물리탐사
Vol. 3, No. 2, 2000, p. 39~47

A Dispersion Analysis for Minimum Grids in the Frequency Domain Acoustic Wave Equation

Jang, Seong-Hyung[1)], Shin, Chang-Soo,[2)] Yoon, Kwang-Jin[2)], Suh, Sang-Young[1)], and Shin, Sung-Ryul[3)]

주파수영역 음향 파동방정식에서 최소 격자수 결정을 위한 격자분산 분석

장성형 · 신창수 · 윤광진 · 서상용 · 신성렬

Abstract : A great deal of computing time and a large computer memory are needed to solve wave equation in a large complex subsurface layers using the finite difference method. The computing time and memory can be reduced by decreasing the number of grid points per minimum wave length. However, the decrease of grids may cause numerical dispersion and poor accuracy. In this study we performed the grid dispersion analysis for several rotated finite difference operators, which was commonly used to reduce grids per wavelength with accuracy in order to determine the solution for the acoustic wave equation in frequency domain. The rotated finite difference operators were to be extended to 81, 121 and 169 difference stars and studied whether the minimum grids could be reduced to 2 or not. To obtain accuracy (numerical errors less than 1%) the following was required: more than 13 grids for conventional 5 point difference stars, 9 grids for 9 difference stars, 3 grids for 25 difference stars, and 2.7 grids for 49 difference stars. After grid dispersion analysis for the new rotated finite difference operators, more than 2.5 grids for 81 difference stars, 2.3 grids for 121 difference stars and 2.1 grids for 169 difference stars were needed. However, in the 169 difference stars, there was no solution because of oscillation of the dispersion curves in the group velocity curves. This indicated that the grids couldn't be reduced to 2 in the frequency acoustic wave equation. According to grid dispersion analysis for the determination of grid points, the more rotated finite difference operators, the fewer grid points. However, the more rotated finite difference operators that are used, the more complex the difference equation terms.

요 약 : 복잡한 지층구조에 대한 파동방정식의 해를 유한 차분법을 이용하여 구하는것은 많은 컴퓨터 계산시간과 기억용량이 필요하다. 컴퓨터 계산시간과 기억용량은 최소 파장당 격자수를 줄이므로써 감소 시킬 수 있지만 수치분산으로 인해 정확도가 떨어지게 마련이다. 본 연구에서는 정확도를 유지하면서 파장당 격자수를 줄이는 방법으로 이용되고 있는 가중평균법을 최대 169점 까지 확장하여 주파수 영역에서 음향파동방정식의 해를 유한차분법으로 구할 때 최소 격자수를 구하기 위한 격자분석을 실시하였다. 지금까지 수치오차가 정확도 1% 내에 존재하기 위해서는 일반적인 5점을 이용하는 경우 파장당 격자수가 13개 이상이 필요하고, 9점의 경우 9개, 25점에서는 3개, 49점에서는 2.7개 이상이 필요하였다. 본 연구에서 정확도를 유지하기 위한 최소격자수를 결정하기 위해 실시된 격자분석 결과 81점에서는 2.5개, 121점에서는 2.3개 그리고 169점에서는 오차 한계를 벗어나 가중평균 계수를 구할 수 없었으며 격자수를 2개까지 줄일 수 없음을 알 수 있었다. 또한 격자분석을 통해 가중평균에 적용되는 격자수가 증가할수록 정확도는 증가하지만 차분식 자체가 증가하여 매우 복잡하게 된다.

Introduction

Explicit schemes that were widely used in the finite difference method for the solution of wave propagation were expensive and required a great deal of computer memory to model exploration scale problems. The methods of reducing the central processing unit (CPU) time and the direct-access memory requirements in finite-difference methods entailed the use of high order finite-difference approximations to spatial and temporal derivatives (Dablain, 1986) as well as the use of the weighted average method which was developed by Jo, *et al.* (1996). The conventional method uses the neighboring 5 points to determine the solution of finite difference modeling of wave propagation.

*2000년 5월 16일 접수
1) Korea Institute of Geology, Mining & Materials (한국자원연구소)
2) Civil, Urban & Geosystem Eng., Seoul National University (서울대학교 지구환경시스템)
3) 한국해양대학교 해양에너지자원공학과

When the grid points per minimum wave length were reduced to save computing time and memory, the grid dispersion became larger and the solution of the wave field may have had more error. To overcome this problem, Jo, *et al.* (1996) developed the new method that reduced the grid points per minimum wave length. This method weighted the finite difference equation after the conventional 5 difference stars were transformed to 45°. The grid points per minimum wave length were reduced to 9 with this method, but the conventional 5 point method required more than 13 grid points. Shin and Sohn (1998) extended this 9 point weighted average method to 25 points which transformed the coordinates to 0°, 45° in 5 points and to 0°, 22.5°, 45°, 62.5° in 25 points. They reduced the grid points perminimum wave length to 3 and maintained accuracy while simultaneously saving computing time and memory. Štekl and Pratt (1998) also applied rotated operators in the viscoelastic media to save computing time and core memory.

According to the weighted average methods it can be deduced that the more finite difference stars, the greater the possible reduction of the grids per minimum wave length with the maintenace of accuracy and resolution. Could the grids be reduced to 2 ? In this study dispersion analysis performed for the weighted average method with 81, 121 and 169 finite difference stars. The study will indicate which weighted average method is optimal and whether the minimum grids can be reduced to 2 or not.

FDM formulation using the weighted average method

In the Cartesian coordinate system, the scalar wave equation in the frequency domain can be written as follows:

$$\frac{\omega^2}{v^2}u+\nabla^2 u=0 \tag{1}$$

where u is the pressure of the wave field, ω is the angular frequency and v is velocity of the medium. The conventional finite difference expression of the eq. (1) by the explicit second-order difference scheme can be written as follows:

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{\Delta z^2}+\frac{\omega^2}{v^2}u_{i,j}=0 \tag{2}$$

where $u_{i,j}$ is compressional field at $x_i=x_0+(i-1)\Delta x$, $z_j=z_0+(j-1)\Delta z$, Δx, Δz is the grid distance and ω is the angular frequency. In order to obtain an accurate solution for eq. (2), 13 grids per minimum wavelength are requireder minimum (Alford et all, 1974). Due to much of grids a great deal of computing time and memory is needed to find out the wavefield for a large and complex geological model. To overcome this problem, the rotated finite difference operator method was developed by Jo, *et al.* (1996). They used the 9 points weighted average finite difference stars that rotated the coordinate system to 45 degrees. After that 25 and 49 points weighted average finite difference stars were developed to reduce the computing time and memory for a complex geological model. This means that if we have a greater finite difference stars in the weighted average method we have the possibility to reduce grids to 2 per minimum wavelength. In this study, the 81, 121 and 169 point weighted average methods were introduced, of which their rotated angles and number of coefficients are shown in Table 1. These methods were to determine whether it is possible to reduce the grids per minimum wavelength to 2 in the frequency domain or not and which is the best optimum method.

Fig. 1 shows the transformed coordinate systems from 9 difference stars to 169. After summing the finite difference equation from 5 points to 169, the Laplacian term was obtained. The new Laplacian term by the weighted average method when we used 169 difference stars is shown below:

$$\begin{aligned}\nabla^2 u=&r_1\nabla^2 u_1+r_2\nabla^2 u_2\\&+r_3\nabla^2 u_3+r_4\nabla^2 u_4\\&\cdots\\&\cdots\\&+r_{41}\nabla^2 u_{41}+r_{42}\nabla^2 u_{42}\end{aligned} \tag{3}$$

Table 1. Transformed coordinates according to several finite difference stars

Finite difference star	5 points	9 points	25 points	49 points	81 points	121 points	169 points
	0	0	0	0	0	0	0
		45	45	45	45	45	45
			26.56	18.43	14.04	11.30	21.8
Coordinate			63.44	33.69	26.57	21.80	18.43
transformation				56.30	36.87	30.96	26.56
angle (degree)				71.56	53.13	38.66	33.69
					63.43	51.34	39.80
					75.96	59.04	50.19
						68.20	56.31
						78.69	63.43
							71.57
							80.75
No. trans. coord.		2	4	6	8	10	12
No. of coeffs.	2	5	13	25	41	61	85

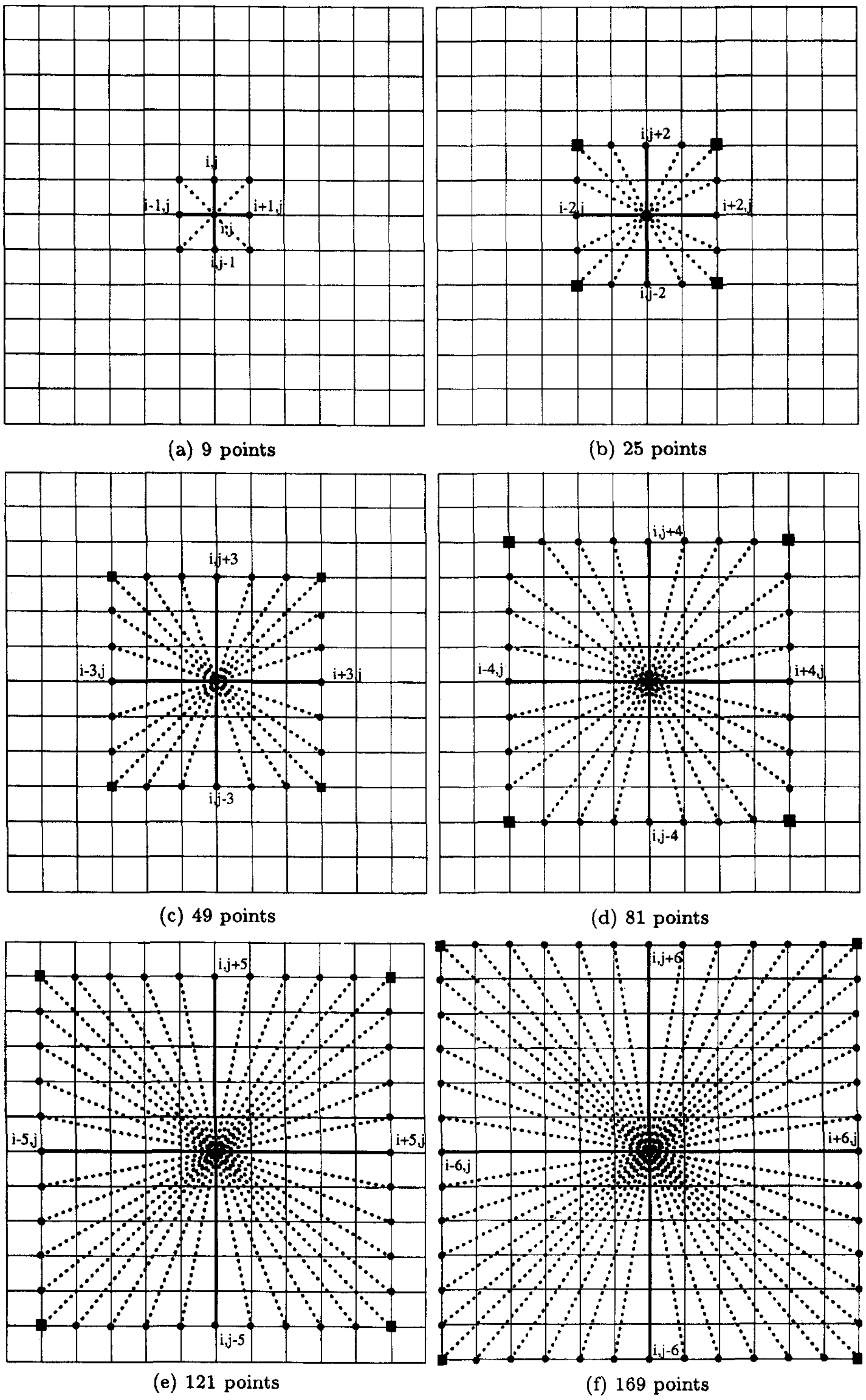

Fig. 1. Computational stars of Laplacian operator which have rotated codrdinate systems from 5 to 169 points.

where r_k refers to the coefficients of difference stars according to each weighted average method, and $\nabla^2 u_k$ is the Laplacian term,

$$\nabla^2 u_1 = \frac{u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}}{\Delta^2}$$

$$\nabla^2 u_2 = \frac{u_{i-1,j+1} + u_{i+1,j-1} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}}{(\sqrt{2}\Delta)^2}$$

$$\cdots$$

$$\cdots$$

$$\nabla^2 u_{42} = \frac{u_{i-6,j+6} + u_{i+6,j-6} - 4u_{i,j} + u_{i,j+6} + u_{i,j-6}}{(\sqrt{36}\Delta)^2}.$$

k, the constant, is 20 for 81 stars and 30 for 121 stars. In order to apply the weighted average method, the mass accelation term of eq. (1) must be expressed with liner combinations of formula wise, that is the lumped mass matrix and consistent mass matrix (Marfurt, 1984). That is

$$[u_{i,j}]=\gamma r^*_{i,j}+(1-\gamma)[u_{i,j}] \tag{4}$$

where γ is the argument that the mass accelation term is determined by the point collocation term and the average term, u^* is the point collocation term, $[u]$ is the average term. The substitution eq. (4) into eq. (1) result in eq. (5):

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}+\gamma\frac{\omega^2}{v^2}u^*+(1-\gamma)\frac{\omega^2}{v^2}[u]=0. \tag{5}$$

The average term is composed with finite difference approximation of the Laplacian term. In 5 point difference stars, the point collocation term which includes the average term can be represented with the weighted average term as shown below:

$$\begin{aligned}[u^*_{i,j}]&=\gamma u^*_{i,j}+(1-\gamma)[u_{i,j}]\\ &=au_{i,j}+b(u_{i+1}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}).\end{aligned} \tag{6}$$

The new mass accelation term for 169 stars by the weighted average method is as follows:

$$\begin{aligned}[u^*_{i,j}]&=\gamma u^*_{i,j}+(1-\gamma)\\ &=a_1u_{i,j}\\ &+a_2(u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1})\\ &\dots\\ &\dots\\ &+a_{43}(u_{i+6,j+6}+u_{i-6,j-6}+u_{i-6,j+6}+u_{i+6,j-6})\end{aligned} \tag{7}$$

where a_k are the weighted average coefficients for the mass term according to each rotated coordinate system. The constant k is 21 for 81 stars and 31 for 121.

Substitution eq. (3) and eq. (7) into eq. (1) yields

$$\sum_{k=1}^{n} r_k\nabla^2u_k+\frac{\omega^2}{v^2}\sum_{k=1}^{n}a_ku^*_{i,j}=0. \tag{8}$$

Substitution of the plane harmonic wave $u=e^{-(k_xx+k_zz)}$, $k_x=$ ksinq and $k_z=k\cos\theta$ into eq. (8), where θ is the propagation angle to the normal, yields the following equation,

$$\frac{\omega^2}{v^2}=-\frac{B}{\Delta^2A} \tag{9}$$

where A and B are shown in the Appendix.

Since phase velocity ($=V_{ph}$) is the function of the wave number and frequency ($=\omega/k$), and group velocity ($=V_{gr}$) is a derivative of phase velocity with respect to the wave number ($=d\omega/dk$), normalized phase velocity for the weighted average method is as follow:

$$\frac{V_{ph}}{V_0}=\frac{1}{2\pi G}\sqrt{\frac{-B}{A}} \tag{10}$$

The normalized group velocity V_{gr} for the weighted average method is

$$\frac{V_{gr}}{V_0}=\frac{1}{4\pi G}\frac{V_0}{V_{ph}}\frac{AB'-BA'}{A^2} \tag{11}$$

where A' and B' are the partial derivative of A and B to the wave number and V_0 is the velocity of the medium.

To determine the coefficients for finite difference stars we set the normalized phase velocity and weighted the average coefficients to 1, then eq. (10) is:

$$\frac{V_{ph}}{V_0}=1 \tag{12}$$

$$\sum_{k=1}^{m} r_k=1 \tag{13}$$

$$a_1+\sum_{k=2}^{n}4a_k=1 \tag{14}$$

This yields eq. (15):

$$4\pi^2G^2A+B=0$$

Substitution eq. (13) and eq. (14) into eq. (15) yields

$$4\pi^2G^2A+C=4 \tag{16}$$

where coefficient C is shown in the Appendix. Since A and C are the function of the number of grid points per wave length, equation (16) can be represented in matrix form.

$$\begin{pmatrix} a_{1,1} & \cdots & a_{n_1} & r_{1,1} & \cdots & r_{n,1}\\ a_{1,2} & \cdots & a_{n_2} & r_{1,2} & \cdots & r_{n,2}\\ \vdots & & & & & \vdots\\ a_{1,m-1} & \cdots & a_{n,m-1} & r_{1,m-1} & \cdots & r_{n,m-1}\\ a_{1,m} & \cdots & a_{n,m} & r_{1,m} & \cdots & r_{n,m}\end{pmatrix}\begin{pmatrix}a_1\\ \vdots\\ a_n\\ r_1\\ \vdots\\ r_n\end{pmatrix}=\begin{pmatrix}4\\ \vdots\\ \vdots\\ 4\end{pmatrix} \tag{17}$$

Eq. (17) is an over-determined matrix which is dependent on the number of grid points and propergation angle.

Dispersion Analysis

In order to determine which weighted average method is optimum the normalized phase and group velocity curves for the weighted average methods were calculated. Letting u in eq. (1) be the plane harmonic wave, the normalized phase and group velocity were obtained.

$$\frac{V_{ph}}{v} = \frac{1}{\pi/G}[\sin^2(\pi G\sin\theta) + \sin^2(\pi G\cos\theta)]^{\frac{1}{2}} \tag{18}$$

$$\frac{V_{gr}}{v} = \frac{\sin\theta\sin\theta(2\pi G\sin\theta) + \cos\theta\sin(2\pi G\cos\theta)}{2[\sin^2(\pi G\sin\theta) + \sin^2(\pi G\cos\theta)]^{\frac{1}{2}}} \tag{19}$$

Dispersion curves for the set parameters defined in eq. (18) and (19) are shown in Fig. 2 and 3. Fig. 2 and Fig. 3 show dispersion curves of phase and group velocity to several weighted average methods. From Fig. 2 and 3, the minimum grids per wavelength for each weighted average method were determined (Table 2).

The numerical errors in the phase velocity of less than 1% for accuracy solution 49 points can be archived at 2.7 grid points and for 81 points at 2.5 grid points. In the group velocity of 49 point error to the vertical propagation direc-

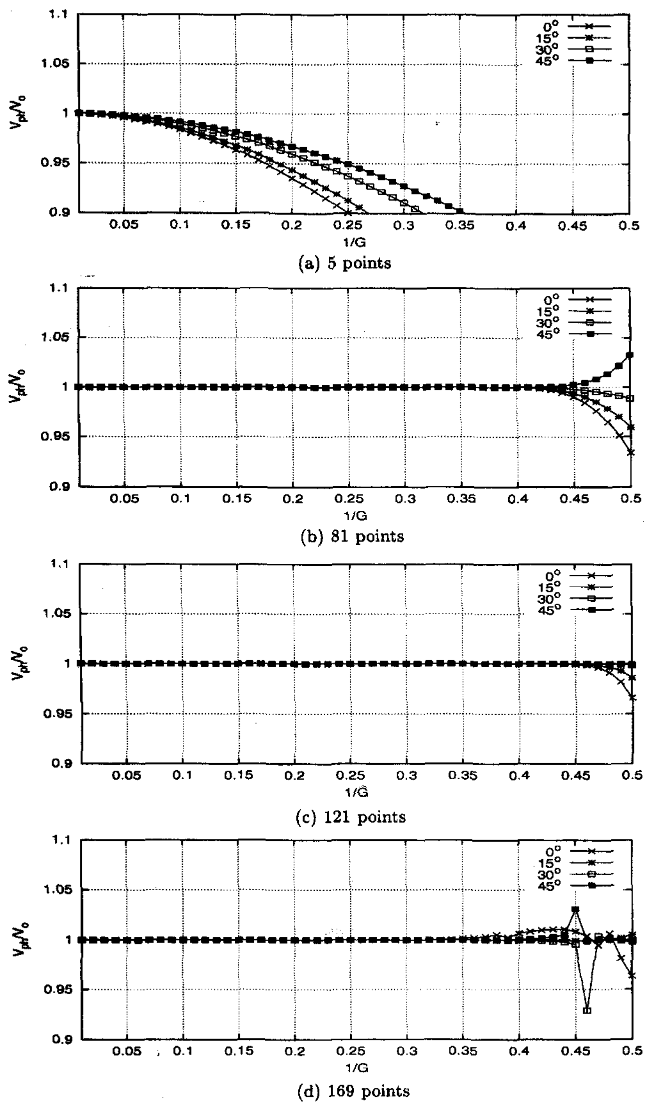

Fig. 2. Normalized phase velocity curves for the finite-difference solution of 2D scalar wave equation in the frequency domain using (a) conventional 5 points finite difference stars, (b) 81 points, (c) 121 points and (d) 169 points.

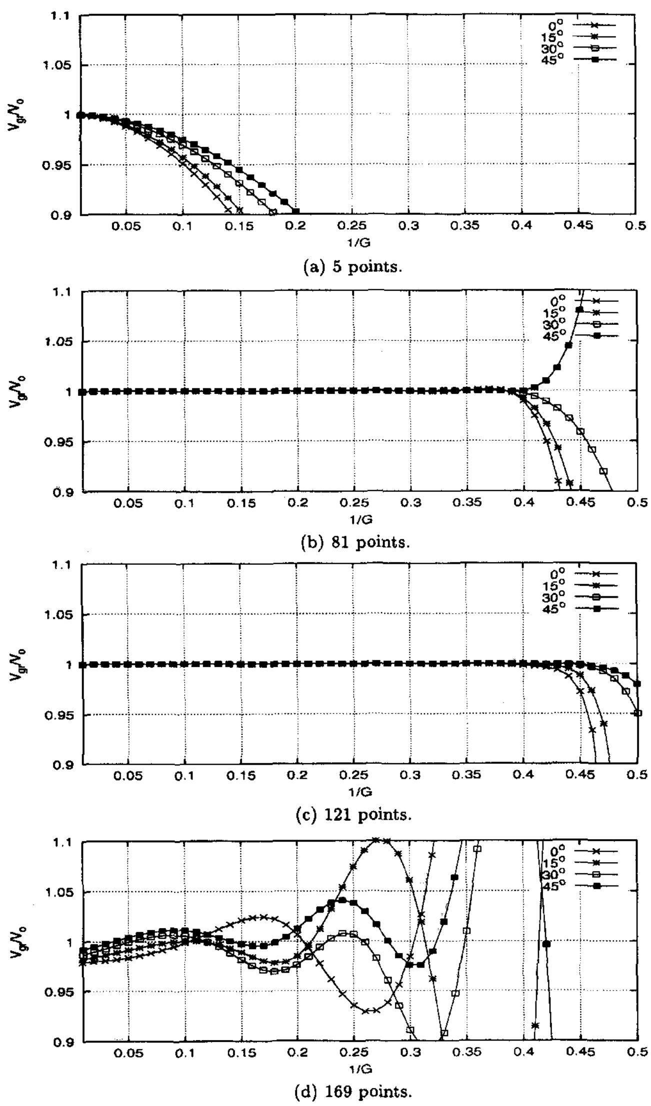

Fig. 3. Normalized group velocity curves for the finite-difference solution of 2D scalar wave equation in the frequency domain using (a) conventional 5 points finite difference star, (b) 81 points, (c) 121 points and (d) 169 points.

Table 2. The number of grids per minimum wave length were required to have less than 1% numerical dispersion error

Diff. stars	5	9	25	49	81	121	169
Grids	13	5	3	2.7	2.5	2.3	N.A

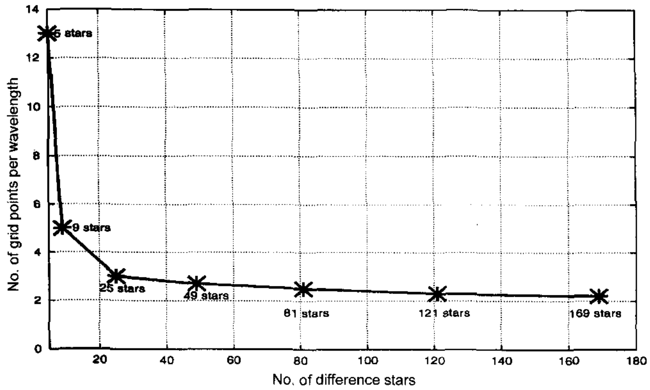

Fig. 4. Relationship between the number of finite difference stars and the grid points per wavelength in phase velocity. It showed that the more finite difference stars, the less grid points per wavelength. However, grids could not be reduced 2 in the frequency domain acoustic wave equation.

tion was more than 3%. However, 81 points had less than 1% numerical errors in both phase and group velocities. In 121 difference stars, only 2.1 grid points were required for accuracy in the phase velocity and 2.3 grid points in the group velocity. However, in the group velocity of 169 points, accurate coefficients were not found. As a result of the dispersion analysis working to 169 points, the more weighted the average points, the less grid points per wavelength (Fig. 4). In the 121 point method, the grid points per wavelength were reduced to 2.3 with an accuracy solution of less than 1% numerical error. This was the minimum grid points per wavelength which were ever studied in the frequency domain finite difference modeling for the acoustic wave equation.

Conclusion

The objective of the study described in this paper was to determine the minimum number of grids per wavelength with accuracy. Dispersion analysis was performed for 81, 121 and 169 difference stars to find out solution for the scalar wave equation in the frequency domain. The conclusions are as follows:

- The new weighted average methods, 81, 121 and 169 points difference stars, were studied. To obtain numerical errors of less than 1% for accuracy solution 81 points 2.5 grid points per wavelength were determined and in 121 points 2.3 grid points resulted. However, in 169 points there was no solution because of the oscillation of the dispersion curves in group velocities.
- The dispersion analysis for the determination of grid points per wavelength showed that the more rotated finite difference operators, the less grid points. However, the more rotated the finite difference operator used, the more complex difference equation terms. In addition it was not possible to reduce grids to 2 in the frequency domain acoustic wave equation.

References

Alford, R. M., Kelly, K. R., and Boor, D. M., 1974, Accuracy of finite difference modeling of the acoustic wave equation: *Geophysics*, **39**, 834-842.

Dablain, M. A., 1986, The application of high-order differencing to the scalar wave equation: *Geophysics*, **51**, 54-66.

Jo, C. H., Shin, C. S., and Suh, J. H., 1996, An optimal 9 point finite difference frequency-space 2D acoustic wave extrapolator: *Geophysics*, **61**, 529-537.

Kelly K. R., Ward, R. W., Treitel Sven, and Alford, R. M., 1974, Synthetic seismograms: A finite difference approach, *Geophysics*, **41**, 2-27.

Kosloff, D. D., and Baysal, E., 1982, Forward modeling by a Fourier method: *Geophysics*, **47**, 1402-1412.

Marfurt, K. J., 1984, Accuracy of finite difference and finite element modeling of the scalar and elastic wave equation, *Geophysics*, **63**, 533-549.

Shin, C. S., and Sohn, H. J., 1998, A frequency space 2-D scalar wave extrapolator using extended 25 points finite difference operators, *Geophysics*, **63**, 289-296.

Štekl, I., and Pratt, R. G., 1998, Accurate viscoelastic modeling by frequency-domain finite difference using rotated operators, *Geophysics*, **63**, 1779-1794.

Appendix

Coefficient A for eq. (9)($\frac{\omega^2}{v^2} = -\frac{B}{\Delta A}$)

$$
\begin{aligned}
A &= a_1 \\
&+ a_2 * (\cos(2\pi G\sin\theta) + \cos(2\pi G\cos\theta)) \\
&+ 4a_3 * \cos(2\pi G\sin\theta) * \cos(2\pi G\cos\theta) \\
&+ 2a_4 * (\cos(2\pi G\cos\theta) + \cos(4\pi G\cos\theta)) \\
&+ 4a_5 * \cos(2\pi G\cos\theta) * \cos(4\pi G\cos\theta) \\
&+ 2a_6 * (\cos(2\pi G\cos\theta) + 2\pi G\cos\theta) + \cos(2\pi G\sin\theta) - 4\pi G\cos\theta)) \\
&+ 2a_7 * (\cos(2\pi G\cos\theta) - 2\pi G\cos\theta) + \cos(2\pi G\sin\theta) + 4\pi G\cos\theta)) \\
&+ 2a_8 * (\cos(6\pi G\sin\theta) + \cos(6\pi G\cos\theta)) \\
&+ 4a_9 * \cos(6\pi G\sin\theta) * \cos(6\pi G\cos\theta) \\
&+ 2a_{10} * (\cos(2\pi G\cos\theta - 6\pi G\cos\theta) + \cos(6\pi G\sin\theta + 4\pi G\cos\theta)) \\
&+ 2a_{11} * (\cos(2\pi G\cos\theta + 6\pi G\cos\theta) + \cos(6\pi G\sin\theta - 4\pi G\cos\theta)) \\
&+ 2a_{12} * (\cos(2\pi G\sin\theta - 6\pi G\cos\theta) + \cos(6\pi G\sin\theta + 2\pi G\cos\theta)) \\
&+ 2a_{13} * (\cos(2\pi G\sin\theta + 6\pi G\cos\theta) + \cos(6\pi G\sin\theta - 2\pi G\cos\theta)) \\
&+ 2a_{14} * (\cos(8\pi G\sin\theta) + \cos(8\pi G\cos\theta)) \\
&+ 4a_{15} * \cos(8\pi G\sin\theta) * \cos(8\pi G\cos\theta) \\
&+ 2a_{16} * (\cos(2\pi G\sin\theta - 8\pi G\cos\theta) + \cos(8\pi G\sin\theta + 2\pi G\cos\theta)) \\
&+ 2a_{17} * (\cos(2\pi G\cos\theta - 8\pi G\cos\theta) + \cos(8\pi G\sin\theta + 4\pi G\cos\theta)) \\
&+ 2a_{18} * (\cos(6\pi G\sin\theta - 8\pi G\cos\theta) + \cos(8\pi G\sin\theta + 6\pi G\cos\theta)) \\
&+ 2a_{19} * (\cos(8\pi G\sin\theta - 6\pi G\cos\theta) + \cos(6\pi G\sin\theta + 8\pi G\cos\theta)) \\
&+ 2a_{20} * (\cos(8\pi G\sin\theta - 4\pi G\cos\theta) + \cos(2\pi G\cos\theta + 8\pi G\cos\theta)) \\
&+ 2a_{21} * (\cos(8\pi G\sin\theta - 2\pi G\cos\theta) + \cos(2\pi G\sin\theta + 8\pi G\cos\theta)) \\
&+ 2a_{22} * (\cos(10\pi G\sin\theta) + \cos(10\pi G\cos\theta)) \\
&+ 4a_{23} * \cos(10\pi G\sin\theta) * \cos(10\pi G\cos\theta) \\
&+ 2a_{24} * (\cos(2\pi G\sin\theta - 10\pi G\cos\theta) + \cos(10\pi G\sin\theta + 2\pi G\cos\theta)) \\
&+ 2a_{25} * (\cos(2\pi G\cos\theta - 10\pi G\cos\theta) + \cos(10\pi G\sin\theta + 4\pi G\cos\theta)) \\
&+ 2a_{26} * (\cos(6\pi G\sin\theta - 10\pi G\cos\theta) + \cos(10\pi G\sin\theta + 6\pi G\cos\theta)) \\
&+ 2a_{27} * (\cos(8\pi G\sin\theta - 10\pi G\cos\theta) + \cos(10\pi G\sin\theta + 8\pi G\cos\theta)) \\
&+ 2a_{28} * (\cos(10\pi G\sin\theta - 8\pi G\cos\theta) + \cos(8\pi G\sin\theta + 10\pi G\cos\theta)) \\
&+ 2a_{29} * (\cos(10\pi G\sin\theta - 6\pi G\cos\theta) + \cos(6\pi G\sin\theta + 10\pi G\cos\theta)) \\
&+ 2a_{30} * (\cos(10\pi G\sin\theta - 4\pi G\cos\theta) + \cos(2\pi G\cos\theta + 10\pi G\cos\theta)) \\
&+ 2a_{31} * (\cos(10\pi G\sin\theta - 2\pi G\cos\theta) + \cos(2\pi G\sin\theta + 10\pi G\cos\theta)) \\
&+ 2a_{32} * (\cos(12\pi G\sin\theta) + \cos(12\pi G\cos\theta)) \\
&+ 4a_{33} * \cos(12\pi G\sin\theta) * \cos(12\pi G\cos\theta) \\
&+ 2a_{34} * (\cos(2\pi G\sin\theta - 12\pi G\cos\theta) + \cos(12\pi G\sin\theta + 2\pi G\cos\theta)) \\
&+ 2a_{35} * (\cos(2\pi G\cos\theta - 12\pi G\cos\theta) + \cos(12\pi G\sin\theta + 4\pi G\cos\theta)) \\
&+ 2a_{36} * (\cos(6\pi G\sin\theta - 12\pi G\cos\theta) + \cos(12\pi G\sin\theta + 6\pi G\cos\theta)) \\
&+ 2a_{37} * (\cos(8\pi G\sin\theta - 12\pi G\cos\theta) + \cos(12\pi G\sin\theta + 8\pi G\cos\theta)) \\
&+ 2a_{38} * (\cos(10\pi G\sin\theta - 12\pi G\cos\theta) + \cos(12\pi G\sin\theta + 10\pi G\cos\theta)) \\
&+ 2a_{39} * (\cos(12\pi G\sin\theta - 10\pi G\cos\theta) + \cos(10\pi G\sin\theta + 12\pi G\cos\theta)) \\
&+ 2a_{40} * (\cos(12\pi G\sin\theta - 8\pi G\cos\theta) + \cos(8\pi G\sin\theta + 12\pi G\cos\theta)) \\
&+ 2a_{41} * (\cos(12\pi G\sin\theta - 6\pi G\cos\theta) + \cos(6\pi G\sin\theta + 12\pi G\cos\theta)) \\
&+ 2a_{42} * (\cos(12\pi G\sin\theta - 4\pi G\cos\theta) + \cos(2\pi G\cos\theta + 12\pi G\cos\theta)) \\
&+ 2a_{43} * (\cos(12\pi G\sin\theta - 2\pi G\cos\theta) + \cos(2\pi G\sin\theta + 12\pi G\cos\theta))
\end{aligned}
$$

Coefficient B for eq. (9)($\frac{\omega^2}{v^2} = -\frac{B}{\Delta A}$)

$$
\begin{aligned}
B &= 2.r_1 * (\cos(2\pi G\sin\theta) + \cos(2\pi G\cos\theta) - 4) \\
&+ 2.r_2 * \cos(2\pi G\sin\theta) * \cos(2\pi G\cos\theta) - 2 \\
&+ r_3/4 * (\cos(4\pi G\sin\theta) + \cos(4\pi G\cos\theta) - 4)
\end{aligned}
$$

$$
\begin{aligned}
&+ \; r_4/2 * cos(4\pi G sin\theta) * cos(4\pi G cos\theta) - 4) \\
&+ \; r_5/5 * (cos(4\pi G sin\theta - 2\pi G cos\theta) + cos(2\pi G sin\theta + 4\pi G cos\theta) - 4) \\
&+ \; r_6/5 * (cos(4\pi G sin\theta + 2\pi G cos\theta) + cos(2\pi G sin\theta - 4\pi G cos\theta) - 4) \\
&+ \; r_7/9 * (cos(6\pi G sin\theta) + cos(6\pi G cos\theta) - 4) \\
&+ \; r_8/18 * cos(6\pi G sin\theta) * cos(6\pi G cos\theta) - 4) \\
&+ \; r_9/13 * (cos(4\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 4\pi G cos\theta) - 4) \\
&+ \; r_{10}/13 * (cos(4\pi G sin\theta + 6\pi G cos\theta) + cos(6\pi G sin\theta - 4\pi G cos\theta) - 4) \\
&+ \; r_{11}/10 * (cos(2\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 2\pi G cos\theta) - 4) \\
&+ \; r_{12}/10(cos(2\pi G sin\theta + 6\pi G cos\theta) + cos(6\pi G sin\theta - 2\pi G cos\theta) - 4) \\
&+ \; r_{13}/16 * (cos(8\pi G sin\theta) + cos(8\pi G cos\theta) - 4) \\
&+ \; r_{14}/32 * cos(8\pi G sin\theta) * cos(8\pi G cos\theta) - 4) \\
&+ \; r_{15}/17 * (cos(2\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 2\pi G cos\theta) - 4) \\
&+ \; r_{16}/20 * (cos(4\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 4\pi G cos\theta) - 4) \\
&+ \; r_{17}/25 * (cos(6\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 6\pi G cos\theta) - 4) \\
&+ \; r_{18}/25 * (cos(8\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 8\pi G cos\theta) - 4) \\
&+ \; r_{19}/20 * (cos(8\pi G sin\theta - 4\pi G cos\theta) + cos(4\pi G sin\theta + 8\pi G cos\theta) - 4) \\
&+ \; r_{20}/17 * (cos(8\pi G sin\theta - 2\pi G cos\theta) + cos(2\pi G sin\theta + 8\pi G cos\theta) - 4) \\
&+ \; r_{21}/25 * (cos(10\pi G sin\theta) + cos(10\pi G cos\theta) - 4) \\
&+ \; r_{22}/50 * cos(10\pi G sin\theta) * cos(10\pi G cos\theta) - 4) \\
&+ \; r_{23}/26 * (cos(2\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 2\pi G cos\theta) - 4) \\
&+ \; r_{24}/29 * (cos(4\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 4\pi G cos\theta) - 4) \\
&+ \; r_{25}/34 * (cos(6\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 6\pi G cos\theta) - 4) \\
&+ \; r_{26}/41 * (cos(8\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 8\pi G cos\theta) - 4) \\
&+ \; r_{27}/41. * (cos(10\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 10\pi G cos\theta) - 4) \\
&+ \; r_{28}/34 * (cos(10\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 10\pi G cos\theta) - 4) \\
&+ \; r_{29}/29 * (cos(10\pi G sin\theta - 4\pi G cos\theta) + cos(4\pi G sin\theta + 10\pi G cos\theta) - 4) \\
&+ \; r_{30}/26 * (cos(10\pi G sin\theta - 2\pi G cos\theta) + cos(2\pi G sin\theta + 10\pi G cos\theta) - 4) \\
&+ \; r_{31}/36 * (cos(12\pi G sin\theta) + cos(12\pi G cos\theta) - 4) \\
&+ \; r_{32}/72 * cos(12\pi G sin\theta) * cos(12\pi G cos\theta) - 4) \\
&+ \; r_{33}/37 * (cos(2\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 2\pi G cos\theta) - 4) \\
&+ \; r_{34}/40 * (cos(4\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 4\pi G cos\theta) - 4) \\
&+ \; r_{35}/45 * (cos(6\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 6\pi G cos\theta) - 4) \\
&+ \; r_{36}/52 * (cos(8\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 8\pi G cos\theta) - 4) \\
&+ \; r_{37}/61 * (cos(10\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 10\pi G cos\theta) - 4) \\
&+ \; r_{38}/61 * (cos(12\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 12\pi G cos\theta) - 4) \\
&+ \; r_{39}/52 * (cos(12\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 12\pi G cos\theta) - 4) \\
&+ \; r_{40}/45 * (cos(12\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 12\pi G cos\theta) - 4) \\
&+ \; r_{41}/40 * (cos(12\pi G sin\theta - 4\pi G cos\theta) + cos(4\pi G sin\theta + 12\pi G cos\theta) - 4) \\
&+ \; r_{42}/37 * (cos(12\pi G sin\theta - 2\pi G cos\theta) + cos(2\pi G sin\theta + 12\pi G cos\theta) - 4)
\end{aligned}
$$

Coefficient A for eq. (20)($4\pi^2 G^2 A + C = 4$)

$$
\begin{aligned}
C \;=&\; r_1(2. * (cos(2\pi G sin\theta) + cos(2\pi G cos\theta))) \\
+&\; r_2(2. + 2. * cos(2\pi G sin\theta) * cos(2\pi G cos\theta)) \\
+&\; r_3(3. + .5 * (cos(4\pi G sin\theta) + cos(4\pi G cos\theta))) \\
+&\; r_4(3.5 + .5 * cos(4\pi G sin\theta) * cos(4\pi G cos\theta)) \\
+&\; r_5(16./5. + .4 * (cos(4\pi G sin\theta - 2\pi G cos\theta) + cos(2\pi G sin\theta + 4\pi G cos\theta))) \\
+&\; r_6(16./5. + .4 * (cos(4\pi G sin\theta + 2\pi G cos\theta) + cos(2\pi G sin\theta - 4\pi G cos\theta))) \\
+&\; r_7(32./9. + 2./9. * (cos(6\pi G sin\theta) + cos(6\pi G cos\theta))) \\
+&\; r_8(68./18. + 4./18. * cos(6\pi G sin\theta) * cos(6\pi G cos\theta)) \\
+&\; r_9(48./13. + 2./13. * (cos(4\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 4\pi G cos\theta))) \\
+&\; r_{10}(48./13. + 2./13. * (cos(4\pi G sin\theta + 6\pi G cos\theta) + cos(6\pi G sin\theta - 4\pi G cos\theta))) \\
+&\; r_{11}(36./10. + .2 * (cos(2\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 2\pi G cos\theta))) \\
+&\; r_{12}(36./10. + .2 * (cos(2\pi G sin\theta + 6\pi G cos\theta) + cos(6\pi G sin\theta - 2\pi G cos\theta))) \\
+&\; r_{13}(60./16. + 2./16. * (cos(8\pi G sin\theta) + cos(8\pi G cos\theta)))
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad r_{14}(124./32. + 4./32. * cos(8\pi G sin\theta) * cos(8\pi G cos\theta)) \\
&+ \quad r_{15}(64./17. + 2./17 * (cos(2\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 2\pi G cos\theta))) \\
&+ \quad r_{16}(76./20. + 2./20. * (cos(4\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 4\pi G cos\theta))) \\
&+ \quad r_{17}(96./25. + 2./25. * (cos(6\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 6\pi G cos\theta))) \\
&+ \quad r_{18}(96./25. + 2./25. * (cos(8\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 8\pi G cos\theta))) \\
&+ \quad r_{19}(76./20. + 2./20. * (cos(8\pi G sin\theta - 4\pi G cos\theta) + cos(4\pi G sin\theta + 8\pi G cos\theta))) \\
&+ \quad r_{20}(64./17. + 2./17. * (cos(8\pi G sin\theta - 2\pi G cos\theta) + cos(2\pi G sin\theta + 8\pi G cos\theta))) \\
&+ \quad r_{21}(96./25. + 2./25. * (cos(10\pi G sin\theta) + cos(10\pi G cos\theta))) \\
&+ \quad r_{22}(196./50. + 4./50. * cos(10\pi G sin\theta) * cos(10\pi G cos\theta)) \\
&+ \quad r_{23}(100./26. + 2./26 * (cos(2\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 2\pi G cos\theta))) \\
&+ \quad r_{24}(112./29. + 2./29. * (cos(4\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 4\pi G cos\theta))) \\
&+ \quad r_{25}(132./34. + 2./34. * (cos(6\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 6\pi G cos\theta))) \\
&+ \quad r_{26}(160./41. + 2./41. * (cos(8\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 8\pi G cos\theta))) \\
&+ \quad r_{27}(160./41. + 2./41. * (cos(10\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 10\pi G cos\theta))) \\
&+ \quad r_{28}(132./34. + 2./34. * (cos(10\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 10\pi G cos\theta))) \\
&+ \quad r_{29}(112./29. + 2./29. * (cos(10\pi G sin\theta - 4\pi G cos\theta) + cos(4\pi G sin\theta + 10\pi G cos\theta))) \\
&+ \quad r_{30}(100./26. + 2./26. * (cos(10\pi G sin\theta - 2\pi G cos\theta) + cos(2\pi G sin\theta + 10\pi G cos\theta))) \\
&+ \quad r_{31}(140./36. + 2./36. * (cos(12\pi G sin\theta) + cos(12\pi G cos\theta))) \\
&+ \quad r_{32}(284./72. + 4./72. * cos(12\pi G sin\theta) * cos(12\pi G cos\theta)) \\
&+ \quad r_{33}(144./37. + 2./37. * (cos(2\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 2\pi G cos\theta))) \\
&+ \quad r_{34}(156./40. + 2./40. * (cos(4\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 4\pi G cos\theta))) \\
&+ \quad r_{35}(176./45. + 2./45. * (cos(6\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 6\pi G cos\theta))) \\
&+ \quad r_{36}(204./52. + 2./52. * (cos(8\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 8\pi G cos\theta))) \\
&+ \quad r_{37}(240./61. + 2./61. * (cos(10\pi G sin\theta - 12\pi G cos\theta) + cos(12\pi G sin\theta + 10\pi G cos\theta))) \\
&+ \quad r_{38}(240./61. + 2./61. * (cos(12\pi G sin\theta - 10\pi G cos\theta) + cos(10\pi G sin\theta + 12\pi G cos\theta))) \\
&+ \quad r_{39}(204./52. + 2./52. * (cos(12\pi G sin\theta - 8\pi G cos\theta) + cos(8\pi G sin\theta + 12\pi G cos\theta))) \\
&+ \quad r_{40}(176./45. + 2./45. * (cos(12\pi G sin\theta - 6\pi G cos\theta) + cos(6\pi G sin\theta + 12\pi G cos\theta))) \\
&+ \quad r_{41}(156./40. + 2./40. * (cos(12\pi G sin\theta - 4\pi G cos\theta) + cos(4\pi G sin\theta + 12\pi G cos\theta))) \\
&+ \quad r_{42}(144./37. + 2./37. * (cos(12\pi G sin\theta - 2\pi G cos\theta) + cos(2\pi G sin\theta + 12\pi G cos\theta)))
\end{aligned}
$$