{j=1}^{k}(-1)^{k+j}x_{(\gamma_0,\cdots,\hat{\gamma}_j,\cdots,\gamma_{k+1},\gamma'_{k+1},\cdots,\gamma'_n)} \\ &+ \sum_{k=0}^{n-1}\sum_{j=k+1}^{n-1}(-1)^{k+j}x_{(\gamma_0,\cdots,\gamma_k,\gamma'_k,\cdots,\hat{\gamma}'_j,\cdots,\gamma'_n)} \\ &+ \sum_{k=0}^{n-1}(-1)^{k+n}x_{(\gamma_0,\cdots,\gamma_k,\gamma'_k,\cdots,\gamma'_{n-1})},\end{aligned}$$

where $\hat{\gamma}_j$ means the deletion of γ_j. Next, we obtain

(B)
$$\begin{aligned}(D\delta x)_{(\gamma_0,\cdots,\gamma_n)} &= \sum_{k=0}^{n}(-1)^k(\delta x)_{(\gamma_0,\cdots,\gamma_k,\gamma'_k,\cdots,\gamma'_n)} \\ &= g_{\gamma_0\gamma'_0}(x_{(\gamma'_0,\cdots,\gamma'_n)}) \\ &+ \sum_{k=1}^{n}(-1)^k g_{\gamma_0\gamma_1}(x_{(\gamma_1,\cdots,\gamma_k,\gamma'_k,\cdots,\gamma'_n)}) \\ &+ \sum_{k=2}^{n}\sum_{j=1}^{k-1}(-1)^{k+j}x_{(\gamma_0,\cdots,\hat{\gamma}_j,\cdots,\gamma_k,\gamma'_k,\cdots,\gamma'_n)} \\ &+ \sum_{k=1}^{n}x_{(\gamma_0,\cdots,\gamma_{k-1},\gamma'_k,\cdots,\gamma'_n)} - \sum_{k=0}^{n-1}x_{(\gamma_0,\cdots,\gamma_k,\gamma'_{k+1},\cdots,\gamma'_n)} \\ &+ \sum_{k=0}^{n-1}\sum_{j=k+2}^{n-1}(-1)^{k+j}x_{(\gamma_0,\cdots,\gamma_k,\gamma'_k,\cdots,\hat{\gamma}'_j,\cdots,\gamma'_n)} \\ &+ \sum_{k=0}^{n-1}(-1)^{k+n+1}x_{(\gamma_0,\cdots,\gamma_k,\gamma'_k,\cdots,\gamma'_{n-1})} - x_{(\gamma_0,\cdots,\gamma_n)}.\end{aligned}$$

We see that the fourth term and the fifth term of (B) add up to 0. And we also can check that the first, the second, the third and the fourth terms of (A) and the second, the third, the sixth and the seventh terms of (B) add up to 0. Thus we have

$$\begin{aligned}(\delta Dx)_{\bar{\gamma}} + (D\delta x)_{\bar{\gamma}} &= g_{\gamma_0\gamma_0'}(x_{(\gamma_0',\cdots,\gamma_n')}) - x_{(\gamma_0,\cdots,\gamma_n)} \\ &= (s^\sharp r^\sharp x)_{(\gamma_0,\cdots,\gamma_n)} - x_{(\gamma_0,\cdots,\gamma_n)}\end{aligned}$$

which shows that D is a cochain homotopy between $s^\sharp r^\sharp$ and the identity $1_{C^\sharp(\mathfrak{G})}$. So our proof is complete.

References

1. S. Araki and Z. I. Yoshimura, *A spectral sequence associated with a cohomology thoery of infinite CW-complexes*, Osaka J. Math. **9** (1972), 351-365.
2. M. Huber and W. Meier, *Cohomology theories and infinite CW-complexes*, Comment Math. Helvetici **53** (1978), 239-257.
3. P. C. Kainen, *Universal coefficient theorems for generalized homology and stable cohomotopy*, Pacific jour. Math **37(2)** (1971), 397-407.
4. S. Mardešić, *A note on strong homology of inverse systems*, Tsukuba J. Math. **11(1)** (1987), 177-197.
5. S. Mardešić and J. Segal, *Shape theory*, North-Holland Publ. Co. Amsterdam , New York, 1982.
6. C. A. McGibbon, *Phantom maps, Handbook of algebraic topology*, North-Holland, New York, 1995.
7. C. A. McGibbon and R. Steiner, *Some questions about the first derived functor of the inverse limit*, J. Pure Appl. Algebra **103** (1995), 325-340.
8. L. Mdzinarishvili and E. Spanier, *Inverse limits and cohomology*, Glasnik Math. **28(48)** (1993), 167-176.
9. G. Nöbeling, *Über die derivierten des inversen und des derekten limes einer modulefamile*, Topology **1** (1961), 47-61.
10. T. Watanabe, *An elementary proof of the invariance of* $\lim^n$ *on pro-abelian groups*, Glasnik Math. **26** (1991), 177-208.