Honam Mathematical J. 22(2000), No. 1,pp. 9–16

CATENARY MODULES II

S. NAMAZI AND H. SHARIF

Dept. of Mathematics, Shiraz University,
Shiraz 71454, Iran.
E-mail : namazish@sums.ac.ir.
E-mail : sharif@sun01.susc.ac.ir.

Abstract. An A-module M is catenary if for each pair of prime submodules K and L of M with $K \subset L$ all saturated chains of prime submodules of M from K to L have a common finite length. We show that when A is a Noetherian domain, then every finitely generated A-module is catenary if and only if A is a Dedekind domain or a field. Moreover, a torsion-free divisible A-module M is catenary if and only if the vector space M over $Q(A)$ (the field of fractions of A) is finite dimensional.

0. Introduction

In this paper all rings are commutative with identity and all modules are unitary. Recall that a ring A is called catenary if the following condition is satisfied: for any prime ideals p and p' of A with $p \subset p'$ there exists a saturated chain of prime ideals starting from p and ending at p', and all such chains have the same finite length. In [**8**] we extended this definition to modules and we gave some characterisations of such modules.

In section 1 we show that when A is a Noetherian domain, then every finitely generated A-module is catenary if and only if A is a Dedekind domain or a field. A Noetherian ring A for which A^2 is a catenary A-module must be of dimension at most one. Moreover, we prove that every module of finite length is catenary.

In section 2 we determine the height of some prime submodules of A^2 as an A-module. By using of catenary modules we show that every Noetherian UFD A of dimension 2 has a height one prime ideal p such that A/p is not a Dedekind domain.

Let A be a ring and M be an A-module. A proper submodule K of M is called prime if $am \in K$ implies $m \in K$ or $aM \subseteq K$, for $a \in A$, $m \in M$ (see, for example, [**3**] or [**6**]). A strictly increasing (or decreasing

Received May 16, 2000.

) chain $K_0 \subset K_1 \subset \dots$ of prime submodules of an A-module M is said to be saturated if there do not exist any prime submodule strictly contained between any two consecutive terms. We say that a prime submodule K of M has height n, if there exsits a chain $K = K_0 \supset K_1 \supset \cdots \supset K_n$ of prime submodules $K_i (0 \leq i \leq n)$ of M, but no longer such chain. Otherwise, we say that it has infinite height. We shall denote the height of K by htK. We define the $h\text{-}dim(M)$ to be the supremum of the heights of all prime submodules of M. If M has no prime submodule, we set $h\text{-}dim(M) = -1$. Note that if $M = A$, then the $h\text{-}dim(M)$ is just the Krull dimension $dim(A)$ of A.

An A-module M is called a locall module if it has exactly one maximal submodule.

1. Catenary modules.

DEFINITION. An A-module M is said to be catenary if for each pair K, K' of prime submodules of M, with $K \subset K'$ there exists a saturated chain of prime submodules of M from K to K' and all such chains have the same finite length.

It is proved that any finitely generated module over a Dedekind domain is catenary and also being catenary is a local property [**8**].

In the proof of Theorem 2.12 of [**8**] we can omit the hypothesis $dim(A) = 1$, since one can show that this condition does hold by using the Principal Ideal Theorem. Thus we have the following theorem.

THEOREM 1.1. *Let $(A, \mathcal{M})$ be a Noetherian local domain. Suppose $M = A^2$ is a catenary A-module. Then A is a discrete valuation domain or a field.*

THEOREM 1.2. *Let A be a Noetherian domain which is not a field. Then the following are equivalent:*

(i) *A is a Dedekind domain.*
(ii) *Every finitely generated A-module is catenary.*
(iii) *A^n is a catenary A-module for some $n \geq 2$.*
(iv) *A^2 is a catenary A-module.*

Proof. $(i) \Longrightarrow (ii)$ The proof follows by [**8** , Corollary 2.10 and Lemma 2.2].

$(ii) \Longrightarrow (iii)$ Clear.

$(iii) \Longrightarrow (iv)$ The proof follows by [**8**, Lemma 2.2].

(iv) $\Longrightarrow$ *(i)* For each non zero prime ideal p of A, A_p is a Noetherian local domain and $A_p \oplus A_p$ is a catenary A_p-module by [**8**, Theorem 2.9]. Thus A_p is a discrete valuation ring, by Theorem 1.1.

COROLLARY 1.3. *Let A be a Noetherian ring. Then the following are equivalent:*

(i) *For each prime ideal p of A, A/p is a Dedekind domain or a field.*
(ii) *Every finitely generated A-module is catenary.*
(iii) *A^n is a catenary A-module for some $n \geq 2$.*
(iv) *A^2 is a catenary A-module.*

Proof. *(i)* $\Longrightarrow$ *(ii)* Let M be a finitely generated A-module. For each prime submodule K of M with $(K : M) = p$, M/K is a finitely generated (A/p)-module and A/p is a Dedekind domain or a field. Thus M/K is a catenary (A/p)-module. Now M is a catenary A-module by [**8**, Lemma 2.3].

(ii) $\Longrightarrow$ *(iii)* Clear.

(iii) $\Longrightarrow$ *(iv)* It follows by [**8**, Lemma 2.2].

(iv) $\Longrightarrow$ *(i)* For each prime ideal p of A, $p \oplus p$ is a prime submodule of $A^2 = A \oplus A$. Hence $\frac{A}{p} \oplus \frac{A}{p} = \frac{A \oplus A}{p \oplus p}$ is a catenary (A/p)-module, by [**8**, Lemma 2.3]. Thus A/p is a Dedekind domain or a field.

For a ring A which satisfies each of the equivalent conditions of Corollary 1.3, we have $dim(A) \leq 1$.

LEMMA 1.4. *Let $\varphi : A \longrightarrow A'$ be a ring epimorphism. Let M be an A-module such that $(ker\varphi)M = 0$. Then M is an A'-module and we have that M is a catenary A-module if and only if M is a catenary A'-module.*

Proof. For $a \in A$, $m \in M$ we have $\varphi(a)m = am$. Thus K is a prime A-submodule of M if and only if K is a prime A'-submodule of M.

COROLLARY 1.5. *Let M be an A-module and I be an ideal of A such that $IM = 0$. If M is a catenary $\frac{A}{I}$-module, M is a catenary A-module.*

COROLLARY 1.6. *Let M be a finitely generated A-module.If p is a prime ideal of A such that $pM = 0$ and A/p is a Dedkind domain or a field, then M is a catenary A-module.*

EXAMPLE 1.7. If $A = k[X, Y]$ where k is a field, then $k[X] = \frac{A}{AY}$ is an A-module and $M = k[X] \oplus k[X]$ is a catenary $k[X]$-module. Thus M is a catenary A-module.

EXAMPLE 1.8. Let A be a Noetherian ring and $\mathcal{M}$ be a maximal ideal of A. Then $M = \mathcal{M}/\mathcal{M}^2$ is a catenary $\frac{A}{\mathcal{M}}$-module, hence M is a catenary A-module.

DEFINITION. Let M be an A-module with $h\text{-}dim(M) < \infty$. We say that M is equidimensional if $h\text{-}dim(\frac{M}{K}) = h\text{-}dim(M)$ for every minimal prime submodule K of M. (To see the definition of an equidimensional ring in [**5**, page 250])

We saw in [**8**, Proposition 2.5] that if M is an A-module with $h\text{-}dim(M) < \infty$ and if for each pair $K \subset L$ of prime submodules of M, we have $ht(\frac{L}{K}) = htL - htK$, then M is catenary. Now we show that for a local module M the converse is true if M is equidimensional.

PROPOSITION 1.9. *Let M be a local equidimensional A-module with maximal submodule N. If M is catenary then for each pair $K \subset L$ of prime submodules of M, we have $ht(\frac{L}{K}) = htL - htK$.*

Proof. If we choose a minimal prime submodule $K_0 \subset K$, then $ht(\frac{N}{K_0}) = ht(\frac{N}{K}) + ht(\frac{K}{K_0})$, since M is catenary. Thus $ht(\frac{K}{K_0}) = h\text{-}dim(M) - ht(\frac{N}{K})$, since M is equidimensional. Hence $ht(\frac{K}{K_0})$ is independent of the choice of K_0, so that $htK = ht(\frac{K}{K_0})$. Similarly $htL = ht(\frac{L}{K_0})$. Therefore, $htL - htK = ht(\frac{L}{K_0}) - ht(\frac{K}{K_0}) = ht(\frac{L}{K})$.

PROPOSITION 1.10. *Every finitely generated Artinian A-module M is catenary. That is, if $l_A(M) < \infty$, then M is catenary.*

Proof. For each prime submodule K of M with $(K : M) = p$, M/K is a finitely generated torsion-free Artinian (A/p)-module. Thus the integral domain A/p is a field and hence M/K is catenary. Thus M is catenary, by [**8**, Lemma 2.3].

COROLLARY 1.11. *Let M be an A-module and $l_A(M) < \infty$. If $K_0, K_1, \dots, K_r$ are all minimal prime submodules of M and $(K_i : M) = p_i$, then*

$$h - \dim(M) = \max\{l_{\frac{A}{p_i}}(\frac{M}{K_i}) - 1; i = 0, 1, \dots, r\}.$$

Proof. First note that M has only finitely many minimal prime submodules, by [**7**, Theorem 4.2]. Let $K_i = L_0 \subset L_1 \subset \cdots \subset L_n = L$ be a saturated chain of prime submodules of M such that $1 \leq i \leq r$ and L is a maximal submodule of M. Then $0 \subset \frac{L_1}{L_0} \subset \cdots \subset \frac{L_n}{L_0}$ is a saturated chain of prime submodules of the $(\frac{A}{p_i})$-module $\frac{M}{K_i} = \frac{M}{L_0}$. Now $\frac{M}{K_i}$ is a finitely generated torsion-free Artinian $(\frac{A}{p_i})$-module, thus $(\frac{A}{p_i})$ is a field and $l_{\frac{A}{p_i}}(\frac{M}{K_i}) = n+1$, that is, $n = l_{\frac{A}{p_i}}(\frac{M}{K_i}) - 1$.

LEMMA 1.12. *Let A be an integral domain and M be a divisible A-module. If N is a proper submodule of M, then $(N : M) = 0$.*

Proof. If $a \in (N : M)$, then $aM \subseteq N$. If $a \neq 0$, then for all $m \in M$ we have $m = ax$, for some $x \in M$, by divisiblity of M. Thus $m = ax \in aM \subseteq N$. That is, $M = N$, a contradiction.

PROPOSITION 1.13. *Let A be a domain and Q be the quotient field of A. If M is a torsion-free divisible A-module, then:*

(i) *M is a vector space over Q.*

(ii) *N is a prime A-submodule of M if and only if N is a proper subspace of the vector space M over Q.*

Proof. (*i*) For any $0 \neq b \in A$ and $x \in M$, there exists an element $m \in M$ such that $bm = x$. m is unique because $bm = bm'$ implies that $m = m'$. Define $m = \frac{1}{b}x$. Hence M is a Q-module by $\frac{a}{b}x = \frac{1}{b}(ax) = a(\frac{1}{b}x)$, for all $a, b \in A$, $x \in M$.

(*ii*) If N is a prime submoudle of the A-module M, then N is a divisible torsion-free A-module by Lemma 1.12. Now part (i) implies that N is a subspace of M. Conversely, let N be a proper subspace of the vector space M over Q. If $am \in N$ for $0 \neq a \in A$, $m \in M$, then $m = \frac{1}{a}(am) \in N$. Thus N is a prime submodule of M.

COROLLARY 1.14. *Let A be a domain and Q be the quotient field of A. Suppose that M is a torsion-free divisible A-module. Then M is a catenary A-module if and only if it is a finite dimensional vector space over Q.*

EXAMPLE 1.15. If $\mathbf{Q}$ is the field of rational numbers, then $\mathbf{Q}^n (n \geq 1)$ is a catenary $\mathbf{Z}$-module, by Corollary 1.14. However it is not a finitely generated $\mathbf{Z}$-module.

PROPOSTION 1.16. *Let A be a domain. If M is a finitely generated divisible A-module, then A is a field.*

Proof. For a maximal ideal $\mathcal{M}$ of A, there exists a prime submodule K of M such that $(K : M) = \mathcal{M}$, by [**6**, Theorem 3.3]. But $\mathcal{M} = (K : M) = 0$, by Lemma 1.12. Thus A is a field.

2. On the height of some prime submodules.

LEMMA 2.1. *Let A be an integral domain and $M = A^2$. If K is a non-zero prime submodule of M such that $(K : M) = 0$, then $ht(K) = 1$.*

Proof. Let $S = A - \{0\}$ and $Q = S^{-1}A$ be the quotient field of A. Then $V = S^{-1}M = Q \oplus Q$ is a vector space of dimension 2 and $S^{-1}K$ is a non-zero prime submodule of V (since $K \subset S^{-1}K$). Thus $ht(S^{-1}K) = 1$. By [**2**, Lemma 10], $ht(K) = ht(S^{-1}K) = 1$.

COROLLARY 2.2. *Let A be an integral domain and $M = A^2$. If L is a prime submodule of M such that $(L : M) = p$ and $L \neq p \oplus p$, then $ht(\frac{L}{p \oplus p}) = 1$.*

Proof. Let $A' = \frac{A}{p}$. Then $L' = \frac{L}{p \oplus p}$ is a non-zero prime submodule of $M' = \frac{M}{p \oplus p} = A' \oplus A'$ as an A'-module and $(L' :_{A'} M') = 0$. Hence $1 = ht(L') = ht(\frac{L}{p \oplus p})$, by Lemma 2.1.

PROPOSITION 2.3. *Let A be an integral domain and $M = A^2$. Suppose that for each pair $q \subset q'$ of prime ideals of A with $ht(q'/q) = 1$ we have $ht(\frac{q' \oplus q'}{q \oplus q}) = 1$. Let p be a prime ideal of A with $ht(p) = n$. Then*

(i) $ht(p \oplus p) = n$.

(ii) If L is a prime submodule of M such that $(L : M) = p$ and $L \neq p \oplus p$, then $ht(L) = n + 1$.

Proof. By induction on n. If $n = 0$, then the result follows by Lemma 2.1. Now let for each prime ideal p with $ht(p) \leq n$, (i) and (ii) hold. If $ht(q) = n + 1$, then we show that $ht(q \oplus q) = n + 1$ and for each prime submodule L of M such that $(L : M) = q$ and $L \neq q \oplus q$ we have $ht(L) = n + 2$.

If N is a prime submodule of M and $N \subset q \oplus q$, then we claim that $ht(N) \leq n$. Since $p_1 = (N : M) \subset q$, $ht(p_1) \leq n$. If $N = p_1 \oplus p_1$, then $ht(N) = ht(p_1) \leq n$. If $p_1 \oplus p_1 \subset N \subset q \oplus q$, then $ht(p_1) < n$ because if $ht(p_1) = n$, then $ht(q/p_1) = 1$ and hence $ht(\frac{q \oplus q}{p_1 \oplus p_1}) = 1$, a contradiction. Thus $ht(N) = 1 + ht(p_1) \leq n$. Therefore, $ht(q \oplus q) \leq n + 1$. Since $ht(q) = n + 1$, then $ht(q \oplus q) = n + 1$.

For each prime submodule K of M with $K \subset L$ we have $p' = (K : M) \subset (L : M) = q$ and $ht(p') \leq n$. Thus $ht(K) \leq 1 + ht(p') \leq n + 1$. Hence $ht(L) \leq n + 2$. Since $q \oplus q \subset L$ and $ht(q \oplus q) = n + 1$, $ht(L) = n + 2$.

LEMMA 2.4. *Let A be a UFD and $M = A^2$. If $K \neq 0$ is a prime submodule of M with $(K : M) = p$, then:*

(i) *If $p = 0$, then there exist $a, b \in A$ such that $gcd(a, b) = 1$ and $K = A(a, b)$. In this case $ht(K) = 1$.*

(ii) *If $ht(p) = 1$ and $K = p \oplus p$, then $ht(K) = 1$.*

(iii) *If $ht(p) = 1$ and $K \neq p \oplus p$, then $ht(K) = 2$.*

Proof. (*i*) The result follows by [**2**, Corollary 5].

(*ii*) If there exists a non-zero prime submodule N of M contained in $p \oplus p$, then $(N : M) = 0$. Thus $N = A(a, b)$ for some $a, b \in A$ with $gcd(a, b) = 1$, by (i). Since $ht(p) = 1$, p is generated by a prime element $x \in A$. Now $(a, b) \in N \subset p \oplus p$, implies that $x|a$, $x|b$, a contradiction.

(*iii*) By corollary 2.2, there is no prime submodule of M between $p \oplus p$ and K. By parts (i) and (ii) we have $ht(K) = 2$.

LEMMA 2.5. *Let A be a Noetherian UFD with $dim(A) = 2$ and $M = A^2$. Suppose A/p is a Dedekind domain for each prime ideal p of A with $ht(p) = 1$. If $\mathcal{M}$ is a maximal ideal of A, then:*

(i) *$ht(\mathcal{M} \oplus \mathcal{M}) = 2$.*

(ii) *If N is a prime submodule of M such that $(N : M) = \mathcal{M}$ and $N \neq \mathcal{M} \oplus \mathcal{M}$, then $htN = 3$.*

Proof. (*i*) If p is a prime ideal of A contained in $\mathcal{M}$ with $ht(p) = 1$, then $p \oplus p \subset \mathcal{M} \oplus \mathcal{M}$ and $\frac{\mathcal{M}\oplus\mathcal{M}}{p\oplus p} = \frac{\mathcal{M}}{p} \oplus \frac{\mathcal{M}}{p}$ is a prime submodule of $\frac{M}{p\oplus p} = \frac{A}{p} \oplus \frac{A}{p}$ as an $(\frac{A}{p})$-module. Since A/p is a Dedekind domain, $ht(\frac{\mathcal{M}\oplus\mathcal{M}}{p\oplus p}) = 1$, by [**2**, Corollary 2]. Thus $ht(\mathcal{M} \oplus \mathcal{M}) = 2$, by Lemma 2.4.

(*ii*) $\frac{N}{\mathcal{M}\oplus\mathcal{M}}$ is a non-zero prime submodule of the vector space $\frac{M}{\mathcal{M}\oplus\mathcal{M}}$ over the field $\frac{A}{\mathcal{M}}$. Thus $ht(\frac{N}{\mathcal{M}\oplus\mathcal{M}}) = rank(\frac{N}{\mathcal{M}\oplus\mathcal{M}}) = 1$. By part (i) and Lemma 2.4, $ht(N) = 3$.

COROLLARY 2.6. *If A is a Noetherian UFD with $dim(A) = 2$, then there exists a prime ideal p of A such that $ht(p) = 1$ and A/p is not a Dedekind domain.*

Proof. If for each prime ideal p of A with $ht(p) = 1$, A/p is a Dedekind doman, then $M = A^2$ is a catenary A-module by Lemmas 2.4 and 2.5. But by Corollary 1.3, M is not catenary, because $dim(A) > 1$ as required.

As we saw in [**8** Example 2.14], $p = B(X^3 - Y^2)$ is a prime ideal of $B = k[X, Y]$ (k is a field) of height 1 and $\frac{B}{p}$ is not a Dedekind domain. Also $< X^3 - 4 >$ is a prime ideal of $\mathbf{Z}[X]$ of height 1 for which $\frac{\mathbf{Z}[X]}{<X^3-4>}$ is not a Dedekind domain.

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