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AN AXISYMMETRIC, NONSTATIONARY BLACK HOLE MAGNETOSPHERE

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ABSTRACT

In the earlier papers we analyzed the axisymmetric, nonstationary electrodynamics of the central black hole and a surrounding thin accretion disk in an active galactic nucleus. In this paper we analyze the axisymmetric, nonstationary electrodynamics of the black hole magnetosphere in a similar way. In the earlier papers we employed the poloidal component of the plasma velocity which is confined only to the radial direction of the cylindrical coordinate system. In this paper we employ a more general poloidal velocity and get the Grad-Shafranov equation of the force-free magnetosphere of a Kerr black hole. The equation is consistent with the previous ones and is more general in many aspects as it should be. We also show in more general approaches that the angular velocity of the magnetic field lines anchored on the accreting matter tends to become close to that of the black hole at the equatorial zone of the hole.

Key Words: accretion – black holes – galaxies : nuclei – radiation mechanism

I. INTRODUCTION

Most astrophysicists believe that central engines in active galactic nuclei are related to supermassive black holes. An axisymmetric, stationary electrodynamic engine model was well formulated by Macdonald and Thorne (1982, hereafter MT), which consists of the supermassive black hole surrounded by a magnetized accretion disk.

The magnetosphere of the MT model can be divided into three regions ($\mathbf{E}$, $\mathbf{B}$, ρ_e, and $\mathbf{j}$ have their usual definitions in electrodynamics):

1) *degenerate region*; the event horizon of the black hole and the surface of the magnetized accretion disk are degenerate,

$$\mathbf{E}\cdot\mathbf{B}=0. \tag{1.1a}$$

2) *force-free region*; the zones closest to the black hole are force-free,

$$\mathbf{E}\cdot\mathbf{B}=0 \;\text{ and }\; \rho_e\mathbf{E}+\mathbf{j}\times\mathbf{B}=0. \tag{1.1b}$$

3) *acceleration region*; the zones farther from the black hole are called the acceleration region.

These regions must be filled by highly conducting plasma and in the force-free region, in particular, the equations of magnetohydrodynamics will be well satisfied because condition (1.1b) guarantees infinite conductivity of plasma.

Based on this model we have investigated an axisymmetric, nonstationary model of the central engine of an active galactic nucleus. In the earlier papers (Park and Vishniac 1989a, paper I; Park and Vishniac 1989b, paper II) we investigated the electrodynamics of a black hole and an accretion disk.

In paper I we derived all the fundamental equations in a fully time-dependent manner and investigated the electrodynamics of a black hole. Under the assumption that the mass accretion is confined to the equatorial plane of the black hole our results suggested that, at the equatorial zone of the hole, the angular velocity of the magnetic field lines anchored on the accreting matter must be close to that of the black hole.

In paper II we analyzed the axisymmetric, nonstationary electrodynamics of a surrounding magnetized accretion disk. We found that the power output due to the Blandford-Znajek process can be variable even on short time scales. This may explain the observed short time scale variability of active galactic nuclei.

MT employed the plasma velocity which had only the toroidal component simply because their goal was to establish an axisymmetric, stationary model. This is the reason why the stream equation of the MT model (MT, eq. [6.4]) is too simple to describe astrophysical poloidal phenomena like electrodynamic jets. In paper I and paper II we employed the poloidal component of the plasma which is confined only to the radial direction of the cylindrical coordinate system. If we consider a black hole and the surrounding thin accretion disk only, this is not a bad approach at all, but this cannot fully describe poloidal phenomena, either.

In this paper we analyze the force-free black hole magnetosphere and we employ the poloidal component of the plasma which is not necessarily confined only to the radial direction of the cylindrical coordinate system. We will, therefore, get some more general results which must be related to some poloidal phenomena.

Following paper I and paper II we will summarize the equations of axisymmetric, nonstationary electrodynamics in Section II. The force-free condition will be introduced in Section III. Finally, in Section IV, we will derive the Grad-Shfranov equation of the force-free magnetosphere of a Kerr black hole and the power output equation.

Throughout this paper we define our units such that $c = G \equiv 1$, and the central black hole is assumed to be a Kerr black hole which possesses the total mass M, the angular momentum J, and the angular momentum per unit mass $a(\equiv J/M)$.

II. AXISYMMETRIC, NONSTATIONARY ELECTRODYNAMICS

In this section we will describe the electrodynamics of an axisymmetric, nonstationary accretion disk. Axisymmetric, nonstationary conditions can be represented as (paper I, eq. [3.1]; paper II, eq. [2.1]),

$$\mathbf{m} \cdot \nabla f \equiv 0, \quad \mathcal{L}_{\mathbf{m}} \mathbf{f} \equiv \mathbf{0} \tag{2.1a}$$

and

$$\frac{\partial f}{\partial t} \equiv \dot{f} \neq 0, \quad \frac{\partial \mathbf{f}}{\partial t} \equiv \dot{\mathbf{f}} \neq \mathbf{0}, \tag{2.1b}$$

where $\mathbf{m}$ is a Killing vector of the axisymmety, $\mathcal{L}$ means the Lie derivative, and f and $\mathbf{f}$ are any scalar and vector, respectively.

To describe the spherically symmetric spacetime we use the spherical coordinate system (r, θ, φ) whose unit vectors are expressed as $\mathbf{e}_{\hat{r}}$, $\mathbf{e}_{\hat{\theta}}$, and $\mathbf{e}_{\hat{\varphi}}$, respectively ($\mathbf{e}_{\hat{r}} \times \mathbf{e}_{\hat{\theta}} = \mathbf{e}_{\hat{\varphi}}$). We also use the cylindrical coordinate system (R, φ, z) with the unit vectors $\mathbf{e}_{\hat{R}}$, $\mathbf{e}_{\hat{\varphi}}$, and $\mathbf{e}_{\hat{z}}$ ($\mathbf{e}_{\hat{R}} \times \mathbf{e}_{\hat{\varphi}} = \mathbf{e}_{\hat{z}}$) to describe the axisymmetry of the magnetosphere.

Throughout this paper $\mathbf{m}$ has the same magnitude as $\tilde{\omega}$, the separation between the symmetric axis of the black hole and a Zero-Augular-Momentum-Observer (ZAMO; see MT)

$$\tilde{\omega} \equiv \frac{\Sigma}{\rho} \sin\theta, \tag{2.2a}$$

where

$$\rho^2 \equiv r^2 + a^2 \cos^2\theta \tag{2.2b}$$

and

$$\Sigma^2 \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta, \tag{2.2c}$$

with

$$\Delta \equiv r^2 + a^2 - 2Mr. \tag{2.2d}$$

(a) Outside the Horizon

Let ∂A be an $\mathbf{m}$-loop, A be any surface bounded by ∂A but not intersecting the event horizon of the black hole, and $d\boldsymbol{\Sigma}$ be the normal vector on an infinitesimal area on A. Then we can define the total electric current passing downward through A, $I(t, \mathbf{x})$, the total magnetic flux passing upward through A, $\Psi(t, \mathbf{x})$, and the total electric flux passing upward through A, $\Phi(t, \mathbf{x})$, as (paper I, eq. [3.2]; paper II, eq. [2.3]),

$$I(t, \mathbf{x}) \equiv -\int_A \alpha \mathbf{j} \cdot d\boldsymbol{\Sigma}, \tag{2.3a}$$

$$\Psi(t, \mathbf{x}) \equiv \int_A \mathbf{B} \cdot d\mathbf{\Sigma}, \tag{2.3b}$$

and

$$\Phi(t, \mathbf{x}) \equiv \int_A \mathbf{E} \cdot d\mathbf{\Sigma}, \tag{2.3c}$$

where α is the lapse function of the ZAMO. The value of α is given by

$$\alpha = \frac{\rho}{\Sigma}\sqrt{\Delta}. \tag{2.4}$$

In terms of these the electromagnetic fields described by the ZAMO are given by (paper I, eqs. [3.3], [3.5], and [3.6]; paper II, eq. [2.5]),

$$\mathbf{E}^T = -\frac{2}{\alpha\tilde{\omega}}\left(\frac{\dot{\Psi}}{4\pi}\right)\mathbf{e}_{\hat{\varphi}}, \tag{2.5a}$$

$$\mathbf{E}^P = \mathbf{E} - \mathbf{E}^T, \tag{2.5b}$$

$$\mathbf{B}^T = -\frac{2}{\alpha\tilde{\omega}}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{e}_{\hat{\varphi}}, \tag{2.5c}$$

and

$$\mathbf{B}^P = -\frac{\mathbf{e}_{\hat{\varphi}} \times \nabla\Psi}{2\pi\tilde{\omega}}, \tag{2.5d}$$

where T, P denote the toroidal and poloidal components respectively.

Let $\mathbf{S}_J$ and $\mathbf{S}_E$ be the Poynting vector of the angular momentum flow and the energy flow, respectively. In this case they are given by (paper I, eq. [3.7]; paper II, eq. [2.6]),

$$\mathbf{S}_J^P = \frac{1}{2\pi\alpha}\left\{\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{B}^P + \left(\frac{\dot{\Psi}}{4\pi}\right)\mathbf{E}^P\right\} \tag{2.6a}$$

and

$$\mathbf{S}_E^P = \frac{\alpha}{4\pi}(\mathbf{E} \times \mathbf{B})^P + \omega\mathbf{S}_J^P, \tag{2.6b}$$

where ω is the ZAMO's angular velocity,

$$\omega \equiv \frac{2aMr}{\Sigma^2}. \tag{2.6c}$$

If we fix the observer as the ZAMO at the given point around a Kerr black hole, Maxwell equations satisfy the condition (2.1) are given by (paper I, eq. [2.9]),

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e, \tag{2.7a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.7b}$$

$$\nabla \times (\alpha\mathbf{E}) = -[\dot{\mathbf{B}} - (\mathbf{B} \cdot \nabla\omega)\mathbf{m}], \tag{2.7c}$$

and

$$\nabla \times (\alpha\mathbf{B}) = \dot{\mathbf{E}} - (\mathbf{E} \cdot \nabla\omega)\mathbf{m} + 4\pi\alpha\mathbf{j}. \tag{2.7d}$$

(b) Boundary Conditions at the Horizon

The event horizon exists where $\alpha = 0$, i.e., $\Delta = 0$ in (2.4). Here $\mathbf{e}_{\hat{\lambda}}$, the unit vector of the poloidal (from the north pole toward the equator) direction, and $\mathbf{e}_{\hat{\varphi}}$ form the local orthonormal frame with the unit vector perpendicular to the horizon, $\mathbf{n}$ as $\mathbf{e}_{\hat{\lambda}} \times \mathbf{e}_{\hat{\varphi}} = \mathbf{n}$. In black hole electrodynamics the horizon behaves as if endowed with surface charge σ^H, electric current $\mathbf{j}^H$, resistance R^H, electric field $\mathbf{E}^H$, magnetic field $\mathbf{B}^H$, (Hawking) temperature T^H, entropy S^H, and so on. At the horizon, the electromagnetic fields must satisfy (paper I, eq. [2.10]),

$$\mathbf{E} \cdot \mathbf{n}(\equiv E_\perp) \to 4\pi\sigma^H, \tag{2.8a}$$

$$\alpha \mathbf{B}_{\parallel} \to \mathbf{B}^H = 4\pi \mathbf{j}^H \times \mathbf{n}, \tag{2.8b}$$

$$\alpha \mathbf{E}_{\parallel} \to \mathbf{E}^H = R^H \mathbf{j}^H = 4\pi \mathbf{j}^H, \tag{2.8c}$$

and

$$\mathbf{E}^H = \mathbf{n} \times \mathbf{B}^H, \tag{2.8d}$$

where $\perp$, $\parallel$ mean perpendicular, parallel to the horizon, respectively.

At the horizon, we also have

$$\alpha \to 0, \quad \omega \to \Omega^H, \tag{2.9}$$

where Ω^H is the angular velocity of the black hole. From equations (2.5c), (2.5d), (2.8b), and (2.9), we get (paper I, eq. [3.9])

$$\mathbf{B}^H = -\frac{2}{\tilde{\omega}}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{e}_{\hat{\varphi}} - \frac{1}{2\pi\tilde{\omega}}(\alpha \mathbf{n} \cdot \nabla\Psi)\mathbf{e}_{\hat{\lambda}}, \tag{2.10}$$

where the second term does not vanish for $\alpha \to 0$ in this case. We also have (paper I, eq. [3.10])

$$\mathbf{B} \cdot \mathbf{n} (\equiv B_{\perp}) = \frac{1}{2\pi\tilde{\omega}} \mathbf{e}_{\hat{\lambda}} \cdot \nabla\Psi, \tag{2.11a}$$

$$\mathbf{E}^H = \frac{2}{\tilde{\omega}}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{e}_{\hat{\lambda}} - \frac{1}{2\pi\tilde{\omega}}(\alpha \mathbf{n} \cdot \nabla\Psi)\mathbf{e}_{\hat{\varphi}}, \tag{2.11b}$$

and

$$\mathbf{j}^H = \frac{1}{4\pi}\left[\frac{2}{\tilde{\omega}}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{e}_{\hat{\lambda}} - \frac{1}{2\pi\tilde{\omega}}(\alpha \mathbf{n} \cdot \nabla\Psi)\mathbf{e}_{\hat{\varphi}}\right]. \tag{2.11c}$$

Using equations (2.8a), (2.11b), and (2.11c), we get the torque per unit area, entropy increase by Joule heating, and the flux of redshifted energy (paper I, eq. [3.11])

$$-\alpha \mathbf{S}_J \cdot \mathbf{n} \to \frac{\Delta \dot{J}_-}{\Delta\Sigma} = -\frac{B_\perp}{2\pi}\left(I - \frac{\dot{\Phi}}{4\pi}\right) - \frac{E_\perp}{8\pi^2}(\alpha \mathbf{n} \cdot \nabla\Psi), \tag{2.12a}$$

$$T^H \frac{\Delta \dot{S}^H}{\Delta\Sigma} \to \frac{1}{\pi\tilde{\omega}^2}\left[\left(I - \frac{\dot{\Phi}}{4\pi}\right)^2 + \frac{1}{16\pi^2}(\alpha \mathbf{n} \cdot \nabla\Psi)^2\right], \tag{2.12b}$$

and

$$-\alpha \mathbf{S}_E \cdot \mathbf{n} \to \frac{\Delta \dot{M}_-}{\Delta\Sigma}\left(\equiv -\frac{\Delta P}{\Delta\Sigma}\right) = \frac{1}{\pi}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\left[\frac{1}{\tilde{\omega}^2}\left(I - \frac{\dot{\Phi}}{4\pi}\right) - \frac{\Omega^H B_\perp}{2}\right]$$
$$+ \frac{1}{4\pi^2}(\alpha \mathbf{n} \cdot \nabla\Psi)\left[\frac{1}{4\pi\tilde{\omega}^2}(\alpha \mathbf{n} \cdot \nabla\Psi) - 2\Omega^H E_\perp\right], \tag{2.12c}$$

where $\Delta\Sigma$ is the area of the tube at the horizon, $\dot{J}_-$ and $\dot{M}_-$ are the rate of the angular momentum loss and the mass loss of the black hole, respectively (see figure 1 in Park and Vishniac 1988), and P is the power output due to the mass extraction.

III. FORCE-FREE MAGNETOSPHERE

As mentioned in the introduction MT employed the plasma velocity (MT, eq. [5.2] and [5.3]),

$$\mathbf{v}^F \equiv -\frac{\omega - \Omega^F}{\alpha}\mathbf{m}, \tag{3.1a}$$

which has only the toroidal component and satisfies the relation

$$\mathbf{E}^P = -(\mathbf{v}^F) \times \mathbf{B}^P, \tag{3.1b}$$

where Ω^F is the angular velocity of the magnetic field lines. Notice that the field lines are anchored on the conducting plasma and they are assumed to move together.

In paper I and paper II we defined the velocity as (paper I, eq. [4.2])

$$\mathbf{v}^F \equiv -\frac{\omega - \Omega^F}{\alpha}\mathbf{m} + v(t, \mathbf{x})_{inf}\mathbf{e}_{\hat{R}}, \tag{3.2a}$$

such that

$$\mathbf{E}^P = -(\mathbf{v}^F)^T \times \mathbf{B}^P - (\mathbf{v}^F)^P \times \mathbf{B}^T \tag{3.2b}$$

and

$$\mathbf{E}^T = -(\mathbf{v}^F)^P \times \mathbf{B}^P, \tag{3.2c}$$

where v_{inf} is the radial-infall velocity of the magnetic field lines.

In this paper we consider $\mathbf{v}$ (without the suffix F)

$$\mathbf{v} \equiv -\frac{\omega - \Omega^F}{\alpha}\mathbf{m} + \mathbf{v}^P, \tag{3.3a}$$

such that

$$\mathbf{E}^P = -\mathbf{v}^T \times \mathbf{B}^P - \mathbf{v}^P \times \mathbf{B}^T \tag{3.3b}$$

and

$$\mathbf{E}^T = -\mathbf{v}^P \times \mathbf{B}^P. \tag{3.3c}$$

Notice here that $\mathbf{v}^P$ is not confined only to the radial direction of the cylindrical coordinate system.

(a) Outside the Horizon

Now, from equations (3.3*a*) and (3.3*b*), we get the poloidal component of the electric field (see paper I, eq. [4.3]),

$$\mathbf{E}^P = \frac{\omega - \Omega^F}{2\pi\alpha}\nabla\Psi + \frac{2}{\alpha\tilde{\omega}^2}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{v}^P \times \mathbf{m}, \tag{3.4}$$

which was not specified in equation (2.5*b*).

¿From equation (3.3*c*), we get the toroidal component of the electric field

$$\mathbf{E}^T = \frac{1}{2\pi\tilde{\omega}^2}(\mathbf{v}^P \cdot \nabla\Psi)\mathbf{m}, \tag{3.5a}$$

which, setting equal to equation (2.5*a*), gives a useful relation such that

$$\alpha\mathbf{v}^P \cdot \nabla\Psi = -\dot{\Psi}. \tag{3.5b}$$

Substituting equations (3.4) and (3.5*a*) into equations (2.6*a*) and (2.6*b*), we get

$$\mathbf{S}_J^P = \frac{1}{2\pi\alpha}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{B}^P + \frac{1}{2\pi\alpha}\left(\frac{\dot{\Psi}}{4\pi}\right)\left\{\frac{\omega - \Omega^F}{2\pi}\nabla\Psi + \frac{2}{\alpha\tilde{\omega}^2}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{v}^P \times \mathbf{m}\right\}, \tag{3.6a}$$

and

$$\mathbf{S}_E^P = \frac{\Omega^F}{2\pi\alpha}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{B}^P + \frac{1}{\pi\alpha\tilde{\omega}^2}\left(I - \frac{\dot{\Phi}}{4\pi}\right)^2\mathbf{v}^P$$

$$+\frac{\omega}{2\pi\alpha}\left(\frac{\dot{\Psi}}{4\pi}\right)\left\{\frac{\omega - \Omega^F}{2\pi}\nabla\Psi + \frac{2}{\alpha\tilde{\omega}^2}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{v}^P \times \mathbf{m} + \frac{\alpha}{\omega\tilde{\omega}^2}\mathbf{B}^P \times \mathbf{m}\right\}. \tag{3.6b}$$

The time-component of the 4-vector potential $(A_0, \mathbf{A})$ is given by (paper I, eq. [3.4])

$$\nabla A_0 = \alpha \mathbf{E} + \dot{\mathbf{A}} - \frac{\omega}{2\pi}\nabla\Psi. \tag{3.7a}$$

Substituting equations (3.4) and (3.5a) in equation (3.7a), we get

$$\nabla A_0 = -\frac{\Omega^F}{2\pi}\nabla\Psi - \frac{2}{\tilde{\omega}^2}\left(\frac{\dot{\Psi}}{4\pi}\right)\mathbf{m} + \frac{2}{\tilde{\omega}^2}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{v}^P \times \mathbf{m} + \dot{\mathbf{A}}. \tag{3.7b}$$

¿From equation (2.7d) we have

$$4\pi \mathbf{j} = \frac{1}{\alpha}\nabla \times (\alpha \mathbf{B}) - \frac{1}{\alpha}\dot{\mathbf{E}} + \frac{1}{\alpha}(\mathbf{E}\cdot\nabla\omega)\mathbf{m}. \tag{3.8a}$$

and, substituting equations (2.5c), (2.5d), and (3.7a) into equation (3.8a), we get

$$4\pi j^T = 4\pi \mathbf{j}\cdot\frac{\mathbf{m}}{\tilde{\omega}}$$

$$= -\frac{\tilde{\omega}}{2\pi\alpha}\nabla\cdot\left(\frac{\alpha}{\tilde{\omega}^2}\nabla\Psi\right) + \frac{\tilde{\omega}}{2\pi\alpha^2}\nabla\omega\cdot(\omega - \Omega^F)\nabla\Psi + \frac{2}{\alpha^2\tilde{\omega}}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\nabla\omega\cdot\mathbf{v}^P\times\mathbf{m}$$

$$-\frac{2}{\alpha^2\tilde{\omega}}\left(\frac{\dot{\Psi}}{4\pi}\right)\nabla\omega\cdot\mathbf{m} + \frac{\tilde{\omega}}{\alpha^2}\nabla\omega\cdot\dot{\mathbf{A}} - \frac{1}{\alpha\tilde{\omega}}\dot{\mathbf{E}}\cdot\mathbf{m}. \tag{3.8b}$$

¿From equations (2.7a) and (3.4) we also have

$$4\pi\rho_e = \nabla\cdot\mathbf{E}^P = \nabla\cdot\left\{\frac{\omega - \Omega^F}{2\pi\alpha}\nabla\Psi + \frac{2}{\alpha\tilde{\omega}^2}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{v}^P\times\mathbf{m}\right\}. \tag{3.9}$$

(b) Boundary Conditions at the Horizon

¿From equations (2.5a), (2.8c), (2.9), and (3.4), we get

$$\mathbf{E}^H = -\frac{2}{\tilde{\omega}}\left(\frac{\dot{\Psi}}{4\pi}\right)\mathbf{e}_{\hat{\varphi}} + \left\{\frac{\Omega^H - \Omega^F}{2\pi}\mathbf{e}_{\hat{\lambda}}\cdot\nabla\Psi - \frac{2}{\tilde{\omega}}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{n}\cdot\mathbf{v}^P\right\}\mathbf{e}_{\hat{\lambda}}, \tag{3.10}$$

which must be equal to equation (2.11b). Therefore, we get,

$$\alpha\mathbf{n}\cdot\nabla\Psi = \dot{\Psi}, \tag{3.11a}$$

and (see MT, eq. [5.10]),

$$I - \frac{\dot{\Phi}}{4\pi} = \frac{\Omega^H - \Omega^F}{2(1 + \mathbf{n}\cdot\mathbf{v}^P)}\tilde{\omega}^2 B_\perp, \tag{3.11b}$$

where equation (2.11a) was also used. Equations (3.10) and (3.11) are more general forms of paper I equations (4.7) and (4.8).

Equation (3.11a) enables us to write equations (2.10) and (2.11b) as,

$$\mathbf{B}^H = -\frac{2}{\tilde{\omega}}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{e}_{\hat{\varphi}} - \frac{2}{\tilde{\omega}}\left(\frac{\dot{\Psi}}{4\pi}\right)\mathbf{e}_{\hat{\lambda}}, \tag{3.12a}$$

and

$$\mathbf{E}^H = -\frac{2}{\tilde{\omega}}\left(\frac{\dot{\Psi}}{4\pi}\right)\mathbf{e}_{\hat{\varphi}} + \frac{2}{\tilde{\omega}}\left(I - \frac{\dot{\Phi}}{4\pi}\right)\mathbf{e}_{\hat{\lambda}}. \tag{3.12b}$$

Therefore, $\mathbf{E}^H$ and $\mathbf{B}^H$ have rotated by the angle χ (see figure 1 of paper I),

$$\chi \equiv \arctan \frac{\dot{\Psi}/4\pi}{I - (\dot{\Phi}/4\pi)}. \tag{3.13}$$

Equations (3.12) and (3.13) have not been changed from paper I equations (4.9) and (4.10).

In paper I we assumed that a small fraction of the field lines can still penetrate the strong field region anchored on the accreting matter, and employed the boundary condition (paper I, eq. [4.6]),

$$v_{inf} \to -\tilde{\delta}, \tag{3.14}$$

where $\tilde{\delta} = 1$ at $\theta \approx \pi/2$, and 0, otherwise. Condition (3.14) meant that, if possible, the field lines can penetrate the strong field region only through the equatorial plane and that they will do so by being dragged by freely falling matter.

In a sense condition (3.14) was too artificial and confined naturally to the equatorial plane of the black hole. In this paper we introduce a much more general condition

$$\mathbf{n} \cdot \mathbf{v}^P \to -1, \tag{3.15}$$

which is natural because everything near the horizon will freely fall toward the hole with speed close to that of light. Moreover, equation (3.5*b*) and (3.11*a*) strongly imply that the condition (3.15) is satisfied.

With condition (3.15), equation (2.12) becomes,

$$-\alpha \mathbf{S}_J \cdot \mathbf{n} \to \frac{\Delta \dot{J}_-}{\Delta \Sigma} = -\frac{\Omega^H - \Omega^F}{4\pi(1 + \mathbf{n} \cdot \mathbf{v}^P)} (\tilde{\omega} B_\perp)^2 - \frac{E_\perp}{2\pi} \left(\frac{\dot{\Psi}}{4\pi} \right), \tag{3.16a}$$

$$T^H \frac{\Delta \dot{S}^H}{\Delta \Sigma} \to \frac{(\Omega^H - \Omega^F)^2}{4\pi(1 + \mathbf{n} \cdot \mathbf{v}^P)^2} (\tilde{\omega} B_\perp)^2 + \frac{1}{\pi \tilde{\omega}^2} \left(\frac{\dot{\Psi}}{4\pi} \right)^2, \tag{3.16b}$$

and

$$-\alpha \mathbf{S}_E \cdot \mathbf{n} \to \frac{\Delta \dot{M}_-}{\Delta \Sigma} \left(\equiv -\frac{\Delta P}{\Delta \Sigma} \right) = -\frac{\Omega^H - \Omega^F}{4\pi(1 + \mathbf{n} \cdot \mathbf{v}^P)} (\tilde{\omega} B_\perp)^2 \left(\Omega^H - \frac{\Omega^H - \Omega^F}{1 + \mathbf{n} \cdot \mathbf{v}^P} \right)$$
$$+ \left(\frac{\dot{\Psi}}{4\pi} \right) \left\{ \frac{1}{\pi \tilde{\omega}^2} \left(\frac{\dot{\Psi}}{4\pi} \right) - 2\Omega^H \left(\frac{E_\perp}{4\pi} \right) \right\}, \tag{3.16c}$$

which, again, are more general forms of paper I equations (4.11) and (4.12). Notice that if condition (3.15) holds, the denominators in equation (3.16) become 0, which strongly requires that

$$\Omega^F \to \Omega^H \tag{3.17}$$

for all the variables to be finite.

Naturally, if $\mathbf{v}^P = \mathbf{0}$ and all the time-derivative terms set to be equal to 0, all the equations become identical with MT equations (MT, eqs. [5.12], [5.13], and [5.14]),

$$-\alpha \mathbf{S}_J \cdot \mathbf{n} \to \frac{\Delta \dot{J}_-}{\Delta \Sigma} = -\frac{\Omega^H - \Omega^F}{4\pi} (\tilde{\omega} B_\perp)^2, \tag{3.18a}$$

$$T^H \frac{\Delta \dot{S}^H}{\Delta \Sigma} \to \frac{(\Omega^H - \Omega^F)^2}{4\pi} (\tilde{\omega} B_\perp)^2, \tag{3.18b}$$

and

$$-\alpha \mathbf{S}_E \cdot \mathbf{n} \to \frac{\Delta \dot{M}_-}{\Delta \Sigma} \left(\equiv -\frac{\Delta P}{\Delta \Sigma} \right) = \Omega^F \frac{\Delta \dot{J}_-}{\Delta \Sigma}. \tag{3.18c}$$

IV. CONCLUSIONS

In this section we will derive the Grad-Shfranov equation of the force-free magnetosphere of a Kerr black hole and the power output equation.

(a) The Grad-Shafranov Equation

As mentioned in paper I, we cannot get a time-dependent 'stream' equation because all the variables are not functions of Ψ only. In this case, for example, the relation like

$$\mathbf{B}^P \cdot \nabla \Omega^F = 0 \tag{4.1}$$

cannot be satisfied and the Ferraro's law of isorotation breaks down. The magnetic field lines, therefore, will be twisted and wound up in the magnetosphere and form jets in a much more complicated way. We, therefore, analyze a *quasi-stationary* axisymmetric magnetosphere in this subsection.

The definitions, $\nabla I = -2\pi \mathbf{m} \times (\alpha \mathbf{j}^P)$ and $\nabla \Psi = 2\pi \mathbf{m} \times \mathbf{B}^P$ have never been altered and the relation (MT, eq. [6.2]),

$$\mathbf{j}^P = -\frac{1}{\alpha}\frac{dI}{d\Psi}\mathbf{B}^P \tag{4.2}$$

is still valid in this paper. Here we *assume*

$$\mathbf{v}^P \simeq \kappa \mathbf{B}^P \tag{4.3}$$

as in Lovelace *et al.* (1986) and Zhang(1989). Equation (4.3) means that we set $\mathbf{E}^T \simeq \mathbf{0}$ in equation (3.3*c*) in our *quasi-stationary* approach. Notice again that the field lines are anchored on the conducting plasma and move together in this paper.

Then, from equations (1.1*b*), (3.3*b*), (4.2), and (4.3), we have

$$\mathbf{j}^T = \rho_e \mathbf{v}^T - \frac{1}{\alpha}\frac{dI}{d\Psi}\mathbf{B}^T - \rho_e \kappa \mathbf{B}^T. \tag{4.4}$$

Substituting equations (3.8*b*), (3.9), and (4.3) into equation (4.4) and setting time-derivative terms to be equal to zero, we finally reach at

$$\nabla \cdot \left\{ \frac{\alpha}{\tilde{\omega}^2}\left(1 - \frac{\tilde{\omega}^2 G^2}{\alpha^2}\right) \nabla \Psi \right\} + \frac{G}{\alpha}\nabla(G-\omega)\cdot\nabla\Psi + \frac{16\pi^2 I}{\alpha\tilde{\omega}^2}\frac{dI}{d\Psi} = 0, \tag{4.5a}$$

where

$$G \equiv (\omega - \Omega^F) - \frac{2\kappa I}{\tilde{\omega}^2}. \tag{4.5b}$$

The specific derivation of equation (4.5) is summarized in Park(1999).

Equation (4.5) is the Grad-Shafranov equation in this case. It is a useful, true stream equation of the force-free magnetosphere and describes the fully-relativistic electrodynamics of a *quasi-stationary*, axisymmetric magnetosphere around a Kerr black hole. The general form of equation (4.5) was obtained by Nitta *et al.* (1991) for the first time in the 4-dimensional spacetime formulation. Notice, however, that our equation is in a more familiar form in the 3+1-spacetime formulation. The general form of equation (4.5) in the 3+1-spacetime formulation can be found in Beskin and Okamoto (2000) and references therein.

For the magnetosphere of a Schwarzschild black hole, we have $\omega \to 0$ and equation (4.5) becomes

$$\nabla \cdot \left\{ \frac{\alpha}{\tilde{\omega}^2}\left(1 - \frac{\tilde{\omega}^2 G^2}{\alpha^2}\right) \nabla \Psi \right\} + \frac{G}{\alpha}\nabla G\cdot\nabla\Psi + \frac{16\pi^2 I}{\alpha\tilde{\omega}^2}\frac{dI}{d\Psi} = 0, \tag{4.6a}$$

where

$$G \equiv -\Omega^F - \frac{2\kappa I}{\tilde{\omega}^2}. \tag{4.6b}$$

In the weak field limit, we have $\alpha \to 1$, $\tilde{\omega} \to R$, and $\omega \to 0$, and equation (4.5) becomes

$$\nabla \cdot \left\{ \left(\frac{1}{R^2} - G^2 \right) \nabla\Psi \right\} + G\nabla G \cdot \nabla\Psi + \frac{16\pi^2 I}{R^2}\frac{dI}{d\Psi} = 0, \tag{4.7a}$$

where

$$G \equiv -\Omega^F - \frac{2\kappa I}{R^2}. \tag{4.7b}$$

Equation (4.7) can be transformed into

$$(1 - R^2G^2)\Delta^*\Psi - \frac{1}{2R^2}\nabla(R^4G^2) \cdot \nabla\Psi + H\frac{dH}{d\Psi} = 0, \tag{4.8a}$$

where

$$\Delta^*\Psi \equiv R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right) + \frac{\partial^2\Psi}{\partial z^2} \tag{4.8b}$$

and

$$H \equiv RB^{\hat{\phi}}. \tag{4.8c}$$

Equation (4.8) is nothing other than Lovelace *et al.* (1986) equation (87).

Substituting

$$G \equiv \omega - \Omega^F \tag{4.9}$$

into equation (4.5a), we naturally can get the original MT stream equation (MT, eq. [6.4]),

$$\nabla \cdot \left[\frac{\alpha}{\tilde{\omega}^2} \left\{ 1 - \frac{(\omega - \Omega^F)^2\tilde{\omega}^2}{\alpha^2} \right\} \nabla\Psi \right] - \frac{\omega - \Omega^F}{\alpha}\frac{d\Omega^F}{d\Psi}(\nabla\Psi)^2 + \frac{16\pi^2 I}{\alpha\tilde{\omega}^2}\frac{dI}{d\Psi} = 0, \tag{4.10}$$

which was solved numerically by Macdonald(1984).

(b) The Power-Output Equation

Now we return to our axisymmetric, nonstationary model. Another result of this paper is related to the power output through the Blandford-Znajek process (Blandford and Znajek 1977). Consider again an annular tube intersecting the black hole (region A in figure 1 in Park and Vishniac 1988). Let the magnetic flux and the electric flux at the horizon be $\Delta\Psi$ and $\Delta\Phi$, respectively. ¿From equation (3.16c), we get

$$\Delta P = \frac{\Omega^H - \Omega^F}{4\pi(1 + \mathbf{n}\cdot\mathbf{v}^P)}\tilde{\omega}^2 B_\perp \Delta\Psi \left(\Omega^H - \frac{\Omega^H - \Omega^F}{1 + \mathbf{n}\cdot\mathbf{v}^P} \right)$$

$$+2\Omega^H \left(\frac{\Delta\Phi}{4\pi}\right)\left(\frac{\Delta\dot{\Psi}}{4\pi}\right) - \frac{\Delta\Sigma}{\pi\tilde{\omega}^2}\left(\frac{\Delta\dot{\Psi}}{4\pi}\right)^2. \tag{4.11}$$

Notice that the condition

$$\Omega^F \sim \Omega^H, \tag{4.12}$$

is necessary again as in paper I for the total power output generated by equation (4.11) to be finite. Equation (3.15) also suggests condition (4.12) to be satisfied, at least, at the event horizon of the central black hole.

Naturally and again, if $\mathbf{v}^P = \mathbf{0}$ and all the time-derivative terms set to be equal to 0, equation (4.11) becomes identical with (MT, eq. [7.2]),

$$\Delta P = \frac{\Omega^F(\Omega^H - \Omega^F)}{4\pi}\tilde{\omega}^2 B_\perp \Delta\Psi, \tag{4.13}$$

which is maximized for

$$\Omega^F \sim \frac{1}{2}\Omega^H. \tag{4.14}$$

Condition (4.14) seems less realistic than condition (4.12) because, in the case of a Kerr black hole, near and inside the ergosphere the dragging of inertial frames will swing the infalling gas into orbital rotation about the hole. As the accreting matter approaches the horizon, Ω^F must approach Ω^H naturally (see Novikov and Thorne 1973).

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