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ON THE GEODESIC CURVATURE OF A SIMPLE CLOSED CURVE IN $\mathbb{R}^2$, THE UNIT SPHERE S^2 OR THE HYPERBOLIC PLANE H^2

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Abstract. We establish a sufficient condition for a simple closed curve in the Euclidean plain $\mathbb{R}^2$, the unit sphere S^2 or the hyperbolic plane H^2 to be the boundary of a metric ball.

1. Introduction

In 1990, L. Coghlan and Y. Itokawa [2] proved

THEOREM. *Let $f : M \to M^{d+1}(c)$ be an isometric immersion of a compact Riemannian manifold of dimension $d \geq 2$ into a simply connected space form of constant curvature c and let K be the sectional curvature of M. Suppose that $f(M)$ is contained in a metric ball B_r of radius r. Assume either one of the following conditions:*

- $c = 0$ *and* $\sup K = \dfrac{1}{r^2}$,
- $c > 0$, $r \leq \dfrac{\pi}{2\sqrt{c}}$ *and* $\sup K = c\csc^2(\sqrt{c}\,r)$,
- $c < 0$ *and* $\sup K = -c\,csch^2(\sqrt{-c}\,r)$.

Then $f(M)$ is the boundary of the metric ball.

Let M be the Euclidean plane $\mathbb{R}^2$, the unit sphere S^2 or the hyperbolic plane H^2. And let B_r be the closed normal ball of radius r in M. Let $\alpha : I \to M$ be a simple closed c^∞-curve in M with the same orientation

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as ∂B_r. Suppose that on M an orientation is given such that the geodesic curvature of ∂B_r is positive. Then, the purpose of this paper is to prove the following two analogous results of the above Theorem for the geodesic curvature κ_g of a simple closed curve α in M.

THEOREM 1. *Let $\alpha : I \to M$ be a simple closed c^∞-curve parametrized by arc-length. Suppose that $\alpha(I)$ is contained in a closed normal ball B_r of radius r in M. Assume either one of the following conditions:*

- $M = \mathbb{R}^2$ *and* $\sup \kappa_g = \frac{1}{r}$,
- $M = S^2$, $r \leq \frac{\pi}{2}$ *and* $\sup \kappa_g = \cot r$,
- $M = H^2$ *and* $\sup \kappa_g = \coth r$.

Then $\alpha(I)$ coincides with ∂B_r.

THEOREM 2. *Let $\alpha : I \to M$ be a simple closed c^∞-curve parametrized by arc-length. Suppose that the domain D bounded by α contains a closed normal ball B_r of radius r with center p. Assume either one of the following conditions:*

- $M = \mathbb{R}^2$ *and* $\inf \kappa_g = \frac{1}{r}$,
- $M = S^2$, $\alpha(I) \subset B_{\frac{\pi}{2}}(p)$ *and* $\inf \kappa_g = \cot r$,
- $M = H^2$ *and* $\inf \kappa_g = \coth r$.

Then $\alpha(I)$ coincides with ∂B_r.

2. Preliminary Results

Throughout this paper, M will be as described in the first paragraph. Let $\overline{\nabla}$ and $\langle\, , \rangle$ be the connection and metric tensor respectively of M. If $\alpha : I \to M$ is a unit speed c^∞-curve in a surface M oriented by a frame field E_1 and E_2, then $T = \alpha'$ is the *unit tangent vector field* of α. On α, we put

$$U = E_1 \times E_2 \quad \text{and} \quad N = U \times T.$$

Then N is called the *principal normal vector field* of α. The *geodesic curvature* κ_g of α is the real valued function on I defined by

$$\kappa_g = \langle \overline{\nabla}_{\alpha'} \alpha', N \rangle.$$

Suppose α has the same orientation as ∂B_r such that under it, the geodesic curvature of ∂B_r is positive.

Let f be a smooth function on M and Δ the Laplacian on α. Then we have

$$\Delta(f|\alpha) = Hess.\, f|\alpha(\alpha', \alpha') = Hess.\, f(\alpha', \alpha') + \langle(\overline{\nabla}_{\alpha'}\alpha')^{\perp}, grad\, f\rangle. \quad (1)$$

We use the following lemma (cf. [1]) to prove Theorem 1.

LEMMA 2.1. *Let p be a point in M and let $l(x)$ be the distance function from p to x in M. Then on M*

$$\begin{cases} (a) & Hess.\, l^2 = 2\langle\,,\rangle & \text{if } M = \mathbb{R}^2, \\ (b) & Hess.\,(-\cos l) = \cos l\,\langle\,,\rangle & \text{if } M = S^2, \\ (c) & Hess.\,\cosh l = \cosh l\,\langle\,,\rangle & \text{if } M = H^2. \end{cases}$$

Proof. Let $\partial/\partial l, V$ be perpendicular vector fields on a neighborhood of $q \in S^2$ such that $|V| = \sin l$, that $\overline{\nabla}_V V$ is parallel to $\partial/\partial l$, and that V is a Jacobi field along any geodesic through p. Clearly $\overline{\nabla}_V\, \partial/\partial l$ is parallel to V and $[V, \partial/\partial l] = 0$. Hence

$$\langle\overline{\nabla}_V \frac{\partial}{\partial l}, V\rangle = \langle\overline{\nabla}_{\partial/\partial l} V, V\rangle = \frac{1}{2}\frac{\partial}{\partial l}\langle V, V\rangle = \sin l \cos l,$$

and

$$\overline{\nabla}_V \frac{\partial}{\partial l} = \overline{\nabla}_{\partial/\partial l} V = \cot l\, V.$$

So

$$\langle\overline{\nabla}_V V, \frac{\partial}{\partial l}\rangle = -\langle V, \overline{\nabla}_V \frac{\partial}{\partial l}\rangle = -\sin l \cos l.$$

Hence

$$\overline{\nabla}_V V = -\sin l \cos l\, \frac{\partial}{\partial l}.$$

Therefore

$$\begin{aligned} Hess.\, \cos l\,(V, V) &= VV\cos l - \overline{\nabla}_V V \cos l \\ &= -\sin^2 l \cos l = -\cos l\,\langle V, V\rangle. \end{aligned}$$

Moreover,

$$\begin{aligned} Hess.\ \cos l\,(\frac{\partial}{\partial l}, V) &= \frac{\partial}{\partial l} V \cos l - (\overline{\nabla}_{\partial/\partial l} V) \cos l \\ &= -\cot l\, V \cos l = 0 = -\cos l\,\langle \frac{\partial}{\partial l}, V\rangle. \end{aligned}$$

Also,

$$\begin{aligned} Hess.\ \cos l\,(\frac{\partial}{\partial l}, \frac{\partial}{\partial l}) &= \frac{d^2}{dl^2} \cos l - (\overline{\nabla}_{\partial/\partial l} \frac{\partial}{\partial l}) \cos l \\ &= -\cos l = -\cos l\,\langle \frac{\partial}{\partial l}, \frac{\partial}{\partial l}\rangle. \end{aligned}$$

Thus we have the proof for S^2. Similar proofs hold for $\mathbb{R}^2$ and H^2.

3. Proof of Theorem 1

Let p be the center of the closed normal ball $B_r \subset M$ that contains $\alpha(I)$. Let $l(x)$ be the distance function from p to x in M and let $q \in \alpha(I)$ be a local maximum point for the function $l|\alpha$. Consider the unique normal geodesic $\gamma : [0, l(q)] \to B_r$ from $\gamma(0) = p$ to $\gamma(l(q)) = q$ $(l(q) \le r)$.

The following two lemmas have analogies with results in [2] which are proved by using a function $l^2/2$. But we find difficulty in using the function in the case H^2.

LEMMA 3.1. *Let q be defined as previously noted. Then,*

$$l(q) = r.$$

Proof. Since q is a local maximum point for $l|\alpha$,

$$\langle \alpha', grad\, l\rangle_q = \alpha'(q)(l) = (l \circ \alpha)'(s) = 0.$$

Hence

$$\alpha'(q) \perp grad\, l(q) \quad \text{and} \quad grad\, l(q) = -N(q). \tag{2}$$

Here, we define a function $f : M \to \mathbb{R}$ by

$$f(x) = \begin{cases} l^2(x) & \text{if } M = \mathbb{R}^2, \\ -\cos l(x) & \text{if } M = S^2, \\ \cosh l(x) & \text{if } M = H^2. \end{cases}$$

Since q is a local maximum point for $l|\alpha$, it is also a local maximum point for $f|\alpha$. Hence

$$Hess.\, f|\alpha(q)(\alpha', \alpha') \leq 0.$$

From (1), Lemma 2.1 and (2), we obtain

In the case $M = S^2$,

$$\begin{aligned} 0 &\geq Hess.\, f|\alpha(q)(\alpha',\, \alpha') \\ &= Hess.\, f(q)(\alpha',\, \alpha') + \langle(\overline{\nabla}_{\alpha'}\alpha')^{\perp},\, grad\, f\rangle_q \\ &= \cos l(q)\, \langle\alpha',\, \alpha'\rangle + \sin l(q)\, \langle(\overline{\nabla}_{\alpha'}\alpha'),\, grad\, l\rangle_q \\ &= \cos l(q) - \sin l(q)\, \kappa_g(q). \end{aligned}$$

So, we get

$$\kappa_g(q) \geq \cot l(q).$$

From the hypothesis $\sup \kappa_g = \cot r$, we have

$$l(q) = r.$$

In the case $M = \mathbb{R}^2$ or H^2, we can prove it in the same way.

LEMMA 3.2. *Let $q \in \alpha(I)$ be a point such that $l(q) = r$. Then there is a neighborhood U of q in α such that $f|\alpha$ is subharmonic on it.*

Proof. We define an auxiliary support function $s : \alpha(I) \to [-1, 1]$ by

$$s(x) = \langle N,\, grad\, l\rangle_x.$$

Since $\langle(\overline{\nabla}_{\alpha'}\alpha')^{\perp},\, N\rangle_x = \langle\overline{\nabla}_{\alpha'}\alpha',\, N\rangle_x = \kappa_g(x)$, we have

$$\begin{aligned} \langle(\overline{\nabla}_{\alpha'}\alpha')^{\perp},\, grad\, l\rangle_x &= \langle\kappa_g N,\, grad\, l\rangle_x \\ &= \kappa_g(x)\, s(x). \end{aligned}$$

Let $U = \{x \in \alpha(I) |\, s(x) < 0 \quad \text{and} \quad \kappa_g(x) > 0\}$. Since $s(q) = -1$ and $\kappa_g(q) > 0$, U is a neighborhood of q in α. We claim that on U, $\Delta(f|\alpha) \geq 0$. Let us prove it in the case $M = H^2$. At each $x \in U$ we have

$$\begin{aligned} \Delta(f|\alpha)(x) &= Hess.\, f(x)(\alpha',\, \alpha') + \langle(\overline{\nabla}_{\alpha'}\alpha')^{\perp},\, grad\, f\rangle_x \\ &= \cosh l(x)\, \langle\alpha',\, \alpha'\rangle + \sinh l(x)\, \langle(\overline{\nabla}_{\alpha'}\alpha'),\, grad\, l\rangle_x \\ &= \cosh l(x) + \sinh l(x)\, \kappa_g(x)\, s(x) \\ &\geq \cosh l(x) - \sinh l(x)\, \kappa_g(x) \\ &\geq \cosh l(x) - \sinh l(x)\, \coth r \\ &\geq \cosh l(x) - \sinh l(x)\, \coth l(x) = 0. \end{aligned}$$

Similar proofs hold for $\mathbb{R}^2$ and S^2.

Lemma 3.2 implies by the maximum principle for subharmonic functions that $f|\alpha(x) \equiv f|\alpha(q)$ on U, and so $l(x) \equiv l(q) = r$ on U. But this implies that if $q \in \alpha(I)$ has $l(q) = r$, then $l \equiv r$ in a neighborhood of q. Let $V = \{q \in \alpha(I) \,|\, l(q) = r\}$. Then $V \neq \emptyset$ and V is a closed and open subset of $\alpha(I)$. Since $\alpha(I)$ is connected, $V = \alpha(I)$ and $\alpha(I)$ coincides with ∂B_r, which completes the proof of Theorem 1.

4. Proof of Theorem 2

If φ is an angle function from E_1 to α' along α, then it is well known [3] that

$$\kappa_g = \frac{d\varphi}{ds} + \omega_{12}(\alpha'),$$

and

$$\begin{aligned}\int_\alpha \kappa_g \, ds &= 2\pi + \int_\alpha \omega_{12} = 2\pi + \int_D d\omega_{12} \\ &= 2\pi + \int_D (-K\,dM) = 2\pi - K\,A, \end{aligned} \tag{3}$$

where A is the area of D.

Let L be the length of α. Then since the domain D bounded by α contains B_r,

$$L \geq \text{ the length of } \partial B_r \quad \text{and} \quad A \geq \text{the area of } B_r. \tag{4}$$

(a) In the case $M = S^2$, since $K = 1$, (3) and the hypothesis $\inf \kappa_g = \cot r$ imply

$$2\pi - A = \int_\alpha \kappa_g \, ds \geq \int_\alpha \cot r \, ds = \cot r \, L.$$

Here, since the area of $B_r = 2\pi\,(1 - \cos r)$,

$$A \geq 2\pi\,(1 - \cos r).$$

Hence, we have

$$\begin{aligned} L &\leq \frac{1}{\cot r}\,(2\pi - A) \\ &\leq \tan r\,\{2\pi - 2\pi\,(1 - \cos r)\} \\ &= 2\pi\,\sin r. \end{aligned} \tag{a.1}$$

And, since the length of $\partial B_r = 2\pi \sin r$, (4) implies

$$L \geq 2\pi \sin r. \tag{a.2}$$

From (a.1) and (a.2), $L = 2\pi \sin r$ and this means that $\alpha(I)$ coincides with ∂B_r.

(b) In the case $M = H^2$, since $K = -1$, the hypothesis $\inf \kappa_g = \coth r$ implies

$$2\pi + A = \int_\alpha \kappa_g \, ds \geq \int_\alpha \coth r \, ds = \coth r \, L.$$

Hence, we have

$$L \leq \frac{1}{\coth r} (2\pi + A) = \tanh r \, (2\pi + A). \tag{b.1}$$

And, since the length of $\partial B_r = 2\pi \sinh r$, (4) implies

$$L \geq 2\pi \sinh r. \tag{b.2}$$

Since the area of $B_r = 2\pi (\cosh r - 1)$, (4) implies

$$A \geq 2\pi (\cosh r - 1). \tag{b.3}$$

If we put $A_1 = A -$(the area of B_r), then

$$A = 2\pi (\cosh r - 1) + A_1. \tag{b.4}$$

Hence, from (b.1) and (b.4)

$$L \leq \tanh r \, (2\pi \cosh r + A_1) = 2\pi \sinh r + A_1 \tanh r. \tag{b.5}$$

Given a domain D bounded by a curve α in H^2, E. Schmidt [4] in 1940 proved that the area A of D and the length L of α are related by the isoperimetric inequality

$$4\pi A \leq L^2 - A^2.$$

Suppose that $A_1 \neq 0$, namely, $\alpha(I) \neq \partial B_r$. Then, from (b.4) and (b.5) we have

$$\begin{aligned}(L^2 - A^2) - 4\pi A \leq& (2\pi \sinh r + A_1 \tanh r)^2 - \{2\pi (\cosh r - 1) + A_1\}^2 \\ &- 4\pi \{2\pi (\cosh r - 1) + A_1\} \\ =& - 4\pi A_1 \frac{1}{\cosh r} - {A_1}^2 (1 - \tanh^2 r) < 0.\end{aligned}$$

This contradicts to the isoperimetric inequality. Hence $\alpha(I)$ coincides with ∂B_r.

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