Journal of Natural Science
Pai Chai University, Korea
Vol. 11, No. 1 : 1 5, 1998

Conditional Yeh-Wiener Integral

Sung Soo Kim
Department of Applied Mathematics, Pai Chai University

조건부 Yeh-Wiener 적분

김 성 수
배재대학교 응용수학과

We introduce the conditional Yeh-Wiener integral, verify the existence of Yeh-Wiener integral, and obtain inversion formulae for conditional Yeh-Wiener integral. Using these, we can evaluate conditional Yeh-Wiender integral.

Yeh-Wiener 측도공간에서 조건부 Yeh-Wiener와 적분이 존재하고 역변환 공식을 얻었다. 이를 이용하면 조건부 Yeh-Wiener 적분을 계산할 수 있다.

Key words : Yeh-Wiener measure space, B - measurable, P_x- integrable, conditional Yeh-Wiener integral.

Ⅰ. Introduction

Let R^2 is two-dimensional Euclidean space, p and q some fixed positive real numbers. Let Q be the set defined by

$$Q = \{(s,t) \in R^2 : 0 \le s \le p,\ 0 \le t \le q\}$$

And Let $C_2(Q)$ be the collection of all real valued continuous functions x defined on Q such that $x(0,t) = x(x,0) = 0$.

We can construct the Yeh-Wiener measure $(C_2(Q), y, m_y)$ space such that a set of points

$$\Pi = \{(s_0, t_0), (s_0, t_1), \cdots, (s_0, t_n), \cdots, (s_m, t_n)\}$$

satisfying $0 = s_0 < s_1 < \cdots < s_m \le p$ and $0 = t_0 < t_1 < \cdots < t_n \le q$.

Let B (R^{mn}) be the σ- algebra of Borel set E in the mn-dimensional Euclidean space R^{mn}, $\mathscr{F}_{mn}$ σ- algebra of sets.

$\{x \in C_2(Q) : (x(s_1, t_1), \cdots, x(s_m, t_n)) \in E\}$,
where, $E \in B(R^{mn})$.

$\mathscr{F} = \bigcup \mathscr{F}_{mn}$, where the union over all partitions of Q. This $\mathscr{F}$ is an algebra of subsets in $C_2(Q)$.

We obtain the σ-algebra y of Caratheodory measurable subsets of $C_2(Q)$ with respect to the outer measure induced by the probability measure on F.

A real valued functional $\mathscr{F}$ on $C_2(Q)$ is said to be Yeh-Wiener measurable if it is y-measurable (measurable trans formation from $(C_2(Q), y)$ into $(R^1, \mathrm{B}(R^1))$). Its integral with respect to m_y, if it exists, is called its Yeh-Wiener integral which is denoted by $E^y(\mathscr{F})$, we write

$$E^y(F) = \int_{C_2(Q)} \mathscr{F}(x)\,dm_y(x)$$

We say that $\mathscr{F}$ is Yeh-Wiener integrable or m_y-integrable when the Yeh-Wiener integral of $\mathscr{F}$, $E^y(\mathscr{F})$, exists and infinite.

The Yeh-Wiener measurability and Yeh-Wiener integrability of a complex valued functional on $C_2(Q)$ are defined in terms of its real and imaginary parts.

Definition 1.1 Let $(C_2(Q), y, m_y)$ be Yeh-Wiener measure space, X a real valued Yeh-Wiener measurable functional on $(C_2(Q), y, m_y)$. Define a probability measure P_x on $(R^1, \mathrm{B}(R^1))$ determined by x as follows :

$$P_x(\mathrm{B}) = m_y(X^{-1}(\mathrm{B}))$$

for every B in $\mathrm{B}(R^1)$.

This P_x is called the probability distribution of X.

Theorem 1.1 Let X and Z be the real valued Yeh-Wiener measurable functions on the Yeh-Wiener space $(C_2(Q), y, m_y)$ with $E^y(|Z|) < \infty$. Then there exists a $\mathrm{B}(R^1)$-measurable and P_x-integrable function f on R^1 such that

$$\int_{X^{-1}(B)} Z(x)\,dm_y(x) = \int_B f(w)\,dP_x(w)$$

for every B in $\mathrm{B}(R^1)$.

If f and g are $\mathrm{B}(R^1)$-measurable and P_x-integrable functions on $(R^1, \mathrm{B}(R^1), P_x)$ which satisfy this expression then $f(w) = g(w)$ for a.e. w in $(R^1, \mathrm{B}(R^1), P_x)$.

(Proof) Let a set function μ on $\mathrm{B}(R^1)$ be

$$\mu(\mathrm{B}) = \int_{X^{-1}(B)} Z(x)\,dm_y(x)$$

for every B in $\mathrm{B}(R^1)$. Then μ is the finite signed measure on $\mathrm{B}(R^1)$ and μ is absolutely continuous with respect to P_x. There exists a real valued $\mathrm{B}(R^1)$-measurable and P_x-integrable function f such that

$$\mu(B) = \int_B f(w)\,dP_x(w)$$

for every B in $\mathrm{B}(R^1)$. Hence

$$\int_{X^{-1}(B)} Z(x)\,dm_y(x) = \int_B f(w)\,dP_x(w)$$

for every B in $\mathrm{B}(R^1)$.

By the hypothesis,

$$\int_B f(w)dP_x(w) = \int_B g(w)dP_x(w)$$

for every B in $\mathcal{B}(R^1)$, and hence

$f(w) = g(w)$ for a.e. w in $(R^1, \mathcal{B}(R^1), P_x)$.

II. Conditional Yeh-Wiener Integral

We introduce the definition of conditional Yeh-Wiener integral and evaluate conditional Yeh-Wiener integral for two Yeh-Wiener integrable functional.

Definition 2.1 Let X and Z be the real valued Yeh-Wiener measurable functions on the Yeh-Wiener measure space $(C_2(Q), y, m_y)$ with $E^y(|Z|)<\infty$. The equivalence class of $\mathcal{B}(R^1)$-measurable and P_x-integrable functions f on R^1 satisfying

$$\int_{X^{-1}(B)} Z(x)dm_y(x) = \int_B f(w)dP_x(w)$$

for every $B \in \mathcal{B}(R^1)$ is called the conditional Yeh-Wiener integral of Z given x and is denoted by $E^y(Z \mid x)$.

We use $E^y(Z \mid X)$ to mean either equivalence class of all functions f or a version in it depending on the contex t.

Thus we have

$$\int_{X^{-1}(B)} Z(x)dm_y(x) = \int_B E^y(Z \mid x)(w)dP_x(w)$$

for every B in $\mathcal{B}(R^1)$.

The notation " $\overset{*}{=}$ " means that the existence of one side in an equality implies that of the other as well as the equality of the two.

Proportion 2.1 Suppose that X and Y are two measurable transformations from $(C_2(Q), y)$ into $(R^1, \mathcal{B}(R^1))$ with $E^y(|Y|)<\infty$. Then for an arbitrary measurable transformation g from $(R^1, \mathcal{B}(R^1))$ into itself, we have

$$E^y((g \cdot x)Y) \overset{*}{=} \int_{R^1} g(w)E^y(Y \mid X)(w)dP_x(w)$$

(Proof) Let μ be a set function on y defined by

$$\mu(B) = \int_B Y(x)dm_y(x)$$

for every $B \in y$.

For the real valued random variables $g \cdot X$ an Y on $(C_2(Q), y, m_y)$

$$\begin{aligned} E^y((g \cdot x)Y) &= \int_{C_2(Q)} g(X(x))Y(x)dm_y(x) \\ &\overset{*}{=} \int_{C_2(Q)} g(X(x))Y(x)d\mu(x) \\ &\overset{*}{=} \int_{R^1} g(w)d\mu \cdot X^{-1}(w) \end{aligned}$$

For every $A \in \mathcal{B}(R^1)$

$$\begin{aligned} \mu \cdot X^{-1}(A) &= \mu(X^{-1}(A)) \\ &= \int_{X^{-1}(A)} Y(x)dm_y(x) \\ &= \int_A E^y(Y \mid X)(w)dP_x(w) \end{aligned}$$

Hence,

$$\frac{d\mu \cdot X^{-1}(w)}{dP_x} = E^y(Y \mid X)(w)$$

We have

$$E^{y}((g\cdot x)Y) \overset{*}{=} \int_{R^1} g(w)E^{y}(Y\mid X)(w)dP_x(w)$$

for every $u\in R^1$.

Now we introduce some properties for the proof of following theorems.

Suppose that X and Y are two measurable transformations from $(C_2(Q), y)$ into $(R^1, \mathcal{B}(R^1))$ with $E^{y}(|Y|)<\infty$, and P_x is absolutely continuous with respect to m on $(R^1, \mathcal{B}(R^1))$.

Let J_{α}^{w} be a function on R^1 defined by

$$J_{\alpha}^{w}(\xi) = \begin{cases} 1/2\alpha, & \text{for } \xi\in(w-\alpha, w+\alpha) \\ 0, & \text{for } \xi\in(w-\alpha, w+\alpha) \end{cases}$$

for every $w\in R^1$, $\alpha>0$.

Substituting J_{α}^{w} for g in proposition 2.1, we have

$$\lim_{\alpha\to 0} E^{y}((J_{\alpha}^{w}\cdot X)Y) = \lim_{\alpha\to 0}\int_{R^1} J_{\alpha}^{w}(x)E^{y}(Y\mid X)(x)\frac{dP_x(x)}{dm}dm(x)$$

Here $E^{y}(Y\mid X)(x)\frac{dP_x(x)}{dm}$ is m- integrable on R^1.

Then there exists a version of $E^{y}(Y\mid X)\frac{dP_x}{dm}$ such that

$$E^{y}(Y\mid X)(w)\frac{dP_x(w)}{dm} = \lim_{\alpha\to 0} E^{y}((J_{\alpha}^{w}\cdot X)Y)$$

for a.e. $w\in(R^1, \mathcal{B}(R^1), m)$. And, by the linearity of E^{y},

$$E^{y}(e^{iux}Y) = E^{y}((\cos ux)Y) + iE^{y}((\sin ux)Y)$$

Applying proposition 2.1, we have

$$E^{y}(e^{iux}Y) = \int_{R^1} e^{iuw}E^{y}(Y\mid X)(w)dP_x(w)$$

Theorem 2.1 Suppose that X and Y are measurable transformations from $(C_2(Q), y, m_y)$ into $(R^1, \mathcal{B}(R^1))$ such that $E^{y}(|Y|)<\infty$, and P_x is absolutely continuous with respect to m on $(R^1, \mathcal{B}(R^1))$. Then there exists a version of $E^{y}(Y\mid X)\frac{dP_x}{dm}$ such that

$$E^{y}(Y(X))(w)\frac{dP_x(w)}{dm} = \lim_{n\to 0}\frac{1}{2}\pi\int_{(-n,n)}\left(1-\frac{|u|}{n}\right)e^{-iuw}E^{y}(e^{iux}Y)dm(u))$$

for a.e. $w\in(R^1, \mathcal{B}(R^1), m)$.

(Proof) For every $u\in R^1$,

$$E^{y}(e^{iux}Y) = \int_{R^1} e^{iuw}E^{y}(Y\mid X)(w)\frac{dP_x(w)}{dm}$$

$E^{y}(Y\mid X)(w)\frac{dP_x(w)}{dm}$ is m- integrable on R^1.

Here $E^{y}(e^{iux}Y)$ is Fourier transform of the m- integrable function $E^{y}(Y\mid X)(w)\frac{dP_x(w)}{dm}$.

Thus the proof is established.

Theorem 2.2 The measurable transformations X, Y, and P_x are as in theorem 2.1. And suppose that $E^{y}(e^{iux}Y)$ is an m- integrable function of $u\in R^1$. Then there exists $E^{y}(Y\mid X)\frac{dP_x}{dm}$ such that

$$E^{y}(Y\mid X)(w)\frac{dP_x(w)}{dm} = \frac{1}{2\pi}\int_{R^1} e^{-iuw}E^{y}(e^{iux}Y)dm(u)$$

for a.e. $w \in (R^1, \mathcal{B}(R^1), m)$.

(Proof) For every $u \in R^1$

$$E^y(e^{iux}Y) = \int_{R^1} e^{iuw} E^y(Y \mid X)(w) \frac{dP_x(w)}{dm} dm(w)$$

Let μ be a set function on $\mathcal{B}(R^1)$ defined by

$$\mu(B) = \int_B E^y(Y \mid X)(w) dP_x(w)$$

for every $B \in \mathcal{B}(R^1)$.

Since, $E^y(Y \mid X)$ is P_x- integrable on R^1, μ is a finite signed measure on $(R^1, \mathcal{B}(R^1))$ which is absolutely continuous with respect to P_x.

Since, $E^y(e^{iux}Y)$ is an m- integrable function of u on R^1, we have

$$\frac{d\mu(w)}{dm} = \frac{1}{2\pi} \int_{R^1} e^{-iuw} E^y(e^{iux}Y) dm(u))$$

for a.e. $w \in (R^1, \mathcal{B}(R^1), m)$.

And then

$$\frac{d\mu(w)}{dm} = E^y(Y \mid X)(w) \frac{dP_x(w)}{dm}$$

We can evaluate conditional Yeh-Wiener integral for Yeh-Wiener integrable function conditioned by $X(x) = x(s,t)$ for $x \in C_2(Q)$ where $Q = [0,s] \times [0,t]$ for some fixed positive real numbers s and t.

Let $Y(x) = \int_0^t \int_0^s \{x(u,v)\}^2 du\, dv$, we can evaluate $E^y[J_\alpha^w(x(s,t))\{x(u,v)\}^2]$, $E^y(Y \mid X)(w)$.

III. Acknowledgment

This work was supported by Paichai University (Central Research Fund, 1996).

IV. References

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